When do we have $a^b > b^a$?
The "mutuabola" and Euler's number e

Oswald Riemenschneider, Hamburg

Some elementary Notes, June 4, 2019, version of October 22, 2019

Introduction

In the following, let a and b always denote *positive* real numbers, i.e. $a, b \in \mathbb{R}_+^*$. As stated in the title, we ask when $a^b > b^a$. In fact an old question, since it was already attacked by Leonhard Euler, at least in the form $a^b = b^a$, and later by many others. In this elementary note, we attempt to give a survey on some geometric visualizations and a few applications of nonelementary analysis.

In the text we will frequently replace the condition $a^b > b^a$ by one of the following *equivalent* conditions (we denote by $\log$ the *natural* logarithm which usually is called $\ln$):

$$\begin{aligned} b \log a &> a \log b \quad , \\ \frac{1}{a} \log a &> \frac{1}{b} \log b \quad , \\ \sqrt[a]{a} &> \sqrt[b]{b} \quad , \\ M(a, b) := a^b - b^a &> 0 \quad . \end{aligned}$$

Clearly, $M(b, a) = -M(a, b)$, such that the inequalities above remain equivalent if one replaces everywhere the $>$–sign by the $<$–sign, and the same holds true for equality. In pictures, we attach to any point (a, b) a $+$-sign, if $a^b > b^a$ or equivalently $M(a, b) > 0$, and similarly with the opposite signature $-$. Since M is a continuous function, the sign is constant by the intermediate value theorem in any domain $G \subset \mathbb{R}_+^* \times \mathbb{R}_+^*$ in which M has no zeros.

1 Some nice geometric proofs without words

In [3], Chakraborty gave a nice visual proof for $\pi^e < e^\pi$ such that - in our notation - the point (e, π) lies in the $+$-region. Although the appearance of the circle number π looks spectacular, it is in fact completely *artificial*, i.e. the same figure shows that $b^e < e^b$ for all $b > e$. This follows just by integrating the monotone decreasing function $1/x$ over the interval $[e, b]$:

$$\log b - 1 = \int_e^b \frac{dx}{x} < \frac{1}{e}(b - e) = \frac{b}{e} - 1 , \quad \text{i.e.} \quad e \log b < b \log e .$$

Similarly, integrating from b to e for $b < e$ gives

$$1 - \log b = \int_b^e \frac{dx}{x} > \frac{1}{e}(e - b) = 1 - \frac{b}{e} , \quad \text{i.e. again} \quad e \log b < b \log e .$$

In particular, we always have $\log b < \sqrt[e]{b}$ for all $b \geq 1$. For example, $\log 20 = 2,99573227...$ and $\sqrt[e]{20} = 3,0103860252...$.

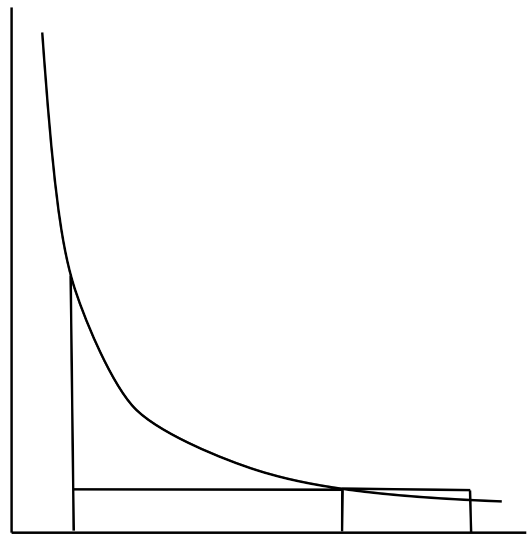

Figure 1.1

It is well known that one can derive the same result(s) from the (also visually evident) inequality $e^x > x + 1$, $x \neq 0$, by setting $x = b/e - 1$. Or, take the equivalent inequality

$$\log x < x - 1 , \quad x \neq 1 ,$$

and set $x = b/e \neq 1$ which leads again to $e \log b < b = b \log e$.[1]

