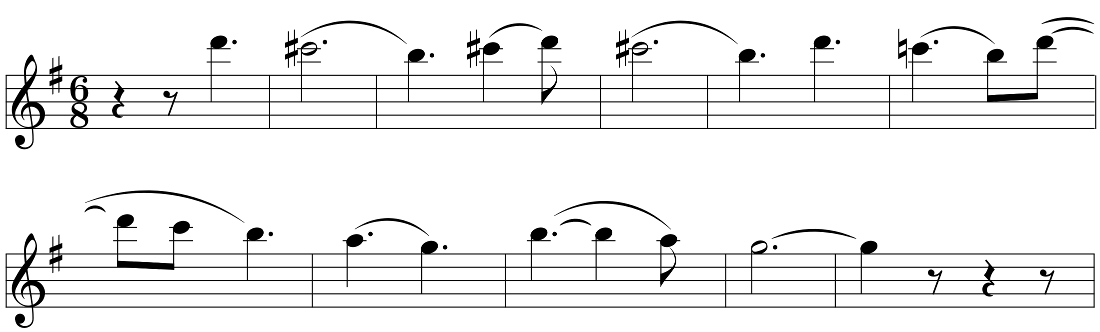
Chapter 19

– Non, dit Rambert avec amertume, vous ne pouvez pas comprendre. Vous parlez le langage de la raison, vous êtes dans l'abstraction.
…
Mais là oû les uns voyaient l'abstraction, d'autres voyaient la vérité.

(Albert Camus, *La peste*)



Chapter 19

Supplement: A short introduction to complex analytic spaces

The main results of former Chapters including the existence of resolutions for all normal surface singularities have been accomplished - among others - by reference to the theory of branched coverings of complex analytic spaces. This and more will be developed here in full generality (but, of course, without complete proofs) thereby giving us the opportunity to fix our vocabulary concerning the language of coherent sheaves that is also heavily used in more advanced parts of the book. The experienced reader may omit reading these notes or consult them if the main text should not be clear enough.

19.1 Holomorphic functions on analytic subsets

As we already have remarked at several occasions in the previous Chapters, we must take into our considerations not only the *geometry* of singularities but also their *function–theoretic* properties. Therefore, we endow a closed analytic subset X of an open set $U \subset \mathbb{C}^n$ (or of any complex analytic manifold) not only with the relative topology coming from U but also with a *complex analytic structure* by calling a function $f : V \to \mathbb{C}$ on an open set $V \subset X$ *holomorphic*, if it is locally induced from a holomorphic function on $\mathbb{C}^n$: *for all* $x \in V$ *there exists a neighborhood* W *of* x *in* U *and a function* $F \in H^0(W, \mathcal{O}_{\mathbb{C}^n})$ *such that* $f_{|X\cap W} = F_{|X\cap W}$.

Under the natural addition and multiplication of complex–valued functions, the set of all holomorphic functions on V forms a ring containing the constants $c \in \mathbb{C}$, hence a $\mathbb{C}$–algebra. We denote it by

$$H^0(V, \mathcal{O}_X) \ \text{or by} \ \mathcal{O}_X(V) \ .$$

If we want to refer to *continuous* functions only, we use the symbols $\mathcal{C}_X$, $\mathcal{C}_X(V)$ and so on. In particular,

$$\mathcal{O}_X(V) \subset \mathcal{C}_X(V)$$

as $\mathbb{C}$–algebras.

By definition, we have a canonical $\mathbb{C}$–algebra–homomorphism

$$H^0(U, \mathcal{O}_{\mathbb{C}^n}) \longrightarrow H^0(X, \mathcal{O}_X)$$

which in general is neither surjective (for $n \geq 2$) nor injective. It is an epimorphism, if $U \subset \mathbb{C}^n$ is a *domain of holomorphy* (in particular, a polydisk or a ball) or more generally a *Stein manifold* (c.f. Chapter 7).

If X is defined as the common zero–set of finitely many holomorphic functions $f_1, \ldots, f_r \in H^0(U, \mathcal{O}_{\mathbb{C}^n})$, then it is clear that each holomorphic function f contained in the ideal $(f_1, \ldots, f_r)$ generated by the elements $f_1, \ldots, f_r$ in $H^0(U, \mathcal{O}_{\mathbb{C}^n})$ vanishes on X. But in general, there exist functions f not belonging to this ideal, yet vanishing on X: just notice that, given $g_1, \ldots, g_s$, we can define $Y = N(g_1, \ldots, g_s)$ also by $Y = N(g_1^{\ell_1}, \ldots, g_s^{\ell_s})$ for any $\ell_j \geq 2$, and not always will g_1 be contained in the ideal $(g_1^{\ell_1}, \ldots, g_s^{\ell_s})$.

The *Rückert Nullstellensatz* (or better: one of its versions; see Section 2 and 7) asserts that there are locally no other functions vanishing on X than those having a power which is contained in $(f_1, \ldots, f_r)$. For a precise statement, we must have the theory of sheaves at our disposal which we begin to develop in the next Section. It is clear from the preceding remarks that the notion of the *radical* of an ideal $\mathfrak{a}$ in a ring R will play the crucial role:

$$\operatorname{rad} \mathfrak{a} = \{ f \in R : \text{it exists } \ell \in \mathbb{N}^* \text{ with } f^\ell \in \mathfrak{a} \}.$$

In the algebraic category, there is an analogous global result called *Hilbert's Nullstellensatz*:

***Theorem 19.1** *Let $P_1, \ldots, P_r$ be complex polynomials in n variables, and denote by X the common zero–set of $P_1, \ldots, P_r$. Then*

$$\{ P \in \mathbb{C}[x_1, \ldots, x_n] : P_{|X} = 0 \} = \operatorname{rad}(P_1, \ldots, P_r)\,\mathbb{C}[x_1, \ldots, x_n].$$

19.2 Germs of holomorphic functions

For local considerations, it is convenient to go over to *germs* of functions. This concept is a special manifestation of forming *inductive limits* that can be explained in a few words as follows:

Let $(A_\iota)_{\iota \in I}$ be a system of sets A_ι indexed by a partially ordered set I, where the ordering is denoted by $\leq$. Thus, $(I, \leq)$ satisfies by definition the two axioms

$$\begin{cases} \iota \leq \iota & \text{for all} \quad \iota \in I \\ \iota \leq \kappa,\ \kappa \leq \lambda \implies \iota \leq \lambda & \text{for all} \quad \iota, \kappa, \lambda \in I. \end{cases}$$

Further, assume that for all $\iota, \kappa \in I$ with $\iota \leq \kappa$ there exists a map $\alpha_{\kappa\iota} : A_\iota \to A_\kappa$ such that the collection $(\alpha_{\kappa\iota})$ is subject to the conditions

$$\begin{cases} \alpha_{\iota\iota} = \mathrm{id}_{A_\iota} & \text{for all} \quad \iota \in I \\ \alpha_{\lambda\kappa} \circ \alpha_{\kappa\iota} = \alpha_{\lambda\iota} & \text{for all} \quad \iota, \kappa, \lambda \in I \quad \text{with} \quad \iota \leq \kappa \leq \lambda. \end{cases}$$

Then the system $(A_\iota, \alpha_{\kappa\iota})$ is called an *inductive system of sets*. Such a collection demands the name of an *inductive system of groups*, if all A_ι have a group structure and the $\alpha_{\kappa\iota}$ are group homomorphisms (and similarly for all other categories).

The *inductive limit* of such a system is defined by an equivalence relation in the disjoint union of the sets A_ι. Call $a_\iota \in A_\iota$ and $a_\kappa \in A_\kappa$ *equivalent*, if there exists a $\lambda \in I$ with $\iota \leq \lambda$, $\kappa \leq \lambda$, and such that

$$\alpha_{\lambda\iota}(a_\iota) = \alpha_{\lambda\kappa}(a_\kappa).$$

When the partially ordered set I satisfies the condition that each pair of elements admits a common larger element, then the definition establishes an *equivalence relation*. Denoting the set of equivalence classes $[a_\iota]$ by

$$\varinjlim_{\iota \in I} A_\iota = A$$

(suppressing the system of maps $\alpha_{\kappa\iota}$ only for economical reasons), we get the inductive limit A together with canonical maps

$$\alpha_\iota : \begin{cases} A_\iota \longrightarrow A \\ a_\iota \longmapsto [a_\iota]. \end{cases}$$

It is absolutely straightforward how to equip the inductive limit A with the structure of a group (ring, algebra, etc.) for inductive systems of groups (rings, algebras, etc.) such that all maps α_ι are homomorphisms.

As an *Example*, which is of paramount importance for the sequel, look at an arbitrary topological space X and associate to any open set $U \subset X$ the $\mathbb{C}$–algebra of continuous functions $\mathcal{C}_X(U)$. The system $\mathcal{U}_x$ of open neighborhoods of a point $x \in X$ is partially ordered by

$$U \leq V \quad \Longleftrightarrow \quad U \supset V$$

and satisfies the extra axiom mentioned above since $U, V \in \mathcal{U}_x$ implies $U \leq U \cap V$, $V \leq U \cap V$. This ordering induces $\mathbb{C}$–algebra homomorphisms

$$r_U^V : \mathcal{C}_X(U) \longrightarrow \mathcal{C}_X(V)$$

by restricting $h \in \mathcal{C}_X(U)$ to V. So, by setting $A_U = \mathcal{C}_X(U)$, $\alpha_{VU} = r_U^V$, we get an inductive system whose limit is usually denoted by

$$\mathcal{C}_{X,x} = \varinjlim_{x \in U} \mathcal{C}_X(U)$$

and whose elements $f_x := [\, f\,]$ are precisely the *germs* of continuous functions $f \in \mathcal{C}_X(U)$ at x.

Since restrictions of *holomorphic* functions $f \in \mathcal{O}_X(U)$ on an open subset U in an analytic subset X to open sets $V \subset U$ are also holomorphic on V by definition, the same procedure as above gives us the *algebra of germs of holomorphic functions* on X at x:

$$\mathcal{O}_{X,x} \,.$$

The analogous local morphism corresponding to the map studied in the second half of Section 1 is now the canonical epimorphism

$$\varepsilon : \mathcal{O}_{\mathbb{C}^n,x} \longrightarrow \mathcal{O}_{X,x}$$

induced by restrictions of representatives of holomorphic function germs in n variables near x to X. If X can be written, in a neighborhood U of x, as the zero–set of functions $f_1, \ldots, f_r \in H^0(U, \mathcal{O}_{\mathbb{C}^n})$ and if I_x denotes the ideal generated by the germs $f_{1,x}, \ldots, f_{r,x}$ in $\mathcal{O}_{\mathbb{C}^n,x}$, then we can state *Rückert's Nullstellensatz* as follows:

***Theorem 19.2** $\quad \ker \varepsilon = \operatorname{rad} I_x$.

In other words: if $g \in H^0(U, \mathcal{O}_{\mathbb{C}^n})$ vanishes on X, then there exists for all $x \in U$ an open neighborhood $V \subset U$ of x, holomorphic functions $h_1, \ldots, h_r \in H^0(V, \mathcal{O}_{\mathbb{C}^n})$ and a positive integer ℓ such that

$$g^\ell(x) = \sum_{\rho=1}^{r} h_\rho(x)\, f_\rho(x) \ \text{ for all } \ x \in V \,.$$

19.3 Presheaves and sheaves on topological spaces

It is quite obvious that there is a unifying concept behind the constructions in Section 2: This procedure works for all systems $S(U)$, $U \subset X$ open, of sets (groups, rings, algebras) together with given maps (homomorphisms of groups, rings, algebras, resp.)

$$r_U^V : S(U) \longrightarrow S(V)\,, \quad V \subset U$$

such that

$$r_U^U = \mathrm{id}_{S(U)} \ \text{ and } \ r_V^W \circ r_U^V = r_U^W\,, \quad W \subset V \subset U\,.$$

Such systems are usually called *presheaves* (of sets, groups, rings, algebras) and are denoted by the symbol S (which is regarded as a functor). By the same recipes, it is clear how to define a presheaf S of *modules* over a presheaf of *rings* R. Although the *restriction maps* r_U^V play an important role, they will in general not be mentioned explicitly, since they are in most examples canonically given (by “obvious” restrictions of objects on U to V).

For any presheaf S on a topological space X, we can form the inductive limit

$$S_x = \varinjlim_{x \in U} S(U)$$

which is also called the *stalk* of the presheaf S at x (having the *germs* of elements in $S(U)$, $x \in U$, as elements). The restriction map $S(U) \to S_x$ is then usually denoted by r_U^x, and, as a rule, we write s_x for the image $r_U^x(s)$, $s \in S(U)$. It goes almost without saying that these stalks carry the same algebraic structure as the defining presheaf does, and that the restriction maps r_x are homomorphisms in the respective category.