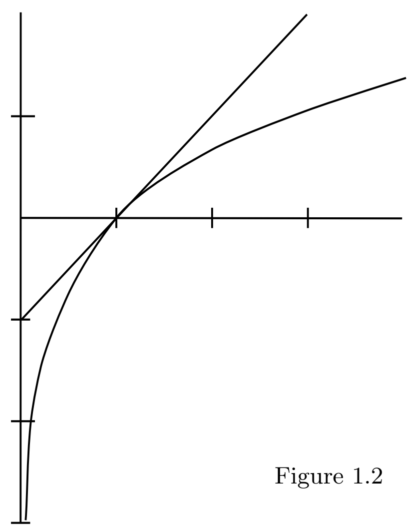

Figure 1.2

The rôle of Eulers number e, on the other hand, is by no means artificial. *If a is a positive real number such that $a^b > b^a$ for all positive real numbers $b \neq a$, then $a = e$.* This follows, e. g., from a simple analysis of the function $x \longmapsto (\log x)/x$ (see, for more details, the next section). This analysis leads moreover to the insight that the equation $a^b = b^a$ has nontrivial solutions $a \neq b$ only inside

$$\{1 < a < e,\ b > e\} \cup \{1 < b < e,\ a > e\} .$$

[1] Note added July 2019. See also: Ananda Mukherjee and Bikash Chakraborty: *Yet Another Visual Proof that $\pi^e < e^\pi$*. The Mathematical Intelligencer, Volume 41, Number 2, Summer 2019, p. 60.

The complement in the octant $b > a$ decomposes into a convex wedge $B_1^+ := \{(a, b) : e \leq a < b\}$ and a starshaped part B_0^+. Since $(1, 2) \in B_0^+$ and $1^2 < 2^1$, we have $a^b < b^a$ for all $(a, b) \in B_0^+$.

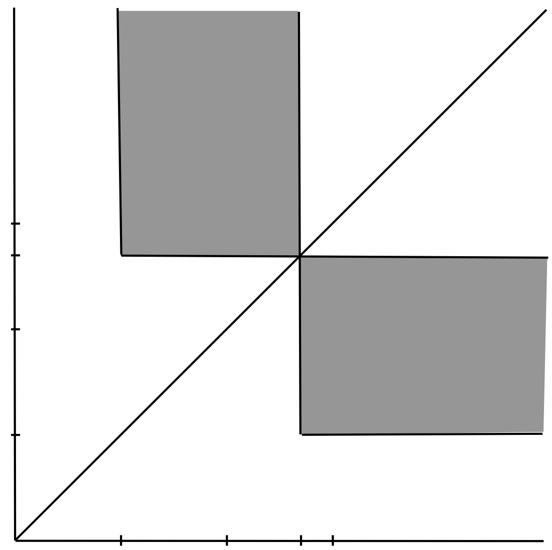

Figure 1.3

On the other hand, we have $(3, 4) \in B_1^+$ and $3^4 = 81 > 64 = 4^3$ and consequently, without further calculations, $a^b > b^a$ for each (a, b) in B_1^+. - Hence, we have the following situation:

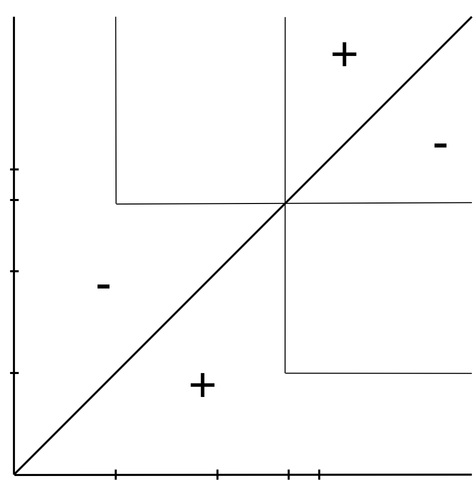

Figure 1.4

There is a nice visualization of $a^b > b^a$ for $e < a < b$ by comparing the slopes of the lines through the origin in $\mathbb{R}^2$ and $(a, \log a)$ resp. $(b, \log b)$ (see [7]).