Besides the presheaves $\mathcal{C}_X$ on a topological space X and $\mathcal{O}_X$ on an analytic set X, we met implicitly other ones in connection with holomorphic vector bundles $\pi : E \to M$ on a complex analytic manifold M by associating to an open set $U \subset M$ the $\mathcal{O}_M(U)$–module of holomorphic sections over U in E:

$$H^0(U, \mathcal{O}(E)) = \{ s : U \to E \text{ holomorphic with } \pi \circ s = \mathrm{id}_U \} .$$

There are canonical restriction maps r_U^V making this system into a presheaf $\mathcal{O}(E)$ or $\mathcal{O}_M(E)$; we hope that the reader will not be confused by the fact that we use $\mathcal{O}_M(U)$ for the *algebra* of holomorphic functions on U and $\mathcal{O}_M(E)$ for the *presheaf* of holomorphic sections in E. It should be evident that the stalk $\mathcal{O}_M(E)_x$ has a canonical structure of a finitely generated free $\mathcal{O}_{M,x}$–module.

All of these examples satisfy the additional properties of sheaves. A presheaf S is called a *sheaf*, if the following two axioms are fulfilled for S (here $U, U_j, j \in J$, denote open sets in X with $U = \cup U_j$):

i) if $s, t \in S(U)$ and $r_U^{U_j} s = r_U^{U_j} t$ for all $j \in J$, then $s = t$;

ii) if $s_j \in S(U_j)$ is a system of elements with $r_{U_j}^{U_j \cap U_k} s_j = r_{U_k}^{U_k \cap U_j} s_k$ for all $j, k \in J$, then there exists an element $s \in S(U)$ with $r_U^{U_j} s = s_j$ for all j.

Let us briefly sketch the method for associating a sheaf $\check{S}$ to a given presheaf S having the same stalks (and the same algebraic structure). Denote by $\check{S}$ the disjoint union of all stalks S_x, $x \in X$, and engrave a *topology* on $\check{S}$ by using the fundamental system of open sets

$$\{ s_x : x \in U \} , \quad s \in S(U) .$$

Then the canonical projection $\pi : \check{S} \to X$ sending $s_x \in S_x$ to $x \in X$ is a continuous (locally homeomorphic) map, and we can form the set of (continuous) sections

$$\check{S}(U) = \{ \sigma : U \to \check{S} \text{ continuous with } \pi \circ \sigma = \mathrm{id}_U \}$$

which - together with the obvious restriction maps - builds up a sheaf $\check{S}$. Moreover, $\check{S}(U)$ can be equipped with additional algebraic structures if those are given on $S(U)$.

There exist canonical maps (homomorphisms)

$$S(U) \longrightarrow \check{S}(U) , \quad U \subset X \text{ open} ,$$

which are easily seen to be bijective for all U, if and only if S is a sheaf. Therefore, we sometimes identify a sheaf S with the topological space $\check{S} \to X$, and we call $S(U)$ simply the set (group, etc.) of *sections* in S over U, denoting it also by $H^0(U, S)$.

19.4 Analytic sheaves

We are now returning to analytic sets X together with their associated structure sheaves $\mathcal{O}_X$ of germs of holomorphic functions. We also have sometimes to make use of the sheaf $\mathcal{O}_X^*$ of invertible elements in $\mathcal{O}_X$, i.e. of the sheaf of germs of nowhere vanishing holomorphic functions. Of course, $\mathcal{O}_X^*$ has the structure of a sheaf of (multiplicatively written) abelian groups. Further, we will consider the sheaf of *locally constant functions* with values in a *ring* R which is usually denoted by the same symbol R, that is

$$H^0(U, R) = \{ g : U \to R ,\ g \text{ locally constant on } U \} .$$

The latter (especially for $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$) appear in topological arguments.

In analytic contexts, we will be concerned most of the time with *sheaves* S *of modules* over $\mathcal{O}_X$, that is with *analytic sheaves*. So, for all open sets $U \subset X$, the abelian group $S(U)$ of sections in S over U has always the additional structure of an $\mathcal{O}_X(U)$–module. (Notice that $\mathcal{O}_X(U)$ has always a unit, namely the function being identically equal to 1; we therefore assume tacitly that the module structure of $S(U)$ is *unitary*, i.e. $1 \cdot s = s$ for all $s \in S(U)$) . Examples of such analytic sheaves are the sheaf of holomorphic sections in a holomorphic vector bundle $E \to M$ on a complex manifold or *ideal sheaves* I associated to elements $f_1, \ldots, f_r \in H^0(X, \mathcal{O}_X)$ via the presheaf assigning to U the ideal generated in $\mathcal{O}_X(U)$ by the restrictions $f_{\rho|U}$.

The last example is a very special type of sheaves that can be constructed by using the concept of *sheaf homomorphisms*. Such a homomorphism $\varphi : S \to \widetilde{S}$ of sheaves (of abelian groups, say) is a collection of group homomorphisms $\varphi_U : S(U) \to \widetilde{S}(U)$ making all diagrams

$$\begin{tikzcd} S(U) \arrow[r, "\varphi_U"] \arrow[d, "r_U^V"'] & \widetilde{S}(U) \arrow[d, "\widetilde{r}_U^V"] \\ S(V) \arrow[r, "\varphi_V"] & \widetilde{S}(V) \end{tikzcd}$$

commutative. We always denote the canonically induced homomorphism

$$\varinjlim_{x \in U} \varphi_U : S_x = \varinjlim_{x \in U} S(U) \longrightarrow \varinjlim_{x \in U} \widetilde{S}(U) = \widetilde{S}_x$$

by φ_x.

For a sequence

$$(*) \qquad S' \xrightarrow{\varphi} S \xrightarrow{\psi} S''$$

of sheaf homomorphisms, *exactness* at S is measured by the exactness of the sequences of all group homomorphisms

$$S'_x \xrightarrow{\varphi_x} S_x \xrightarrow{\psi_x} S''_x , \quad x \in X .$$

In other words: $(*)$ is *exact* at S, if and only if $\ker \psi_x = \operatorname{im} \varphi_x$ for all $x \in X$.

Denoting by 0 the trivial sheaf of abelian groups, we call φ a *monomorphism*, if

$$0 \longrightarrow S' \xrightarrow{\varphi} S$$

is an exact sequence at S' (the left arrow being self explanatory).

ψ is an *epimorphism*, if the sequence

$$S \xrightarrow{\psi} S'' \longrightarrow 0$$

is exact at S''.

As in the case of groups, exactness of any sequence longer than $(*)$ means exactness at all places where the definition makes sense. In particular, a *short exact sequence* is a sequence of type

$$0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0$$

where exactness holds at S', S and S''.

One of the crucial facts about exact sequences of sheaves is the following

Lemma 19.3 *Let* $0 \to S' \xrightarrow{\varphi} S \xrightarrow{\psi} S''$ *be an exact sequence of sheaves of abelian groups. Then, for all open sets* $U \subset X$, *the associated sequence of groups of sections*

$$0 \longrightarrow H^0(U, S') \longrightarrow H^0(U, S) \longrightarrow H^0(U, S'')$$

is exact.

Regarding $H^0(U, \cdot)$ as a functor assigning abelian groups to sheaves of abelian groups, Lemma 3 is usually phrased by saying that $H^0(U, \cdot)$ is *left–exact*. Before we prove this result, we give an *Example* which shows that $H^0(U, \cdot)$ is in general not right–exact (and hence not exact): Associate to any holomorphic function $f \in H^0(U, \mathcal{O}_X)$ the function $\exp f$ defined by

$$\exp f(x) = e^{f(x)}, \quad x \in U .$$

Then $\exp f \in H^0(U, \mathcal{O}_X^*)$, and the morphisms of abelian groups

$$H^0(U, \mathcal{O}_X) \longrightarrow H^0(U, \mathcal{O}_X^*)$$

are compatible with restrictions. It is clear that the corresponding sequence of sheaves of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

is exact. (Here, $\cdot 2\pi i$ means that locally constant $\mathbb{Z}$–valued functions are multiplied by $2\pi i$). The main point is that nonvanishing holomorphic functions have locally well–defined branches of logarithms. However, if U is not simply connected, it is in general not possible to patch these local logarithms together to get a global logarithm on U. Hence,

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}^*$$

is surjective for all $x \in X$, whereas

$$H^0(U, \mathcal{O}_X) \longrightarrow H^0(U, \mathcal{O}_X^*)$$

is not always an epimorphism.

Proof of Lemma 3. (a) Let $s' \in H^0(U, S')$, and let $s = \varphi_U(s')$ be zero in $H^0(U, S)$. Then $\varphi_x(s'_x) = s_x = 0_x \in S_x$ for all $x \in U$ such that $s'_x \in \ker \varphi_x = (0_x) \subset S_x$. By the first axiom for sheaves, s' must be the zero element.

(b) Using the same argument as in (a) yields $\psi_U \circ \varphi_U = 0$, i.e. $\operatorname{im} \varphi_U \subset \ker \psi_U$. If, on the other hand, $s \in \ker \psi_U$, then $s_x \in \ker \psi_x = \operatorname{im} \varphi_x$ for all $x \in U$. Therefore, we can find for all x a neighborhood $U_x \subset U$ of x and an element $s'_{U_x} \in H^0(U_x, S')$ with $\varphi_{U_x}(s'_{U_x}) = s_{|U_x} := r_U^{U_x} s$. Applying (a) to $U_x \cap U_y$ leads to $s'_{U_x|U_x\cap U_y} = s'_{U_y|U_x\cap U_y}$ for all $x, y \in U$. Hence, the second axiom for sheaves guarantees the existence of an element $s' \in H^0(U, S')$ with $\varphi_U(s') = s$. □

For studying morphisms $G \xrightarrow{\varphi} \widetilde{G}$ of groups (vector spaces, modules, etc.), it is sometimes very useful to go over to the canonical short exact sequences

$$0 \longrightarrow \ker \varphi \longrightarrow G \longrightarrow \operatorname{im} \varphi \longrightarrow 0 ,$$
$$0 \longrightarrow \operatorname{im} \varphi \longrightarrow \widetilde{G} \longrightarrow \operatorname{coker} \varphi \longrightarrow 0 .$$

For a sheaf morphism $\varphi : S \to \widetilde{S}$, these constructions are a little bit more complicated. In view of Lemma 3, only the definition of $\ker \varphi$ is straightforward: just put

$$(\ker \varphi)(U) := \ker \varphi_U ,$$

show that the system $(\ker\,\varphi)\,(U)$ is compatible with the restriction maps r_U^V of S and that this collection of groups and homomorphisms satisfies all axioms of a sheaf which we then call $\ker\,\varphi$. The functorial exact sequences

$$0 \longrightarrow \ker\,\varphi_U \xrightarrow{i_U} S\,(U) \xrightarrow{\varphi_U} \widetilde{S}\,(U)$$

can be viewed as a sequence of sheaves

$$0 \longrightarrow \ker\,\varphi \xrightarrow{i} S \xrightarrow{\varphi} \widetilde{S}$$

which is exact, since forming inductive limits is an exact functor. In particular, we have $(\ker\,\varphi)_x = \ker\,\varphi_x$ for all x.

In order to define $\mathrm{im}\,\varphi$ correctly, we have to do so in such a way that $\mathrm{im}\,\varphi = \widetilde{S}$ for an epimorphism φ. Therefore, it is not correct to define the image by $(\mathrm{im}\,\varphi)\,(U) = \mathrm{im}\,\varphi_U$, which in general is only a presheaf (together with the obvious restriction maps coming from $\widetilde{r}_U^V : \widetilde{S}\,(U) \to \widetilde{S}\,(V)$). However, it is an easy exercise to prove that the associated sheaf, which we call $\mathrm{im}\,\varphi$, has the following natural description:

$$(\mathrm{im}\,\varphi)\,(U) = \left\{ \widetilde{s} \in \widetilde{S}\,(U) : \begin{array}{l} \text{for all } x \in U \text{ there exists a neighborhood} \\ V \subset U \text{ such that } \widetilde{r}_U^V(\widetilde{s}) \text{ is the image} \\ \text{of an element } s_V \in S\,(V) \text{ under } \varphi_V \end{array} \right\}.$$

In particular, $\mathrm{im}\,\varphi_U \subset (\mathrm{im}\,\varphi)\,(U)$ such that there exists a canonical functorial factorization

$$\begin{tikzcd} S\,(U) \arrow[rr, "\varphi_U"] \arrow[dr, "\overline{\varphi}_U"'] & & \widetilde{S}\,(U) \arrow[dl, "j_U"] \\ & (\mathrm{im}\,\varphi)\,(U) & \end{tikzcd}$$

where j_U denotes the natural inclusion $(\mathrm{im}\,\varphi)\,(U) \subset \widetilde{S}\,(U)$. Therefore, we get exact sequences of sheaves

$$0 \longrightarrow \mathrm{im}\,\varphi \xrightarrow{j} \widetilde{S}$$

$$0 \longrightarrow \ker\,\varphi \xrightarrow{i} S \xrightarrow{\overline{\varphi}} \mathrm{im}\,\varphi\,.$$

But, by the definition of $\mathrm{im}\,\varphi$, it is plain that $(\mathrm{im}\,\varphi)_x = \mathrm{im}\,\varphi_x$ and $\overline{\varphi}_x$ is equal to the natural map $\varphi_x : S_x \to \mathrm{im}\,\varphi_x$. Thus, $\overline{\varphi}$ (which most of the time will be denoted by φ, too) is an epimorphism.