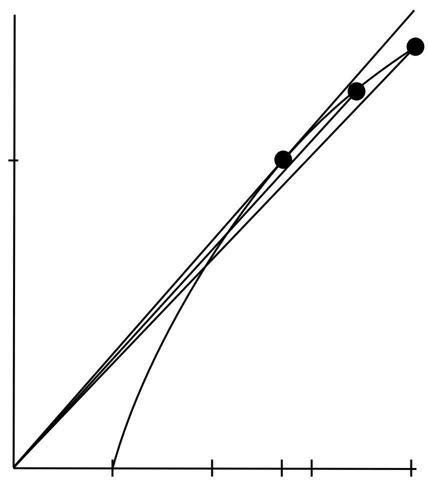

Figure 1.5

As a consequence, we must have nontrivial solutions of $a^b = b^a$ on each line $b = t\,a$, $t \neq 1$. This "Ansatz" is exactly what Euler was doing (see Section 3).

2 The function $(\log x)/x$ and some consequences

The function $f(x) := (\log x)/x$, $x > 0$, has the derivative

$$f'(x) := (1 - \log x)/x^2 ,$$

which is positive on $0 < x < e$ and negative on $x > e$. Hence, f is strictly increasing in the first interval and strictly decreasing in the second interval and thus attains an absolute maximum at e with value $1/e$. A rough sketch of its graph looks as follows:

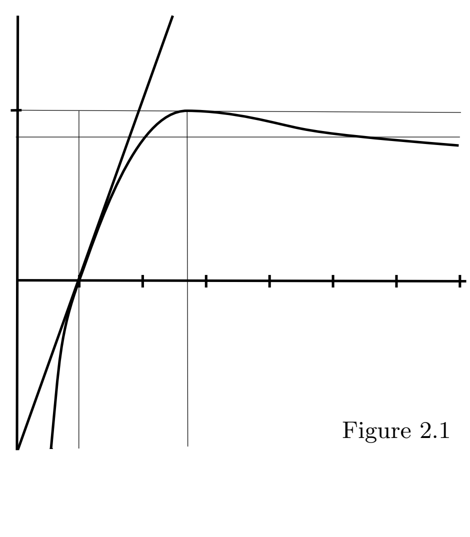

Figure 2.1

Clearly, by L'Hospitals rule,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1/x}{1} = 0 .$$

Therefore, we can state the following Theorem which finally justifies Figures 1.3 and 1.4.

Proposition 2.1 *The equation*

$$f(x) = c, \quad x > 0,$$

has

i) *no solution, if* $c > 1/e$;

ii) *exactly one solution, if* $c = 1/e$, *namely* $x = e$;

iii) *exactly one solution, if* $c < 0$, *namely in* $x < 1$;

iv) *exactly two solutions* a *and* b, *when* $0 \leq c < 1/e$, *with* $1 < a < e$ *and* $b > e$. *In that case*

$$\lim a = e \quad \text{if and only if} \quad \lim b = e$$

and

$$\lim a = 1 \quad \text{if and only if} \quad \lim b = \infty .$$

3 The "mutuabola", and how Euler did it

Of course, since the function f possesses in the interval $(0, 1/e]$ a strongly decreasing inverse coming from $+\infty$ and going down to e, the closure of the set of nontrivial solutions of the equation $a^b = b^a$ is the graph of a monotonically decreasing continuous function $\mu : (1, \infty) \longrightarrow (1, \infty)$ with

$$\lim_{a \searrow 1} \mu(a) = \infty, \quad \lim_{a \to \infty} \mu(a) = 1, \quad \mu(e) = e .$$

We call it the *mutuabola* in accordance with [9].

A nice parametric description goes back already to Euler [5]. He puts (in principle) $b = t\,a$, $0 < t < \infty$, $t \neq 1$, expecting exactly one solution for any given t. In fact, this works quite well, since from the condition $a^b = b^a$ we conclude

$$t\,a \log a = b \log a = a \log b = a(\log a + \log t)$$

and therefore

$$\log a = \frac{\log t}{t - 1} \quad \text{and} \quad \log b = (\log t)\left\{1 + \frac{1}{t - 1}\right\} = \frac{t \log t}{t - 1}$$

or

$$a = t^{1/(t-1)} \quad \text{and} \quad b = t^{t/(t-1)},$$

which we also write in the form

$$a(t) = \exp\left(\frac{\log t}{t - 1}\right), \quad b(t) = \exp\left(\frac{t \log t}{t - 1}\right) .$$

The reflection $(a, b) \longmapsto (b, a)$ is explicitly given by the transformation $t \longmapsto 1/t$ since clearly

$$a(1/t) = b(t) .$$

Therefore, it is sufficient to study the behaviour of the mutuabola analytically only, e.g., in the interval $0 < t \leq 1$.