The construction of $\mathrm{coker}\,\varphi$ follows the same pattern. Suppose, more generally, we are given a monomorphism

$$0 \longrightarrow S' \xrightarrow{j} \widetilde{S}\,.$$

Then we can identify $S'(U)$ via j_U with a subgroup of $\widetilde{S}\,(U)$ - that is we regard S' as a *subsheaf* of $\widetilde{S}$. The system of quotient groups $\widetilde{S}\,(U)/\,S'(U)$ has a natural structure as a presheaf whose associated sheaf is called $\widetilde{S}/\,S'$. The canonical projections $\widetilde{S}\,(U) \to \widetilde{S}\,(U)/\,S'(U)$ define a sheaf homomorphism $p : \widetilde{S} \to \widetilde{S}/\,S'$ satisfying $\ker\,p_x = S'_x$, $\mathrm{im}\,p_x = (\widetilde{S}/\,S')_x$. Hence,

$$0 \longrightarrow S' \xrightarrow{j} \widetilde{S} \xrightarrow{p} \widetilde{S}/\,S' \longrightarrow 0$$

is an exact sequence, and $(\widetilde{S}/\,S')_x$ is canonically isomorphic to $\widetilde{S}_x/\,S'_x$.

It is clear that for any exact sequence $0 \to S' \xrightarrow{j} \widetilde{S} \xrightarrow{p} S'' \to 0$ the sheaf S'' can be identified with the *quotient sheaf* $\widetilde{S}/\,S'$. Therefore,

$$\mathrm{im}\,\varphi = S/\ker\,\varphi\,,$$

and, by definition,

$$\mathrm{coker}\,\varphi = \widetilde{S}/\,\mathrm{im}\,\varphi\,.$$

The reader may convince himself/herself that these constructions of $\ker \varphi$, $\operatorname{im} \varphi$ and $\operatorname{coker} \varphi$ together with the exact sequences

$$0 \longrightarrow \ker \varphi \longrightarrow S \longrightarrow \operatorname{im} \varphi \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} \varphi \longrightarrow \widetilde{S} \longrightarrow \operatorname{coker} \varphi \longrightarrow 0$$

carry over to sheaves of modules, in particular to analytic module sheaves.

There are other purely algebraic devices to build up new sheaves from old ones. Suppose that S_1 and S_2 are two sheaves of modules over the sheaf R of rings, then

$$(S_1 \oplus S_2)(U) = S_1(U) \oplus S_2(U)$$

defines a new one with $(S_1 \oplus S_2)_x = S_{1,x} \oplus S_{2,x}$, the *direct sum* of S_1 and S_2. This procedure can be extended to more than two summands. In particular, if $S_1 = \cdots = S_r = S$, then $S_1 \oplus \ldots \oplus S_r$ will also be denoted by S^r or $S^{\oplus r}$.

There is also no problem in restricting sheaves S on X to open subsets $V \subset X$. We denote this restriction by $S_{|V}$:

$$S_{|V}(U) = S(U) \text{ for all } U \subset V, \quad U \text{ open}.$$

19.5 Finitely generated sheaves and the permanence principle

Of particular interest in the following are free R–modules R^p. For instance, it is simply checked that the existence of a morphism $\varphi : R^p \to S$ for a sheaf S of modules over R on X is equivalent to selecting p sections $s_1, \ldots, s_p \in S(X)$, namely the φ–images of the basis elements $e_j = (0, \ldots, 1, \ldots, 0)$ in the free $R(X)$–module $R^p(X)$. The subsheaf $S' = \operatorname{im} \varphi \subset S$ is then called the sheaf *generated by the sections* $s_1, \ldots, s_p$. By definition,

$$S'_x = R_x\text{–submodule of } S_x \text{ generated by } s_{1,x}, \ldots, s_{p,x}.$$

We usually write

$$S' = R(s_1, \ldots, s_p);$$

we say that S is *finitely generated*, if $S = R(s_1, \ldots, s_p)$ for some sections $s_1, \ldots, s_p \in H^0(X, S)$.

With the notion of restricting a sheaf, the concept of a sheaf being generated by finitely many sections can easily be *localized*. We say that S is of *finite type*, if it is *locally* finitely generated, i.e. if for all $x^{(0)} \in X$ we can find a neighborhood V of $x^{(0)}$ and an epimorphism

$$R^p_{|V} \longrightarrow S_{|V}$$

(where, of course, p may depend on $x^{(0)}$). Under this assumption, all stalks S_x in a neighbourhood of $x^{(0)}$ are finitely generated R_x–modules. But locally finite generation is more than this: we must be able to choose a system of generators for *all* stalks S_x simultaneously in a “continuous” manner.

The “permanence principle” we have applied already several times in the core part of the manuscript can be formulated in the following way.

Theorem 19.4 *Let S be a sheaf of modules over R of finite type. If $s_1, \ldots, s_p$ are sections in S in a neighbourhood of a point $x \in X$ such that the germs $s_{1,x}, \ldots, s_{p,x}$ generate the stalk S_x over R_x, then $s_{1|V}, \ldots, s_{p|V}$ generate $S_{|V}$ over $R_{|V}$, i.e.: $S_{|V} = R_{|V}(s_1, \ldots, s_p)$, for some neighbourhood V of x.*

Proof. Suppose without loss of generality that all sections $s_1, \ldots, s_p$ are defined on the same neighborhood W of x. Look at the exact sequence

$$0 \longrightarrow R_{|W}(s_1, \ldots, s_p) \longrightarrow S_{|W} \longrightarrow S_{|W}/R_{|W}(s_1, \ldots, s_p) \longrightarrow 0$$

in which

$$\left(S_{|W}/R_{|W}(s_1, \ldots, s_p)\right)_x \cong S_x/(R_x(s_{1,x}, \ldots, s_{p,x})) = 0.$$

Since with S every quotient sheaf S/S' is also of finite type, it remains to show the following Lemma.

Lemma 19.5 *Let S be a sheaf of finite type on X. Then the set of points $x \in X$ in which $S_x = 0_x$ is open in X.*

Remark. The complement of this set is usually called the *support* of the sheaf S (see also Section ??). For sheaves of finite type, this support is hence a *closed* subset of X. For *coherent analytic* sheaves on a complex space X, it is even an *analytic* subset (see Section ??).

Proof of Lemma. Fix an arbitrary point $x \in X$ and a neighbourhood W of x such that S is generated over W by p sections $s_1, \ldots, s_p \in S(W)$. If $S_x = 0_x$, then $s_{1,x} = \cdots = s_{p,x} = 0$ such that there exists an eventually smaller neighbourhood V of x with $s_{1|V} = \cdots = s_{p|V} = 0$. Consequently, $S_{|V} = 0$. □

Remarks. 1. It is not difficult to construct ideals in the sheaf of differentiable functions on $\mathbb{R}$, necessarily not of finite type, that contradict the result of the preceding Theorem.

2. Notice that the theorem and the lemma are obviously equivalent.

19.6 Coherent analytic sheaves

Coherence is a finiteness property of sheaves that has its roots in *Oka's Coherence Theorem* for the sheaf of germs of holomorphic functions on $\mathbb{C}^n$ which can be stated as follows:

***Theorem 19.6** *Let $\varphi : \mathcal{O}^p_{|U} \to \mathcal{O}_{|U}$, $U \subset \mathbb{C}^n$ open, be a sheaf homomorphism. Then the kernel* $\ker \varphi$ *is of finite type.*

Notice that $\ker \varphi_x$ consists of those p–tuples $(g_{1,x}, \ldots, g_{p,x})$ of germs satisfying

$$\sum_{j=1}^{p} g_{j,x} f_{j,x} = 0 ,$$

where the functions $f_j \in H^0(U, \mathcal{O}_{\mathbb{C}^n})$ are the images of the basis elements e_j. In other words: $\ker \varphi$ is equal to the sheaf of germs of *relations* between the generators $f_1, \ldots, f_p$ of the ideal sheaf $\operatorname{im} \varphi$.

Therefore, we say that an analytic sheaf S is of *finite relation type*, if for all homomorphisms

$$\varphi : \mathcal{O}^p_{|U} \longrightarrow S_{|U} , \quad U \subset X ,$$

the kernel $\ker \varphi$ is of finite type. Whereas finite type carries over to homomorphic images, i.e. S/S' is of finite type for all subsheaves $S' \subset S$ if S is of finite type, finite relation type will automatically be transmitted to subsheaves.

An analytic sheaf S is called *coherent*, if it is of finite type and of finite relation type. Oka's Coherence Theorem says that the structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ is coherent (viewed as a sheaf of $\mathcal{O}_{\mathbb{C}^n}$–modules). By the preceding remarks it follows that an ideal sheaf $I \subset \mathcal{O}_{|U}$, $U \subset \mathbb{C}^n$ open, is coherent, if and only if I is of finite type.

Analytic sets in U and coherent ideal sheaves $I \subset \mathcal{O}_{|U}$ are closely related. Suppose that I is such a sheaf, and define the zero–set of I by

$$X = N(I) = \{ x \in U : I_x \neq \mathcal{O}_x \} ,$$

where $\mathcal{O}$ stands for $\mathcal{O}_{|U}$. Then it is immediately checked that on all open sets $V \subset U$, where I_V is generated by functions $f_1, \ldots, f_r \in H^0(V, \mathcal{O})$, we have

$$X \cap V = \{ x \in V : f_1(x) = \cdots = f_r(x) = 0 \} .$$

So, X is an analytic subset of U. On the other hand, if X is analytic in U, it is only locally defined by such coherent ideal sheaves which, however, do not fit together globally in general. Nevertheless, X can always be constructed from a coherent ideal sheaf on U by taking the sheaf of all germs of holomorphic functions vanishing on X. This is a consequence of Rückert's Nullstellensatz and another important Coherence Theorem due to Cartan and Oka:

***Theorem 19.7** *Let $I \subset \mathcal{O}_{|U}$, $U \subset \mathbb{C}^n$ open, be a coherent ideal sheaf. Then its radical* $\operatorname{rad} I$ *is coherent, too.*

Here, $\operatorname{rad} I$ is the sheaf associated to the presheaf of all ideals $\operatorname{rad} I(V)$, $V \subset U$ open, which satisfies the identity

$$(\operatorname{rad} I)_x = \operatorname{rad} I_x \, .$$

Next, we would like to mention *Serre's Coherence Criterion*.

***Theorem 19.8** *If in the exact sequence of $\mathcal{O}_X$–module sheaves*

$$S_1 \longrightarrow S_2 \longrightarrow S \longrightarrow S_3 \longrightarrow S_4$$

the sheaves S_1, S_2, S_3 and S_4 are coherent, then so is S.

Since the trivial sheaf 0 is obviously coherent, this result implies

Corollary 19.9 *In a short exact sequence*

$$0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0$$

all sheaves are coherent, if and only if two of the sheaves S', S, S'' have this property.

In particular, the canonical exact sequence $0 \to S_1 \to S_1 \oplus S_2 \to S_2 \to 0$ shows that direct sums of coherent sheaves are coherent. Moreover, given a morphism $\varphi : S \to \widetilde{S}$ of coherent sheaves, $\operatorname{im} \varphi$ is a subsheaf of $\widetilde{S}$ of finite type, hence coherent, and the short exact sequences for $\ker \varphi$ and $\operatorname{coker} \varphi$ yield the same conclusion for these sheaves.

We want to use these remarks to indicate how one can prove that all *structure sheaves* $\mathcal{O}_X$ of analytic sets $X \subset U \subset \mathbb{C}^n$ are coherent. More generally, we would like to describe the possibility of restricting coherent sheaves S on U to X "coherently": We start with the coherent ideal sheaf $I \subset \mathcal{O}_{|U}$ of all germs of holomorphic functions vanishing on X and form the sheaf

$$I \otimes_{\mathcal{O}_{|U}} S \, ,$$

the tensor product of I and S over $\mathcal{O}_{|U}$, which is associated to the presheaf

$$I(V) \otimes_{\mathcal{O}_{|U}(V)} S(V) \, .$$

Since tensor products commute with inductive limits, we have

$$\bigl(I \otimes_{\mathcal{O}_{|U}} S\bigr)_x \cong I_x \otimes_{|\mathcal{O}_{U,x}} S_x \, .$$

By definition of coherence, S can locally always be written as the cokernel of a map $\mathcal{O}_{|V}^q \to \mathcal{O}_{|V}^p$ such that we get sequences

$$(*) \qquad I_V \otimes_{\mathcal{O}_{|V}} \mathcal{O}_{|V}^q \longrightarrow I_V \otimes_{\mathcal{O}_{|V}} \mathcal{O}_{|V}^p \longrightarrow I_V \otimes_{\mathcal{O}_{|V}} S_{|V} \cong (I \otimes_{\mathcal{O}} S)_{|V} \longrightarrow 0 \, ,$$

where $I_V \otimes_{\mathcal{O}} \mathcal{O}_{|V}^\ell$, $\ell = p, q$, is canonically isomorphic to the ℓ–fold direct sum

$$I_V \oplus \cdots \oplus I_V \, ,$$

hence coherent. Corresponding to $(*)$, we have the following sequence at $x \in V$:

$$(**) \qquad \begin{array}{ccccccc} (I \otimes_{\mathcal{O}} \mathcal{O}^q)_x & \longrightarrow & (I \otimes_{\mathcal{O}} \mathcal{O}^p)_x & \longrightarrow & (I \otimes S)_x & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ I_x \otimes_{\mathcal{O}_x} \mathcal{O}_x^q & \longrightarrow & I_x \otimes_{\mathcal{O}_x} \mathcal{O}_x^p & \longrightarrow & I_x \otimes_{\mathcal{O}_x} S_x & & \end{array}$$

which is induced from the exact sequence

$$\mathcal{O}_x^q \longrightarrow \mathcal{O}_x^p \longrightarrow S_x \longrightarrow 0 \,.$$

Tensoring being a right–exact functor, it follows that the sequence $(**)$ is exact for all $x \in V$. Therefore, the sequence $(*)$ is exact and $I \otimes_{\mathcal{O}_{|U}} S$ is seen to be coherent.