By L'Hospitals rule,

$$\lim_{t \to 1} \frac{\log t}{t - 1} = \lim_{t \to 1} 1/t = 1 \quad \text{and therefore} \quad \lim_{t \to 1} \frac{t \log t}{t - 1} = 1$$

such that

$$\lim_{t \to 1} a(t) = \lim_{t \to 1} b(t) = e ,$$

as we already know. Further,

$$\lim_{t \searrow 0} (t \log t) = \lim_{t \searrow 0} \frac{\log t}{1/t} = - \lim_{t \searrow 0} \frac{1/t}{1/t^2} = 0$$

such that

$$\lim_{t \searrow 0} b(t) = 1 \text{ and } \lim_{t \to \infty} a(t) = 1 .$$

Finally, by similar arguments,

$$\lim_{t \searrow 0} a(t) = \infty \text{ and } \lim_{t \to \infty} b(t) = \infty .$$

Since each line $b = ta$, $t \neq 1$, cuts the mutuabola in exactly one point, it is divided into two parts with definite sign. Because it hits for small resp. large a regions with a known sign, we find the following distribution of signs on exactly four regions.

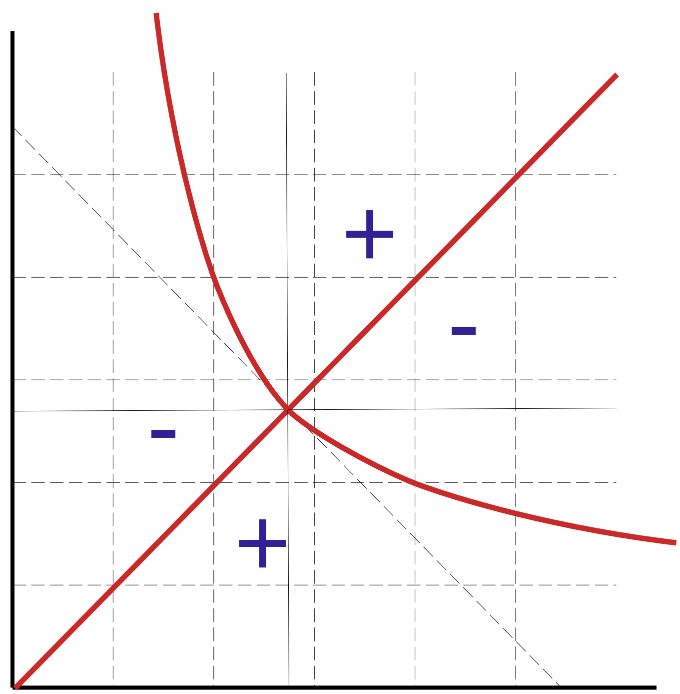

Figure 3.1

For more details, see also [8] and the literature cited therein.

We note two elementary characterizations of Euler's e.

Proposition 3.1 i) *If a is a positive real number such that $a^b > b^a$ for all positive real numbers $b \neq a$, then $a = e$;*

ii) $e = \min \{a > 0 : a^b > b^a \text{ for all } b > a\}$, *more precisely:*

$$\{ a \in \mathbb{R}_+^* : a^b > b^a \text{ for all } b > a \} = [e, \infty) .$$

4 Another geometric interpretation

Starting with the obvious identity

$$\frac{1}{a}\log a - \frac{1}{b}\log b = \left(1 - \frac{a}{b}\right)\left(\frac{1}{a}\log a - \frac{\log b - \log a}{b - a}\right)$$

we get another geometric visualization of the results obtained so far. Assuming without loss of generality that $a < b$, we obtain the correct signature also by looking at

$$\frac{1}{a}\log a - \frac{\log b - \log a}{b - a}\,.$$

If for $a > 1$ we draw a halfline from the origin through $(a, \log a)$ it will cut the graph of the logarithm function in another point $(b_0, \log b_0)$ where $b_0 > a$ if $a < e$, $b_0 < a$ if $a > e$ and $b_0 = a$ if $a = e$. Of course, if $a \neq e$, b_0 is the uniquely determined real number different from a such that $a^{b_0} = {b_0}^a$.