We use the tensor product via the canonical sheaf homomorphism

$$I \otimes_{\mathcal{O}_{|U}} S \longrightarrow S$$

that associates to $\sum f_j \otimes s_j \in I(V) \otimes S(V)$ the element $\sum f_j s_j \in S(V)$. We denote its image sheaf by $I \cdot S$, or $I\,S$ for short, since

$$(I \cdot S)_x \,=\, I_x \cdot S_x \,=\, \text{submodule of } S_x \text{ generated by } I_x \,.$$

By construction, the cokernel

$$S/\,I\,S$$

of this homomorphism is a coherent $\mathcal{O}_{|U}$–module sheaf which is "concentrated" on X:

$$(S/\,I\,S)_x \,=\, S_x/\,I_x S_s \,=\, O_x \text{ for all } x \in U \setminus X \,.$$

In particular, taking $S = \mathcal{O}_{|U}$, we get the coherent $\mathcal{O}_{|U}$–module sheaf $\mathcal{O}_{|U}/\,I$ which can be regarded as the trivial extension of the structure sheaf $\mathcal{O}_X$ to U, since

$$(\mathcal{O}_{|U}/\,I)_x \,\cong\, \begin{cases} \mathcal{O}_{X,x} \,, & x \in X \\ 0_x \,, & x \in U \setminus X \,. \end{cases}$$

It is now an easy matter to derive the coherence of $\mathcal{O}_X$ (as an $\mathcal{O}_X$–module sheaf) from the coherence of $\mathcal{O}_{|U}/\,I$ as $\mathcal{O}_{|U}$–module sheaf. Moreover, for the coherent analytic sheaf S on U, the quotient $S/\,I\,S$ inherits the structure of a coherent sheaf of $(\mathcal{O}_{|U}/\,I)$–modules. It is clear that for any two open sets $W_1,\ W_2 \subset U$ with $W_1 \cap X = W_2 \cap X = V$ the groups of sections

$$H^0(W_1,\, S/\,I\,S) \text{ and } H^0(W_2,\, S/\,I\,S)$$

are identical as $H^0(W_1 \cap W_2,\, \mathcal{O}_{|U}/\,I) = H^0(V,\, \mathcal{O}_X)$–modules. Hence, there exists a sheaf $\overline{S}$ of $\mathcal{O}_X$–modules on X with

$$\overline{S}_x \,=\, S_x/\,I_x S_x \,, \quad x \in X \,,$$

which is called the *analytic restriction* of S to X. As a rule, we denote this analytic sheaf by $S_{|X}$. From the preceding remarks, one can also deduce that $S_{|X}$ is a coherent sheaf on X.

Let us summarize some of the previous results and a few of their immediate consequences in

Theorem 19.10 *Let $X \subset U \subset \mathbb{C}^n$ be a closed analytic subset. Then the structure sheaf $\mathcal{O}_X$ is coherent. Kernels, images and cokernels of homomorphisms between coherent analytic sheaves are coherent. In particular, an analytic sheaf S on X is coherent, if and only if it is locally presentable, i.e. if it coincides with the cokernel of a homomorphism*

$$\mathcal{O}_{|V}^q \longrightarrow \mathcal{O}_{|V}^p \,, \quad V \subset X \ \textit{open} \,.$$

For two coherent analytic sheaves $S_1,\ S_2$ on X, their direct sum $S_1 \oplus S_2$ and their tensor product $S_1 \otimes S_2$ are coherent. Further, if I is a sheaf of ideals in $\mathcal{O}_X$ of finite type and if S is coherent and analytic, then so is $I\,S$.

Only the statement about the tensor product needs verification. However, for this one can follow exactly the same pattern of reasoning as in the case of the tensor product $I \otimes_{\mathcal{O}} S$, since we used there only the coherence of I, S and of the structure sheaf $\mathcal{O}$.

In the rest of our manuscript, we adopt the widespread behaviour to call a sheaf of $\mathcal{O}_X$–modules simply an $\mathcal{O}_X$–module. Similarly, we use the words *ideal* or *ideal sheaf* synonymously for a sheaf of ideals in $\mathcal{O}_X$.

19.7 Analytic sets and supports of coherent analytic sheaves

Since we now have available the notion of holomorphic functions on a closed analytic subset $X \subset U \subset \mathbb{C}^n$, it is straightforward to give a definition for a closed subset A of X to be analytic in X: locally in X, the set A has to be the precise zero–set of finitely many holomorphic functions on X. This condition is, of course, equivalent to the following:

i) A is a closed in U and contained in X;

ii) A is a closed analytic subset of U.

So, as sets, we do not find new objects. But, from the sheaf–theoretical point of view, we must carefully distinguish between ideals $J \subset \mathcal{O}_X$ and $\widetilde{J} \subset \mathcal{O}_{|U}$ defining A. If the set X itself is given by the coherent ideal $I \subset \mathcal{O}_{|U}$ of all holomorphic functions vanishing on X, we obtain such a defining ideal $\widetilde{J} \subset \mathcal{O}_{|U}$ from a defining ideal $J \subset \mathcal{O}_X$ by taking the kernel of the composite map

$$\mathcal{O}_{|U} \longrightarrow \mathcal{O}_{|U}/\, I \xrightarrow{\sim} \mathcal{O}_X \longrightarrow \mathcal{O}_X/\, J \ ,$$

where we identify $\mathcal{O}_X$ with its trivial $\mathcal{O}_{|U}$–coherent extension to U. Hence, denoting by

$$\varepsilon_x : \mathcal{O}_{U,x} \longrightarrow \mathcal{O}_{X,x}$$

the canonical epimorphism, we have for all $x \in X$:

$$\widetilde{J}_x = \varepsilon_x^{-1}(J_x) \ , \quad \varepsilon_x(\widetilde{J}_x) = J_x \ ,$$

and consequently, by an easy exercise:

$$\mathrm{rad}\ \widetilde{J}_x = \varepsilon_x^{-1}(\mathrm{rad}\ J_x) \ , \quad \varepsilon_x(\mathrm{rad}\ \widetilde{J}_x) = \mathrm{rad}\ J_x \ .$$

Using the standard Rückert Nullstellensatz and the Coherence Theorem of Cartan and Oka, these equations imply:

Theorem 19.11 *Let A be a closed analytic subset of $X \subset U \subset \mathbb{C}^n$. Then the ideal $I_A \subset \mathcal{O}_X$ of germs of holomorphic functions on X vanishing on A is coherent and (locally) identical to the radical of any coherent ideal in $\mathcal{O}_X$ defining A.*

It follows from our definitions that an analytic set $A \subset X \subset U$ can be described as the set of points $x \in X$, where we have

$$\mathcal{O}_{X,x}/\, I_{A,x} \cong (\mathcal{O}_X/\, I_A)_x \neq 0_x \ .$$

This is a special case of the *support* of a coherent $\mathcal{O}_X$–module sheaf S which is defined by

$$\mathrm{supp}\ S = \{\, x \in X : S_x \neq 0_x \,\} \ .$$

One of the most important features of coherent analytic sheaves lies in

Theorem 19.12 *The support of a coherent $\mathcal{O}_X$–module S is a closed analytic subset of X.*

Proof. We work locally at a point $x^{(0)} \in X$ and present S in a neighborhood V of $x^{(0)}$ by the cokernel of a map

$$\varphi : \mathcal{O}_{|V}^q \longrightarrow \mathcal{O}_{|V}^p \ ,$$

where φ is given by a $q \times p$ matrix of holomorphic functions $\varphi_{jk} \in H^0(V, \mathcal{O}_X)$. Consequently, we have for $A = \mathrm{supp}\ S$:

$$(*) \qquad \begin{aligned} A \cap V &= \{\, x \in V : \varphi_x : \mathcal{O}_x^q \to \mathcal{O}_x^p \text{ not surjective} \,\} \\ &= \{\, x \in V : \mathrm{rank}\, (\varphi_{jk}(x))_{\substack{j=1,\ldots,q \\ k=1,\ldots,p}} < p \,\} \ . \end{aligned}$$

Now, if $x^{(0)} \notin A$, then there exists a $p \times p$ submatrix of (φ_{jk}) whose determinant at $x^{(0)}$ is not zero. By continuity of holomorphic functions and the determinant, this remains true in a whole neighborhood of $x^{(0)}$ so that A is closed in X. But, again using $(*)$, A is easily seen to be locally analytic, since $A \cap V$ is the set of points $x \in V$, where all $p \times p$ submatrices of $(\varphi_{jk}(x))$ vanish. $\square$

Theorem 10 is only a very special form of what can be called the *permanence principle* for coherent analytic sheaves: For most "good" properties of modules over local rings, it can be proved that, if a stalk $S_{x^{(0)}}$ of a coherent analytic sheaf S has such a property, all neighboring stalks S_x are equally good, and the "bad" points $x \in X$, where S_x does not satisfy the condition, form an analytic subset. For instance, $x^{(0)} \in X$ being a smooth point can be translated into the purely algebraic statement that the stalk of the structure sheaf at $x^{(0)}$ is a *regular* local ring. The permanence principle related to this property of rings implies that the set $\operatorname{sing} X$ of singular points in X is a closed analytic set.

We will come back to such questions later. For the moment, we draw only some simple conclusions from Theorem 10. So, for instance, we can state:

Corollary 19.13 *Let $\varphi : S' \to S''$ be a morphism of coherent analytic sheaves. Then the following sets are closed and analytic in X:*

$$\{ x \in X : \varphi_x \text{ is not surjective} \},$$
$$\{ x \in X : \varphi_x \text{ is not injective} \},$$
$$\{ x \in X : \varphi_x \text{ is not an isomorphism} \}.$$

Moreover, we can generalize Theorem 10 to

Theorem 19.14 *Let S', S'' be coherent submodules of a coherent $\mathcal{O}_X$–module S. Then*

$$\{ x \in X : S'_x \not\subset S''_x \}$$

is a closed analytic subset of X. In particular, if $S'_{x^{(0)}} \subset S''_{x^{(0)}}$ or $S'_{x^{(0)}} = S''_{x^{(0)}}$ for a point $x^{(0)} \in X$, then $S'_{|V} \subset S''_{|V}$ or $S'_{|V} = S''_{|V}$ in a suitable neighborhood V of $x^{(0)}$.

Proof. Since $S'_x \subset S''_x$ if and only if $S'_x = S'_x \cap S''_x$, we can reduce the problem to the investigation of the set $\{ x \in X : S'_x = S''_x \}$. In fact, using the morphism $\psi : S' \oplus S'' \to S$ defined by $(s', s'') \mapsto s' - s''$ for $s' \in S'(V) \subset S(V)$, $s'' \in S''(V) \subset S(V)$, yields the coherence of the sheaf $\ker \psi$ whose stalks can be identified with the intersections $S'_x \cap S''_x$. Consequently, we write $\ker \psi = S' \cap S''$. Therefore, if we replace S'' by the coherent sheaf $S' \cap S''$, we may assume that $S'' \subset S'$, so that $S'_x \not\subset S''_x \Leftrightarrow S'_x \neq S''_x \Leftrightarrow (S'/S'')_x = S'_x/S''_x \neq 0_x$; i.e. $\{ x \in X : S'_x \not\subset S''_x \} = \operatorname{supp}(S'/S'')$. $\square$

It should be noticed that the result of Theorem 12 remains valid when we only assume that S is of finite relation type, whereas the subsheaves S' and S'' are of finite type; because then S', S'' are automatically coherent as subsheaves of S, $\operatorname{im} \psi$ is of finite type and hence coherent for the same reason, and consequently $\ker \psi$ is coherent. We finally remark that the stalks of the coherent $\mathcal{O}_X$–module $\operatorname{im} \psi$ are the sums $S'_x + S''_x$ of S'_x and S''_x in S_x. Therefore, we put $S' + S'' := \operatorname{im} \psi$.

19.8 The annihilator of coherent sheaves and the generalized Rückert Nullstellensatz

Closely related to an analytic sheaf S on X is an ideal sheaf in $\mathcal{O}_X$ whose stalks are the annihilators of the $\mathcal{O}_{X,x}$–modules S_x:

$$\operatorname{Ann} S_x = \{ f_x \in \mathcal{O}_{X,x} : f_x S_x = 0_x \}, \quad x \in X.$$

We leave it as an exercise to the reader to show that there exists indeed an ideal sheaf $\operatorname{Ann} S \subset \mathcal{O}_X$ having these stalks, if S is of finite type (just notice that any local representative f of an annihilator

$f_{x^{(0)}} \in \mathrm{Ann}\, S_{x^{(0)}}$ has then the property to annihilate all nearby stalks: $f_x S_x = 0_x$ for all x close to $x^{(0)}$). Of course, we have $1_x \in (\mathrm{Ann}\, S)_x$, if and only if $S_x = 0_x$. In other words, if we use the fact that $\mathcal{O}_{X,x}$ is a local ring (with maximal ideal $\mathfrak{m} = \mathfrak{m}_{X,x}$ which is the image of the maximal ideal in $\mathcal{O}_{\mathbb{C}^n,x}$ under the natural epimorphism $\mathcal{O}_{\mathbb{C}^n,x} \to \mathcal{O}_{X,x}$) , we have the equivalence:

$$\begin{aligned} x \in \mathrm{supp}\, S \iff & \quad (\mathrm{Ann}\, S)_x \quad \subset \mathfrak{m}_{X,x} \\ \iff & \mathcal{O}_{X,x}/\,(\mathrm{Ann}\, S)_x \neq \quad 0_x \quad . \end{aligned}$$

Therefore, the following is a stronger version of Theorem 10:

***Theorem 19.15** *For any coherent analytic $\mathcal{O}_X$–module S , the annihilator* Ann S *is a coherent ideal in $\mathcal{O}_X$.*

By the very definition of $\mathrm{Ann}\, S_x$, it is quite obvious that each module S_x can also be regarded as an $R_x = \mathcal{O}_{X,x}/\,\mathrm{Ann}\, S_x$–module. Similarly to Section 5, we can think of S as being a sheaf on $A = \mathrm{supp}\, S \subset X$ of modules over the sheaf R of rings on A, and again, it is not difficult to show that S is a coherent R–module. However, R will in general not be the structure sheaf $\mathcal{O}_A$ that we introduced in Section 1. We have seen before that for $f \in (\mathrm{Ann}\, S)\,(V)$ and $x \in (\mathrm{supp}\, S) \cap V$ we must have $f\,(x) = 0$. This implies

$$(\mathrm{Ann}\, S)_x \subset I_{A,x} \ ,$$

if I_A denotes the sheaf of all holomorphic function germs vanishing on A , and hence the existence of an epimorphism

$$R = \mathcal{O}_X/\,\mathrm{Ann}\, S \longrightarrow \mathcal{O}_X/\,I_A = \mathcal{O}_A \ ,$$

which, on the other side, is almost never an isomorphism such that S cannot be regarded as an $\mathcal{O}_A$–module. The precise relationship between Ann S and I_A is the content of the generalized *Rückert Nullstellensatz*:

***Theorem 19.16** *Let S be a coherent analytic sheaf on X . Then*

$$\mathrm{rad}\, \mathrm{Ann}\, S_x = I_{A,x} \ , \quad A = \mathrm{supp}\, S$$

for alle $x \in X$.

Notice that $\mathrm{rad}\, I_{A,x} = I_{A,x}$ and $\mathrm{Ann}\, S_x \subset I_{A,x}$ implies $\mathrm{rad}\, \mathrm{Ann}\, S_x \subset I_{A,x}$. So, Theorem 14 is equivalent to its

Corollary 19.17 *Let S be a coherent analytic sheaf, and let f vanish on $A =$* supp S *(locally near x). Then there exists a number $t \in \mathbb{N}^*$ such that $f_x^t \in \mathrm{Ann}\, S_x$.*

If S is a coherent sheaf of type $\mathcal{O}_X/\,J$ for a finitely generated ideal sheaf J , we have $\mathrm{Ann}\, S_x = J_x$ such that Theorem 14 implies the classical Rückert Nullstellensatz.

19.9 Complex analytic spaces and their reductions

We are now in a position to define ultimately the correct category we want to work in. The last Section together with earlier examples bears evidence of the necessity not to deal completely within the framework of analytic sets carrying their *natural* structure sheaf of holomorphic functions, that is we should allow structure sheaves of type $\mathcal{O}_{|U}/\,J$, $U \subset \mathbb{C}^n$ open, for *arbitrary* ideals J of finite type. On the other hand, we are forced to make this concept to a global abstract one, as we have already seen in the procedure to resolve singularities of curves. The second step uses the same patching idea as in the case of complex analytic manifolds.

To begin with that step, we introduce the notion of a *ringed space* $(X,\, R)$, where X is a topological space (Hausdorff with a countable basis, as we always assume) and R denotes a fixed sheaf of (associative, commutative) rings (with a unit). *Examples* of such spaces are:

1. *Topological* spaces X with $R = \mathcal{C}_X$, the sheaf of germs of *continuous* functions on X.
2. *Differentiable* manifolds X with $R = \mathcal{C}_X^\infty$, the sheaf of germs of *differentiable* functions on X.
3. *Complex analytic* manifolds M with $R = \mathcal{O}_M$, the sheaf of germs of *holomorphic* functions on M.

The space (X, R) is called *locally ringed*, if the stalks R_x, $x \in X$, are local rings (whose maximal ideal is then denoted by $\mathfrak{m}_x$). In most cases, R_x carries the structure of a $\mathbb{C}$–algebra which induces an isomorphism $\mathbb{C} \xrightarrow{\sim} R_x/\mathfrak{m}_x$ from $\mathbb{C}$ to the residue field $R_x/\mathfrak{m}_x$. In particular, $R_x \cong \mathbb{C} \oplus \mathfrak{m}_x$ as $\mathbb{C}$–vector spaces. If (X, R) is a (locally) ringed space, so is $(V, R_{|V})$ for any open subset $V \subset X$.

In order to find the correct notion for morphisms between ringed spaces (X, R) and (X', R'), recall that a continuous map $\varphi : X \to X'$ between complex manifolds is holomorphic, if and only if $f' \circ \varphi$ is holomorphic for any (locally given) holomorphic function f' on X'. But, in our general context, the elements $f' \in H^0(V', R')$ may not be functions; therefore, it makes no sense to speak directly about the composition $f' \circ \varphi$. The idea to circumvent that difficulty is to introduce an abstract sheaf on X' associated to R and φ whose sections are sections in R, and to relate this new sheaf to R' in a functorial way.

To be more precise, we associate to any open set $V' \subset X'$ the ring $R(\varphi^{-1}(V))$. Since the system $\{\varphi^{-1}(V') : V' \text{ open in } X'\}$, is part of the topology of X, it is straightforward to check that $\{R(\varphi^{-1}(V')\}$ defines a sheaf of rings on X. We denote it by

$$\varphi_* R$$

and call it the *direct image* of R under φ. It is clear that locality of R does not transfer to $\varphi_* R$ in general (take, for instance, the constant map $\varphi : \mathbb{C} \to \{0\}$ and observe that $(\varphi_*\mathcal{O}_\mathbb{C})_{\{0\}} = H^0(\mathbb{C}, \mathcal{O}_\mathbb{C})$). However, for all $x \in X$, $x' = \varphi(x) \in X'$, there exists a canonical map $(\varphi_* R)_{x'} \to R_x$ given by

$$(\varphi_* R)_{x'} = \varinjlim_{U' \ni x'} \varphi_* R(U') = \varinjlim_{U' \ni x'} R(\varphi^{-1}(U')) \longrightarrow \varinjlim_{U \ni x} R(U) = R_x .$$

For, e.g., continuous maps $\varphi : X \to X'$ there exists a canonical sheaf homomorphism $\mathcal{C}_{X'} \to \varphi_*\mathcal{C}_X$, namely

$$\begin{cases} \mathcal{C}_{X'}(V') \longrightarrow \mathcal{C}_X(\varphi^{-1}(V')) = (\varphi_*\mathcal{C}_X)(V') \\ \qquad f' \longmapsto f' \circ \varphi . \end{cases}$$

We denote this homomorphism by $\widehat{\varphi}$. In the case of complex manifolds, analyticity of a continuous map $\varphi : X \to X'$ can be expressed by the morphism $\widehat{\varphi}$. Since $\mathcal{O}_X \subset \mathcal{C}_X$, it follows that $\varphi_*\mathcal{O}_X \subset \varphi_*\mathcal{C}_X$. On the other hand, $\widehat{\varphi}$ maps $\mathcal{O}_{X'}$ into $\varphi_*\mathcal{C}_X$, and so, it makes sense to demand that

$$\widehat{\varphi}(\mathcal{O}_{X'}) \subset \varphi_*\mathcal{O}_X .$$

The reader may convince himself that this is precisely the condition for φ to be holomorphic.

In general, $\widehat{\varphi}$ may not be determined by φ. Therefore, we simply assume that $\widehat{\varphi}$ exists. A *morphism of ringed spaces* between (X, R) and (X', R') is a pair $(\varphi, \widehat{\varphi})$ consisting of a continuous map $\varphi : R \to R'$ and a morphism of sheaves of rings $\widehat{\varphi} : R' \to \varphi_* R$. If X and X' are locally ringed spaces, we assume moreover that the composition

$$R'_{x'} \xrightarrow{\widehat{\varphi}_{x'}} (\varphi_* R)_{x'} \longrightarrow R_x , \quad x' = \varphi(x) ,$$

is local, i.e. that it maps the maximal ideal of $R'_{x'}$ into the maximal ideal of R_x. Holomorphic maps between complex manifolds have obviously this property.

We leave it as an exercise to the reader to check that locally ringed spaces together with their morphisms form a category. Isomorphisms are pairs $(\varphi, \widehat{\varphi})$, where φ is a homeomorphism and $\varphi : R' \to \varphi_* R$ is an isomorphism of sheaves of local rings. Moreover, if $R \subset \mathcal{C}_{X'}$, $R' \subset \mathcal{C}_{X'}$, then $\widehat{\varphi}$ is uniquely determined by φ as in the case of holomorphic maps, i.e. $\widehat{\varphi}$ is the restriction of the canonical map $\mathcal{C}_{X'} \to \varphi_*\mathcal{C}_X$ to $R' \subset \mathcal{C}_{X'}$.

To build up complex analytic spaces, we construct *local models* as indicated before: Take any coherent sheaf I_X in the structure sheaf $\mathcal{O}_{|U}$ of an open set $U \subset \mathbb{C}^n$, and denote by X the support of the coherent $\mathcal{O}_{|U}$–module $\mathcal{O}_{|U}/I_X$, which is a closed analytic subset of U. Denote by $\mathcal{O}_X$ the canonical restriction of $\mathcal{O}_{|U}/I_X$ to X satisfying

$$\mathcal{O}_{X,x} = \mathcal{O}_{|U,x}/I_{X,x}, \quad x \in X .$$

Then $\mathcal{O}_X$ is a coherent sheaf of local rings such that the pair $(X, \mathcal{O}_X)$ forms a special example of a locally ringed space: a local model of a complex space. One should notice that this symbol is slightly misleading: it is not true that the set X determines the structure sheaf $\mathcal{O}_X$, since $X_1 = X_2$ for two ideals $I_{X_1}, I_{X_2} \subset \mathcal{O}_{|U}$, if and only if $\operatorname{rad} I_{X_1} = \operatorname{rad} I_{X_2}$ by Rückert's Nullstellensatz.

A *complex analytic space*, or *complex space* for short, is by definition a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a model of a complex space. Since coherence is a local property, the *structure sheaf* $\mathcal{O}_X$ is always coherent. Moreover, each stalk $\mathcal{O}_{X,x}$ is an *analytic algebra*, i.e. a quotient of a convergent power series ring by an ideal contained in the maximal ideal $\mathfrak{m}_n$ of $\mathcal{O}_0^{(n)}$.

The concept of coherent analytic module sheaves can easily be introduced in this new set up. All constructions and results presented so far remain valid mutatis mutandis. In particular, an ideal $I \subset \mathcal{O}_X$ is coherent, if and only if it is of finite type, and the support Z of $\mathcal{O}_X/I$ is a closed analytic subset of X. The pair $(Z, \mathcal{O}_Z)$, where $\mathcal{O}_Z$ is the restriction of $\mathcal{O}_X/I$ to Z, is again a complex space. We call it a *closed subspace* of $(X, \mathcal{O}_X)$. *Open subspaces* are, of course, pairs $(V, \mathcal{O}_{|V})$, where V is open in X and $\mathcal{O}_{|V}$ equals the natural restriction of $\mathcal{O}_X$ to V.

A *holomorphic map* between complex spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is just a morphism $(\varphi, \widehat{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces. In particular, if $y = \varphi(x)$, then the natural map

$$\widehat{\varphi}_x : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

is a local algebra homomorphism.

Any complex manifold M can be considered as a complex space $(M, \mathcal{O}_M)$, and holomorphic maps $\varphi : M \to N$ can be extended to holomorphic mappings $(\varphi, \widehat{\varphi}) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$.