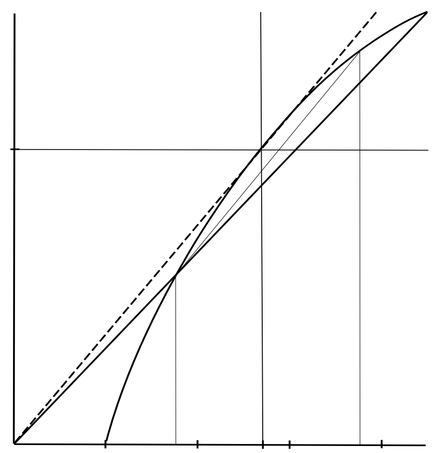

Figure 4.1

Moreover, if $a < b < b_0$, one can directly see that

$$\frac{1}{a}\log a < \frac{\log b - \log a}{b - a}\,,$$

and for $b > b_0$ we have the opposite inequality

$$\frac{1}{a}\log a > \frac{\log b - \log a}{b - a}\,.$$

5 The smoothness of the mutuabola

It is easy to check that the mutuabola is a *smooth* curve at each point different from (e, e): An elementary calculation gives

$$a'(t) = \frac{(t-1) - t\log t}{t\,(t-1)^2}\,a\,(t) \quad \text{and} \quad b'(t) = \frac{(t-1) - \log t}{(t-1)^2}\,b\,(t)\,.$$

Since $a\,(t)/t$ and $b\,(t)$ never vanish, we must have at any point with $a'(t) = b'(t)$ $(= 0)$ that $(t - 1) - t\log t = (t - 1) - \log t$, hence $(t - 1)\log t = 0$ and consequently $t = 1$ which is excluded.

By invoking L'Hospitals rule again, we can establish the smoothness also for the exceptional point. We leave it as an exercise to the reader to prove

$$\lim_{t\to 1} a'(t) = -e/2\,, \quad \lim_{t\to 1} b'(t) = e/2\,.$$

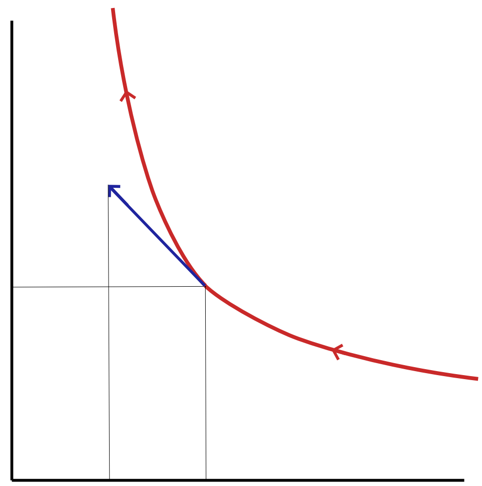

Figure 5.1

There is, however, a conceptual argument in [8] which we want to repeat here. Put

$$F(x, y) = \frac{\log x}{x} - \frac{\log y}{y}$$

such that

$$F_x = \frac{\partial F}{\partial x} = \frac{1 - \log x}{x^2} \quad \text{and} \quad F_y = \frac{\log y - 1}{y^2}\,.$$

So, there is, as we expect, exactly one critical point (at $(x, y) = (e, e)$). This exceptional point, however, is *non-degenerate*: Since $F_{xy} = F_{yx} = 0$, the 2–jet of F at this point is of the form

$$A\,x^2 - A\,y^2 = A\,(x - y)\,(x + y)\,,$$

where

$$A = F_{xx}(e, e) = -F_{yy}(e, e)\,,$$

and it is easily checked that $A \neq 0$. Hence, by the famous "Morse Lemma", the zero set of F is near (e, e) after a local coordinate transformation given by $(x - y)\,(x + y) = 0$ such that the mutuabola is given in this local coordinates by $y = -x$.