We would like to emphasize here that - although we will speak of holomorphic "functions" $f \in H^0(X, \mathcal{O}_X)$ - these sections are not really functions in the usual sense. The main point is that such sections produce indeed continuous functions or - in other words - that there exists a canonical algebra homomorphism

$$\operatorname{red}_X : H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{C}_X) ,$$

but that the map red_X is not always injective.

Let us illustrate this phenomenon by the simplest possible example: We take the structure sheaf $\mathcal{O}_{\mathbb{C}}$ of $\mathbb{C}$ and the coherent ideal I_X generated by the square x^2 of a complex coordinate x; then $X = \operatorname{supp} \mathcal{O}_{\mathbb{C}}/I = \{0\}$, but I_X is not its own radical. In fact, we have $\operatorname{rad} I_X = x\mathcal{O}_{\mathbb{C}} \neq x^2\mathcal{O}_{\mathbb{C}} = I_X$. The structure sheaf of X consists of precisely one stalk $\mathcal{O}_{X,0}$ which is isomorphic to

$$\mathcal{O}_{\mathbb{C},0}/\mathfrak{m}_0^2 \cong \mathbb{C}[x]/x^2 .$$

So, denoting the residue class of x in $\mathcal{O}_{X,0}$ by ε, we have

$$\mathcal{O}_{X,0} = \mathbb{C} \oplus \varepsilon\mathbb{C}$$

with multiplication $(a + \varepsilon b)(c + \varepsilon d) = ac + \varepsilon(bc + ad)$, i.e. $\varepsilon^2 = 0$. We sometimes denote this algebra by $\mathbb{C}[\varepsilon]$ and call it the algebra of *dual numbers*. Clearly, $H^0(X, \mathcal{O}_X) = \mathcal{O}_{X,0}$, $H^0(X, \mathcal{C}_X) \cong \mathbb{C}$, and the canonical map red_X assigns to such a section f_0 the value of any representative $F \in H^0(U, \mathcal{O}_{\mathbb{C}})$ of f_0 at the origin:

$$\mathbb{C}[\varepsilon] \ni a + b\varepsilon \stackrel{\operatorname{red}_X}{\longrightarrow} a \in \mathbb{C} ,$$

such that the kernel of red_X is generated by the nilpotent element ε.

Up to now, we tried hard in the present text to avoid or even to mention such *nonreduced* spaces. The restriction has been accomplished by considering in connection with an analytic subset $A \subset U \subset \mathbb{C}^n$

exclusively the sheaf of *all* holomorphic functions vanishing on A, whereas in the example above the ideal I is not maximal with respect to this property. Although, at first thought, this example might look to be extremely artificial, it is rather natural, if one is studying e.g. the simple map $f : x \mapsto x^2$ in $\mathbb{C}$ and tries to encode the tendency of the one point fiber $f^{-1}(0) = \{ 0 \}$ to split up into two points $f^{-1}(x) = \{ x_1, x_2 \}$ for $x \neq 0$ close to zero, which, of course, cannot be read off the point 0 alone.

We are confronted here with a general paradoxe that flashes up from time to time in every branch of mathematics: The need to enlarge a category C in order to understand what is really going on in C.

To *reduce* a complex space $(X, \mathcal{O}_X)$ means to endow X with a new structure sheaf of *continuous* functions denoted by $\mathrm{red}\, \mathcal{O}_X$ and defined by

$$(\mathrm{red}\, \mathcal{O}_X)(V) = \mathrm{im}\, (H^0(V, \mathcal{O}_X) \stackrel{\mathrm{red}_V}{\longrightarrow} H^0(V, \mathcal{C}_X))\, , \quad V \subset X \text{ open} .$$

Before we can outline the reasons for (X, red_X) being indeed a complex space, we should define the map red_V rigorously: Whenever $f \in H^0(V, \mathcal{O}_X)$ and $x \in X$, we find a germ $F_x \in \mathcal{O}_{\mathbb{C}^n, x}$ which is mapped onto f_x under the epimorphism

$$\mathcal{O}_{\mathbb{C}^n, x} \longrightarrow \mathcal{O}_{\mathbb{C}^n, x} / I_x = \mathcal{O}_{X, x}\ ,$$

if X is realized as the zero–set $N(I)$ locally near x, I a coherent ideal in $\mathcal{O}_{|U}$, $U \subset \mathbb{C}^n$ open. Taking any representative F of F_x near x and restricting F to X gives a continuous function $\overline{f}_x$ on X near x which does not depend on the special choice of F (since the difference of two such representatives vanishes on X). By construction, we have for all functions $\overline{f}_y$, y close to x, the identity

$$\overline{f}_y(y) = \overline{f}_x(y)$$

such that all these local functions $\overline{f}_x$ patch together to a continuous function $\overline{f} = \mathrm{red}_V(f)$ on V, V open in $N(I)$.

It remains to show that red does not depend on the choice of local models. For each complex space X and each point $x \in X$, there exists a canonical sequence of homomorphism

$$\mathbb{C} \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x} / \mathfrak{m}_{X,x}\ .$$

Since the composition is not trivial, it is necessarily an isomorphism. Analytic homomorphisms $\widehat{\varphi}_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ being local $\mathbb{C}$–algebra homomorphisms, the induced field homomorphism in the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[r] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\
\mathbb{C} \cong \mathcal{O}_{Y,y} / \mathfrak{m}_{Y,y} \arrow[r, "\sim"] & \mathcal{O}_{X,x} / \mathfrak{m}_{X,x} \cong \mathbb{C}
\end{tikzcd}$$

is always the identity. Now, for $Y = \mathbb{C}^n$, $y = 0$ and F a representative of a germ F_0, the value of F at 0 can simply be identified with the residue class of F_0 in $\mathcal{O}_{\mathbb{C}^n, 0} / \mathfrak{m}_{\mathbb{C}^n, 0} \cong \mathbb{C}$. Hence,

$$\begin{aligned} \overline{f}\,(x) = \overline{f}_x(x) = F\,(x) &= F_x \bmod \mathfrak{m}_{\mathbb{C}^n, x} \\ &= f_x \bmod \mathfrak{m}_{X,x} =: \overline{f_x}\ , \end{aligned}$$

yielding an intrinsic definition for the map red.

As a next step we want to compute the kernel of red_X. By the last description, $f \in \ker(\mathrm{red}_X)$, if and only if

$$(*) \qquad f_x \in \mathfrak{m}_{X,x} \text{ for all } x \in X\ .$$

Certainly, for each $x \in X$ there exists (locally near x) a finitely generated ideal sheaf $I \subset \mathcal{O}_X$ satisfying $I_x = \mathfrak{m}_{X,x}$ such that

$$x \in \operatorname{supp}(\mathcal{O}_X / I) = N(I) \subset N(f) .$$

Hence, $(*)$ is equivalent to

$$x \in N(f) \text{ for all } x ,$$

which in turn is equivalent to

$$X_x = N(f)_x , \quad x \in X ,$$

where A_x denotes the *germ of a set* $A \subset X$ at x to be defined in exactly the same manner as the germ of a function at x.

On the other hand, the germ of X at x is obviously equal to $N(0)_x$, 0 the trivial function, such that by Rückert's Nullstellensatz we get the necessary and sufficient condition

$$f_x \in \operatorname{rad}(0_x) = \mathfrak{n}_{X,x} , \quad x \in X ,$$

where $\mathfrak{n}_{X,x} = \mathfrak{n}(\mathcal{O}_{X,x})$ is the *nilradical* of $\mathcal{O}_{X,x}$ consisting of all nilpotent elements.

As we have remarked earlier, $\operatorname{rad}(0) =: \mathfrak{n}_X$ is in fact an ideal *sheaf* in $\mathcal{O}_X$ having the stalks $\mathfrak{n}_{X,x}$. By the preceding considerations, it is also clear that

$$(\operatorname{red} \mathcal{O}_X)_x = \mathcal{O}_{X,x} / \mathfrak{n}_{X,x} ,$$

and we have for trivial reasons

$$\operatorname{supp}(\mathcal{O}_X / \mathfrak{n}_X) = X .$$

Thus, $(X, \operatorname{red} \mathcal{O}_X)$ will be a complex analytic space, if the nilradical is of finite type. This, however, is always true according to another classical *Coherence Theorem* to be commented on in the next Section.

19.10 The Coherence Theorem of Cartan and Oka

We formulate the result without proving it in a strong form and will say some words about its consequences.

***Theorem 19.18 (Coherence Theorem of Cartan and Oka)** *Let $(X, \mathcal{O}_X)$ be a complex space, and let $I \subset \mathcal{O}_X$ be a coherent ideal. Then its radical* rad I *is also coherent. In particular, the reduction $(X, \operatorname{red} \mathcal{O}_X)$ exists together with a natural holomorphic map $(X, \operatorname{red} \mathcal{O}_X) \to (X, \mathcal{O}_X)$.*

As an application of Rückert's Nullstellensatz, Theorem 16 is equivalent to

Theorem 19.19 *Let $A \subset X$ be a closed analytic subset of the complex space X. Then the ideal sheaf I_A of all germs of holomorphic functions vanishing on A is coherent.*

Another reformulation is

Theorem 19.20 *A set $A \subset X$ is closed analytic, if and only if to each point $x \in X$ there exists a neighborhood U of x and finitely many elements $f_1, \ldots, f_t \in H^0(U, \mathcal{O}_X)$ such that $A \cap U = N(f_1, \ldots, f_t)$.*

Evidently, a factor ring $A / \mathfrak{a}$ is reduced (in the sense that it has no nontrivial nilpotent elements), if and only if the ideal $\mathfrak{a}$ coincides with its radical $\operatorname{rad} \mathfrak{a}$. This remark implies:
Each closed analytic subset $A \subset X$ carries one (and only one) natural structure $\mathcal{O}_A$ of a reduced complex space:

$$\mathcal{O}_A = (\mathcal{O}_X / I_A)_{|A} .$$

Moreover, the space $(X, \mathcal{O}_X)$ is reduced at all points x, where the canonical homomorphism of coherent $\mathcal{O}_X$–modules

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X / \mathfrak{n}_X$$

is an isomorphism. Thus, we conclude from Corollary 11:

Theorem 19.21 *The set of points, where a complex space $(X, \mathcal{O}_X)$ is not reduced, is a closed analytic subset.*

19.11 The singular locus of complex spaces

We are now in a position to fix our notions of regular and singular points for the rest of the book. A point x in a complex space $(X, \mathcal{O}_X)$ is called *regular* or *smooth* or a *manifold point*, if there exists a neighborhood V of x in X and an open set U in some number space $\mathbb{C}^n$ such that

$$(V, \mathcal{O}_{|V}) \text{ and } (U, \mathcal{O}_{|U})$$

are biholomorphically equivalent (where $\mathcal{O}_{|V}$ denotes the restriction of $\mathcal{O}_X$ to V and $\mathcal{O}_{|U}$ is the restriction of $\mathcal{O}_{\mathbb{C}^n}$ to U), i.e. if there exist holomorphic maps

$$(f, \widehat{f}) : (V, \mathcal{O}_{|V}) \longrightarrow (U, \mathcal{O}_{|U}) , \quad (g, \widehat{g}) : (U, \mathcal{O}_{|U}) \longrightarrow (V, \mathcal{O}_{|V})$$

satisfying

$$g \circ f = \mathrm{id}_V , \quad f \circ g = \mathrm{id}_U$$

and inducing the resp. identical sheaf homomorphisms under the compositions:

$$\underbrace{\mathcal{O}_{|V} \xrightarrow{\widehat{g}} g_*\mathcal{O}_{|U} \xrightarrow{g_*(\widehat{f})} g_*(f_*\mathcal{O}_{|V}) = (g \circ f)_*\mathcal{O}_{|V} = (\mathrm{id}_V)_*\mathcal{O}_{|V}}_{\mathrm{id}_{\mathcal{O}_{|V}}} = \mathcal{O}_{|V}$$

and vice versa. In other words: $f : V \to U$ is a homeomorphism such that for all open sets $W \subset U$ the canonical homomorphism

$$H^0(W, \mathcal{O}_{|U}) \longrightarrow H^0(W, f_*\mathcal{O}_{|V}) \cong H^0(f^{-1}(W), \mathcal{O}_{|V})$$

is a ($\mathbb{C}$–algebra) isomorphism.

Since, by definition, $(U, \mathcal{O}_{|U})$ consists of smooth points only, the set of regular points $\mathrm{reg}\, X$ in $(X, \mathcal{O}_X)$ is a priori an open set. Of course, the *singular locus* is the set of nonsmooth points:

$$\mathrm{sing}\, X = X \setminus \mathrm{reg}\, X .$$

It follows directly from the definition that the structure sheaf must be regular at a smooth point:

$$x \in X \text{ smooth} \implies \mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n,0} =: R_n \text{ for some } n .$$

The opposite is also true as we will outline in the following Section.

Since the algebras R_n are integral domains, they are also reduced. Thus, a complex space is necessarily singular at its nonreduced points. Notice that we have found examples of complex spaces which are nonreduced at all points but smooth (of positive dimension) everywhere after reduction. In particular, we have to be precise about the coherent ideal sheaf I defining an analytic set $A \subset X$ when speaking about regular points of A. It should be clear now that our original definition for a smooth point of an analytic set in $\mathbb{C}^n$ refers to the canonical reduced structure of A (see the remark just before Theorem 19).

Using the same arguments as before it follows at once that $\mathrm{sing}\, X$ is the union of the set of nonreduced points and the singular set of the reduction $\mathrm{red}\, X$. Since analytic subsets of $\mathrm{red}\, X$ are analytic in X (just lift a defining ideal under the epimorphism $\mathcal{O}_X \to \mathcal{O}_X / \mathfrak{n}_X$), it is sufficient to analyze the structure of the singular set in reduced spaces.

Theorem 19.22 *The singular set* $\mathrm{sing}\, X$ *of a reduced complex space* $(X, \mathcal{O}_X)$ *is a nowhere dense closed analytic subset of* X.

Corollary 19.23 *In an arbitrary complex space* X, *the singular locus is closed analytic.*

For the proof of Theorem 20 we will apply a regularity criterion referring to Kähler differentials on complex spaces (see Section 13 and 14).

19.12 Complex analytic singularities

From now on, we use the word (abstract) *complex analytic singularity* as a synonym for a germ of a complex analytic space $(X, \mathcal{O}_X)$ at a point $x \in X$, even if x is a regular point of X (the "regular singularity"). Instead of the correct notation $(X, \mathcal{O}_X, x)$ for such a singularity we use very often the shorter symbol (X, x) or speak about the singularity $x \in X$ tacitly assuming that the structure sheaf $\mathcal{O}_X$ is a priori given. We always think of X as a concrete representative for the germ (X, x) which, however, can be chosen as small as we wish. Mostly, we will assume X to be a *Stein space* (see Chapter 6??).

A *holomorphic map* between singularities (X, x) and (Y, y) is by definition the germ of a holomorphic map

$$(f, \widehat{f}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

with $f(x) = y$. Two singularities (X, x) and (Y, y) are called *biholomorphically equivalent* (or *isomorphic* for short), if there exists a biholomorphic map $(f, \widehat{f})$ for suitably chosen representatives $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. (Follow verbatim the definition of regular points in the preceding Section in order to introduce the concept of biholomorphic maps in full generality).

Whereas the sheaf homomorphism $\widehat{f} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is not determined by the continuous map f underlying a holomorphic map $(f, \widehat{f})$, the opposite is true at least locally: Let X be given by $N(I)$, $I \subset \mathcal{O}_{|U}$ a coherent ideal, U open in $\mathbb{C}^n$, and put similarly $Y = N(J)$, $J \subset \mathcal{O}_{|V}$, $V \subset \mathbb{C}^m$. Let x be a point in X, and let y be its image $f(x)$ such that the local homomorphism $\widehat{f}_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ fits into the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^m,y} \arrow[r, dashed] \arrow[d] & \mathcal{O}_{\mathbb{C}^n,x} \arrow[d] \\ \mathcal{O}_{Y,y} \arrow[r, "\widehat{f}_x"] & \mathcal{O}_{X,x} \end{tikzcd}$$

with surjective vertical arrows. Since $\mathcal{O}_{\mathbb{C}^m,y} \cong \mathbb{C}\; y_1, \ldots, y_m = R_m$ is a *free* algebra in the sense that any local homomorphism $\varphi : R_m \to A$ into an analytic algebra A is completely fixed by the images $\varphi(y_1), \ldots, \varphi(y_m)$, we can easily make the diagram commutative by replacing the dotted arrow by a substitution homomorphism $\widehat{F}_x$ sending y_k to suitable elements $F_k \in \mathcal{O}_{\mathbb{C}^n,x}$, $k = 1, \ldots, m$. Defining $F(x_1, \ldots, x_n) = F(x) = (F_1(x), \ldots, F_m(x))$ and shrinking U if necessary yields a holomorphic map

$$F : U \longrightarrow V \text{ with } F(x) = y ,$$

and if $g_1, \ldots, g_r$ are generators of J near y, then

$$(g_j \circ F)_x = \widehat{F}_x(g_{j,y}) \in I_x ,$$

such that from the coherence of I we may conclude that

$$X = N(I) \subset N(g_1 \circ F, \ldots, g_r \circ F) \subset F^{-1}(N(g_1, \ldots, g_r)) = F^{-1}(Y) .$$

Hence, F restricts to a continuous map $X \to Y$ which we expect to be f.

To prove this, we can regard instead the maps $j \circ F_{|X}$ and $j \circ f$ where $j : Y \to V$ is the natural embedding. In other words: we are reduced to the special case $Y = V$ open in $\mathbb{C}^m$, and all we need is the following Lemma whose proof is left to the reader:

***Lemma 19.24** *Let $(X, \mathcal{O}_X)$ be a complex space, and denote by $\mathrm{Hol}(X, \mathbb{C}^m)$ the set of all holomorphic maps from X to the complex manifold $\mathbb{C}^m$ with the standard complex analytic structure. Then there exists a canonical bijection*

$$\mathrm{Hol}(X, \mathbb{C}^m) \xrightarrow{\sim} H^0(X, \mathcal{O}_X)^m .$$

By way of these considerations we come to the important interplay between local complex analytic geometry and local algebra which can simply be stated as follows.

Theorem 19.25 *By the associations*

$$\left\{ \begin{array}{rcl} \textit{singularity } (X,\, \mathcal{O}_{X,x}) & \longmapsto & \textit{analytic algebra } \mathcal{O}_{X,x} \\ \textit{morphism } (f,\, \widehat{f}) : (X,\, \mathcal{O}_{X,x}) \to (Y,\, \mathcal{O}_{Y,y}) & \longmapsto & \textit{map } \widehat{f}_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \end{array} \right.$$

we get an isomorphism between the categories of (isomorphism classes) of

$$\left\{ \begin{array}{r} \textit{singularities} \\ \textit{with (germs of) holomorphic maps} \\ \textit{as morphisms} \end{array} \right\} \quad \textit{and} \quad \left\{ \begin{array}{l} \textit{analytic algebras} \\ \textit{with local homomorphisms} \\ \textit{as morphisms} \end{array} \right\} .$$

In particular, two singularities $(X,\, x)$ and $(Y,\, y)$ are isomorphic, if and only if $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$. The singularity $(X,\, x)$ is regular, if and only if $\mathcal{O}_{X,x} \cong R_n$ for some $n \geq 0$.

To be sure, the opposite direction $\mathcal{O}_{X,x} \mapsto (X,\, \mathcal{O}_X,\, x)$ is constructed by writing $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^n,0}/\,\mathfrak{a}$, $\mathfrak{a} = (f_{1,0}, \ldots, f_{r,0})$ and putting $X = N\,(f_1, \ldots, f_r)$, $\mathcal{O}_X = \mathcal{O}_{|U}/\,(f_1, \ldots, f_r)$, $f_j \in H^0(U,\, \mathcal{O}_{\mathbb{C}^n})$ representatives for $f_{j,0}$, $j = 1, \ldots, r$.

19.13 Kähler differentials on complex spaces and analyticity of the singular locus

Let us suppose that we are given a complex analytic manifold M with structure sheaf $\mathcal{O}_M$. Our first aim will be to give in that situation different characterizations for the (complex analytic) *tangent bundle* T_M of M. Describing the sheaf of holomorphic sections in T_M in terms of *derivations*, we will be able to define the *tangent sheaf* Θ_X for any complex analytic space $(X,\, \mathcal{O}_X)$ in a purely algebraic way. The sheaf Ω^1_X of *(holomorphic) Kähler differentials of degree 1* shall then be introduced by solving a universal problem with respect to derivatives with values in arbitrary coherent analytic sheaves. The sheaves Ω^p_X of Kähler differentials of degree p are finally easily defined by setting $\Omega^p_X = \Lambda^p \Omega^1_X$. We close the present Section by filling in the remaining step in the proof of analyticity of the singular locus.

So, as we suggested above, let M be an n–dimensional complex analytic manifold. By definition, a *(holomorphic) tangent vector* at the point $x^{(0)} \in M$ is the equivalence class of germs of holomorphic maps

$$v : (\mathbb{C},\, 0) \longrightarrow (M,\, x^{(0)})\,,$$

where equivalence is defined by *equality up to first order*: if locally $(M,\, x^{(0)}) \cong (\mathbb{C}^n,\, 0)$ and $v_j = (v_1^{(j)}, \ldots, v_n^{(j)}) : (\mathbb{C},\, 0) \to (\mathbb{C}^n,\, 0)$, $j = 1,\, 2$, then $v_1 \sim v_2$ if and only if $v_\nu^{(1)} - v_\nu^{(2)} \in \mathfrak{m}_0^2$, $\nu = 1, \ldots, n$, $\mathfrak{m}_0$ the maximal ideal of $\mathcal{O}_{\mathbb{C},0}$. It is a trivial exercise to show that this is an intrinsic definition for M at x. We denote by $T_{M,x^{(0)}}$ the set of these holomorphic tangent vectors to M at $x^{(0)}$. Invoking again a local isomorphism $(M,\, x^{(0)}) \cong (\mathbb{C}^n,\, 0)$, we simply can state that $T_{M,x^{(0)}}$ *carries a natural structure of a complex vector space*. Sending $s \in \mathbb{C}$ to $z = (0, \ldots, 0, s, 0, \ldots, 0)$, the number s in the j–th place, we construct tangent vectors v_j, $j = 1, \ldots, n$, and it is easily seen that $T_{M,x^{(0)}}$ is minimally generated by $v_1, \ldots, v_n$. Hence, $T_{M,x^{(0)}}$ *is a complex vector space of dimension* $n = \dim_{\mathbb{C}} M$.

We next give the (above promised) description of $T_{M,x^{(0)}}$ by algebraic means, i.e. by *derivations*. Whenever $v \in T_{M,x^{(0)}}$ and $f \in \mathcal{O}_{M,x^{(0)}}$, we can form $D_v(f) = f \circ v \in \mathcal{O}_{\mathbb{C},0}$. Clearly, the maps

$$D_v : \left\{ \begin{array}{ccl} \mathcal{O}_{M,x^{(0)}} & \longrightarrow & \mathcal{O}_{\mathbb{C},0} \\ f & \longmapsto & f \circ v \end{array} \right.$$

obey the following conditions:

$$\begin{array}{lll} \text{i)} & D_v(f_1 + f_2) = D_v(f_1) + D_v(f_2) & , \\ \text{ii)} & D_v(cf) = cD_v(f) & , \\ \text{iii)} & D_v(fg) = fD_v(g) + gD_v(f) & , \\ \text{iv)} & D_{v_1+v_2}(f) = D_{v_1}(f) + D_{v_2}(f) & , \\ \text{v)} & D_{cv}(f) = c\,D_v(f) & . \end{array}$$

Consequently, the map

$$(*) \qquad \psi_{x^{(0)}} : T_{M,x^{(0)}} \longrightarrow \mathrm{Der}\,(\mathcal{O}_{M,x^{(0)}}, \mathcal{O}_{M,x^{(0)}})$$

where

$$\mathrm{Der}\,(\mathcal{O}_{M,x^{(0)}}, \mathcal{O}_{M,x^{(0)}}) = \{ D : \mathcal{O}_{M,x^{(0)}} \to \mathcal{O}_{M,x^{(0)}} \text{ satisfying i), ii) and iii)} \}$$

is a linear map of $\mathbb{C}$–vector spaces. Now, it is easy to check that this morphism is injective and

$$\dim_{\mathbb{C}} \mathrm{Der}\,(\mathcal{O}_{M,x^{(0)}}, \mathcal{O}_{M,x^{(0)}}) = n = \dim_{\mathbb{C}} M ,$$

since this $\mathbb{C}$ vector space of derivations is canonically spanned by the partial derivatives

$$\left.\frac{\partial}{\partial x_j}\right|_{x=x^{(0)}} : \mathcal{O}_{M,x^{(0)}} \longrightarrow \mathcal{O}_{M,x^{(0)}} , \quad j = 1, \ldots, n .$$

Therefore, the canonical map in $(*)$ is bijective for all $x^{(0)} \in M$.

Now, defining the *(holomorphic) vector bundle* T_M in the usual way, means precisely the following: Giving a *holomorphic* section $s \in H^0(U, \mathcal{O}_M(T_M))$, the map

$$U \ni x \longmapsto \psi_x(s_x) \in \mathrm{Der}\,(\mathcal{O}_{M,x}, \mathcal{O}_{M,x})$$

is *holomorphic*, i.e. if f is holomorphic in U, then

$$U \ni x \longmapsto ((\psi_x(s_x))\,(f_x))\,(x) \in \mathbb{C}$$

is a holomorphic function on U.