6 Rational points on the mutuabola

If we put

$$t = \left(1 + \frac{1}{u}\right)\,, \quad 0 < u < \infty\,,$$

we get another parametrization of the left branch of the mutuabola, namely:

$$a(u) = \left(1 + \frac{1}{u}\right)^u\,, \quad b(u) = \left(1 + \frac{1}{u}\right)^{u+1}\,.$$

In particular, by setting $u := n \in \mathbb{N}^*$, we come up with the well known rational sequences

$$a_n := \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n := \left(1 + \frac{1}{n}\right)^{n+1}$$

converging to e monotonically from below resp. from above. They were (partly) known to Euler and to Daniel Bernoulli [2] and are in fact, as the pairs (a_n, b_n), *the unique rational points* on this branch. (For more historical information, see [4], [11] and [1]. The last paper presents a demonstration for this claim and gives some evidence that the first complete proof can be found in Flechsenhaar [6]).

Taking this fact for granted, we may conclude the following well known:

Lemma 6.1 *There are exactly two* integer *points on the mutuabola, namely* $(4, 2)$ *and* $(2, 4)$.

This, of course, is also evident from our considerations culminating in Figure 3.1. We give here another *argument* (see [10]). Since the given equation is equivalent to $\sqrt[a]{a} = \sqrt[b]{b}$ and the function $\sqrt[x]{x}$ is strictly decreasing for $x > e$ and tends to 1 as $x \to \infty$, it follows that

$$\sqrt[3]{3} > \sqrt[4]{4} = \sqrt[2]{2} > \sqrt[5]{5} > \sqrt[6]{6} > \cdots > \sqrt[1]{1}\,.$$

Literatur

[1] Michael A. Bennett, Bruce Reznick: *Rational Solutions to $x^y = y^{mx}$: A Number-Theoretic Excursion*. The American Mathematical Monthly, Vol. 111, No. 1 (Jan., 2004), pp. 13 - 21.

[2] D. Bernoulli: *Corresp. Math. Phys.*, Vol. 2, P.-H. Fuss, ed. St. Pétersbourg, 1843.

[3] Bikash Chakraborty: *A Visual Proof that $\pi^e < e^\pi$*. The Mathematical Intelligencer, Volume 41, Number 1, Spring 2019, p. 56.

[4] Leonhard Eugene Dickson: *History of the Theory of Numbers*, Vol. II. Carnegie Institution of Washington, Washington 1920, p. 687. Reprint: Dover Books of Mathematics 2005.

[5] L. Euler: *Introductio in analysin infinitorum*, Vol. 2, Chapter 21. Translated and annotated by Ian Bruce. page 564 ff. See also the translation by J. D. Blanton: *Introduction to Analysis of the Infinite*, Springer-Verlag: New York 1990.

[6] A. Flechsenhaar: *Über die Gleichung $x^y = y^x$*. Unterrichtsblätter für Mathematik und Naturwissenschaften, Vol. 17, 1911, 70 - 73.

[7] Charles D. Gallant: *Proof without Words: Comparing B^A and A^B for $A < B$*. Mathematics Magazine, Vol. 64, No. 1 (February 1991), p.31.

[8] A.M. Gleason, R.E. Greenwood, L.M. Kelly, edts.: *The William Lowell Putnam Mathematical Competition*. The Mathematical Association of America, 1980, pp. 538 - 540.

[9] R. Robinson Rowe: *The Mutuabola*. Journal of Recreational Mathematics, Vol. 3, 1970, pp. 176 - 178.

[10] Waclaw Sierpinsky: *Elementary Theory of Numbers*. PWN- Polish Scientific Publishers. Warszawa 1964, pp. 106 - 107. Reprint: North Holland. Amsterdam-New York-Oxford 1988.

[11] M. Sved: *On rational solutions of $x^y = y^x$*. Mathematics Magazine, Vol.63, 1990, pp. 30 - 33.