Since the opposite is obviously true, we obtain the following

Theorem 19.26 *The $\mathcal{O}_M$–sheaf $\Theta_M = \mathcal{O}_M(T_M)$ of germs of holomorphic sections in the tangent bundle T_M is locally free of rank $n = \dim_{\mathbb{C}} M$. Each stalk $\mathcal{O}_M(T_M)_{x^{(0)}}$ is canonically isomorphic to the n–dimensional free $\mathcal{O}_{M,x^{(0)}}$–module* $\mathrm{Der}\,(\mathcal{O}_{M,x^{(0)}}, \mathcal{O}_{M,x^{(0)}})$.

Thus, any section $\partial \in H^0(U, \mathcal{O}(T_M))$ can in particular be interpreted as an operator

$$\partial : H^0(U, \mathcal{O}_M) \longrightarrow H^0(U, \mathcal{O}_M)$$

satisfying conditions i), ii) and iii) as above. If U is (biholomorphic to) an open subset in $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$, then the partial derivatives

$$\partial_j = \frac{\partial}{\partial x_j} , \quad j = 1, \ldots, n ,$$

are special objects of this kind, and it is evident from the remarks above that every element ∂ can uniquely be written in the form

$$\partial = \sum_{j=1}^{n} a_j\,\partial_j , \quad a_j \in \mathcal{O}_{\mathbb{C}^n}(U) .$$

We call such elements in $H^0(U, \mathcal{O}_M(T_M))$ *holomorphic vector fields* on U.

Now, by definition, the sheaf Ω^1_M of germs of *holomorphic (Kähler) 1–forms* on M is the $\mathcal{O}_M$*–dual* of Θ_M. It is therefore a locally free sheaf of rank n, as well. At every point $x_0 \in M$, the stalk $\Omega^1_{M,x^{(0)}}$ has a basis $dx_1, \ldots, dx_n$, dual to the partial derivatives $\partial_1, \ldots, \partial_n$ at $x^{(0)}$. These germs extend to sections in $H^0(U, \Omega^1_M)$, also denoted by $dx_1, \ldots, dx_n$, in a coordinate system U with holomorphic coordinates $x_1, \ldots, x_n$, and $H^0(U, \Omega^1_M)$ is a free $\mathcal{O}_M(U)$–module with basis $dx_1, \ldots, dx_n$.

We next associate to any germ $f_x \in \mathcal{O}_{M,x}$ a germ of a holomorphic 1–form $(df)_x$, called the *differential* of f, by associating to each tangent vector $v_x \in \mathrm{Der}\,(\mathcal{O}_{M,x}, \mathcal{O}_{M,x}) = \mathcal{O}_M(T_M)_x$ the holomorphic function germ

$$(df)_x(v_x) = v_x(f_x) .$$

Then d is a $\mathbb{C}$–linear sheaf homomorphism

$$d : \mathcal{O}_M \longrightarrow \Omega^1_M$$

satisfying the *Leibniz rule*

$$d(f\,g) = f\,dg + g\,df .$$

In local coordinates $x_1, \ldots, x_n$, we easily calculate

$$(df)(\partial_k) = \partial_k f = \frac{\partial f}{\partial x_k}$$

such that, by definition, the *differential* dx_j of the function x_j coincides with the element dx_j introduced above, and

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_1}\,dx_n$$

for every holomorphic function $f \in \mathcal{O}_M(U)$.

We are now going to generalize these notions to general complex analytic spaces $(X, \mathcal{O}_X)$. If S is any coherent analytic sheaf on X, we can define the *sheaf of germs of derivations* with values in S by the germs of $\mathbb{C}$–linear maps

$$d : \mathcal{O}_X \longrightarrow S$$

satisfying the Leibniz rule $d(f\,g) = f\,dg + g\,df$. Obviously, that sheaf admits the canonical structure of an $\mathcal{O}_X$–module and as such, it is coherent (since each derivation is determined by its values on a parameter system of $\mathcal{O}_{X,x}$). We denote this sheaf by

$$\mathcal{D}er\,(\mathcal{O}, S)$$

and, in the special case $S = \mathcal{O}_X$, by Θ_X:

$$\Theta_X = \mathcal{D}er\,(\mathcal{O}_X, \mathcal{O}_X) .$$

Θ_X is the (analytic) *tangent sheaf* of X.

Next, observe that composing a derivation $d_0 : \mathcal{O}_X \to S_0$ with an $\mathcal{O}_X$–module morphism $\varphi : S_0 \to S$ yields a derivation $d = \varphi \circ d_0 : \mathcal{O}_X \to S$. Thus, we have a natural morphism

$$(*) \qquad \mathcal{H}om\,(S_0, S) \longrightarrow \mathcal{D}er\,(\mathcal{O}_X, S)$$

induced by d_0, and we may ask the following natural question: Does there exist a *universal* object S_0 together with a derivation $d_0 : \mathcal{O}_X \to S_0$ such that $(*)$ is an isomorphism ?

It is not hard to show that S_0 is uniquely determined (up to isomorphism) if it exists, and that, for a complex analytic manifold M, this object is precisely the sheaf Ω^1_M of Kähler 1–forms with d_0 the usual differential. Therefore, we will call S_0 the sheaf Ω^1_X of Kähler 1–forms on X.

By the uniqueness property just stated, it is only necessary to construct Ω^1_X locally. So let X be of the form $(N\,(I), \mathcal{O}_U/\,I)$, $U \subset \mathbb{C}^n$ open, I a coherent ideal. Then set

$$\Omega^1_X = \Omega^1_U/\,(I \cdot \Omega^1_U + \mathcal{O}_U \cdot dI) ,$$

which is a coherent sheaf on X. The differential $d : \mathcal{O}_U \to \Omega^1_U$ yields canonically a derivation $d : \mathcal{O}_X \to \Omega^1_X$, and the pair (Ω^1_X, d) has the desired property:

***Theorem 19.27** *On each complex analytic space X, there exists a uniquely determined coherent analytic sheaf Ω^1_X and a derivation $d: \mathcal{O}_X \to \Omega^1_X$ such that the canonical homomorphism*

$$\mathcal{H}om\,(\Omega^1_X,\, S) \longrightarrow \mathcal{D}er\,(\mathcal{O}_X,\, S)$$

is bijective for all coherent analytic sheaves S on X. In particular, the dual of Ω^1_X is canonically isomorphic to the tangent sheaf Θ_X :

$$(\Omega^1_X)^* \xrightarrow{\sim} \Theta_X\,.$$

Here, a word of warning is in order. Although we introduced Ω^1 on a *manifold* M as the dual of the tangent sheaf Θ, i.e. by $\Omega^1_X \cong \Theta^*_M$, this relation is not satisfied for general spaces X. Or, in other terms: *the canonical morphism*

$$\Omega^1_X \longrightarrow (\Omega^1_X)^{**}$$

is in general neither injective nor surjective.

For a manifold M of dimension n, Θ_M and Ω^1_M are locally free of rank n. It is not known whether the opposite is correct for the tangent sheaf (Zariski's conjecture): *Does $\Theta_{X,x}$ free at a point $x \in X$ imply that x is a regular point of X?*

The corresponding statement for the sheaf of Kähler 1–forms, however, holds true:

Theorem 19.28 *If $\Omega^1_{X,x}$ is free for a point $x \in X$, then $\mathcal{O}_{X,x}$ is a regular analytic algebra.*

The main point of the *proof* lies in the fact that $\Omega^1_{X,x}$ is minimally generated by $e = \operatorname{emb} \mathcal{O}_{X,x}$ elements: Clearly, $\operatorname{cg} \Omega^1_{X,x} \le e$, and the homomorphism $\omega : \Omega^1_{X,x} \to \mathfrak{m}_x/\mathfrak{m}_x^2$ attached to the derivation

$$\delta : \begin{cases} \mathcal{O}_{X,x} \longrightarrow & \mathfrak{m}_x/\mathfrak{m}_x^2 \\ \quad f \quad \longmapsto & f - f(x) \bmod \mathfrak{m}_x^2 \end{cases}$$

via $(*)$ is surjective by Theorem ? and $\mathfrak{m}_x\,\Omega^1_{X,x} \subset \ker \omega$. Because of

$$\dim_{\mathbb{C}} \Omega^1_{X,x}/\,\mathfrak{m}^1_{X,x}\Omega^1_{X,x} = \operatorname{cg} \Omega^1_{X,x} \le e = \dim_{\mathbb{C}} \mathfrak{m}_x/\,\mathfrak{m}_x^2\,,$$

we must have equality.

To prove the Theorem, take a representation $(N(I), \mathcal{O}_U/I)$ of X near x with $\dim U = e = \operatorname{emb} \mathcal{O}_{X,x}$. Then the images $\overline{dx_1}, \ldots, \overline{dx_e}$ of $dx_1, \ldots, dx_e$ generate $\Omega^1_{X,x}$ minimally, and since $\Omega^1_{X,x}$ is locally free at x, they form a basis near x. But, for $f \in I$, we have

$$0 = \overline{df} = \sum_{j=1}^{e} \overline{\left(\frac{\partial f}{\partial x_j}\right)}\, \overline{dx_j}\,;$$

hence $\partial f/\partial x_j \in I$ for all $j = 1, \ldots, e$. Thus, the claim follows from the next Lemma. □

Lemma 19.29 *If $\mathfrak{a}$ is a proper ideal in a regular algebra R_n such that $\partial f/\partial x_j \in \mathfrak{a}$ for all $f \in \mathfrak{a}$, $j = 1, \ldots, n$, then $\mathfrak{a} = 0$.*

Proof. By induction, all partial derivatives of f are in $\mathfrak{a} \subset \mathfrak{m}_n$. Therefore, $\partial^{|\nu|} f/\partial x^\nu = 0$ at $x = 0$, and, by Taylor expansion, $f = 0$. □

We are now ready to conclude the *analyticity of the singular locus of any complex analytic space*. In virtue of Theorem xx, we may assume that the complex analytic space X is reduced and purely d–dimensional. Then $x \in X$ is a regular point of X if and only if the module of Kähler differentials $M_x = \Omega^1_{X,x}$ is free of rank d. Now, the following *criterion for freeness* is very easy to prove:

***Theorem 19.30** *A finitely generated module M over the local ring A is free of rank d if, and only if, the canonical map*

$$\Lambda^d M \otimes \Lambda^d M^* \longrightarrow A$$

is bijective.

Since there is a sheaf version of this homomorphism on complex analytic spaces, we are done. □

19.14 Normal complex spaces

Suppose for the rest of this Chapter that $(X, \mathcal{O}_X)$ denotes a *reduced* complex analytic space. One of the main features of complex analysis are *extension theorems* like *Riemann's Removable Singularity Theorems*. We want to discuss these questions here in more detail.

We begin with a bunch of definitions. For $U \subset X$ open, we call a function $f \in H^0(U \setminus A, \mathcal{O}_X)$ *weakly holomorphic* on U, if $A \subset U$ is an analytically thin subset of U and f is locally bounded at A. It should be evident how the *sheaf* of germs of *weakly holomorphic functions* ought to be defined. We denote this sheaf in the sequel by $\widetilde{\mathcal{O}}_X$. A weakly holomorphic function $f \in H^0(U \setminus A, \mathcal{O}_X)$ is called *continuous* (resp. *holomorphic*) if it admits a (necessarily unique) extension to a continuous (resp. a holomorphic) function on U (also called f in general).

One can show:

***Theorem 19.31** *The following statements are equivalent* :

i) *Every germ* $f \in \widetilde{\mathcal{O}}_{X,x^{(0)}}$ *is continuous at* $x^{(0)}$;

ii) *the germ of* X *at* $x^{(0)}$ *is irreducible* ;

iii) *the ring* $\mathcal{O}_{X,x^{(0)}}$ *is an integral domain.*

Every point $x^{(0)}$ in a *smooth* space X satisfies the conditions ii) and iii) of Theorem ?. Therefore, weakly holomorphic functions on complex analytic manifolds X are continuous. Of course, they are even holomorphic.

The complex space X is called *weakly normal* resp. *normal* at the point $x^{(0)} \in X$, if

$$(*) \qquad \widetilde{\mathcal{O}}_{X,x^{(0)}} \cap \mathcal{C}_{X,x^{(0)}} = \mathcal{O}_{X,x^{(0)}} ,$$

that is, if every germ of a continuous weakly holomorphic function at $x^{(0)}$ is actually holomorphic, resp. if

$$(**) \qquad \widetilde{\mathcal{O}}_{X,x^{(0)}} = \mathcal{O}_{X,x^{(0)}} ,$$

that is, if every germ of a weakly holomorphic function at $x^{(0)}$ is automatically holomorphic.

The space X is called *normal* (*weakly normal*) if it is normal (weakly normal) at every point $x^{(0)} \in X$. Riemann's *first removable singularity theorem* says exactly that a smooth space X is weakly normal. Consequently, since the singular locus $\operatorname{sing} X$ of a reduced space X is analytically thin, it follows from the definitions that the complex structure of a normal space X is completely determined by the underlying topological structure and the complex analytic structure on the regular part $X \setminus \operatorname{sing} X$.

If X is irreducible at $x^{(0)}$, one can show that $\widetilde{\mathcal{O}}_{X,x^{(0)}}$ is contained in the algebraic closure of $\mathcal{O}_{X,x^{(0)}}$ in its *quotient field* $\mathcal{M}_{X,x^{(0)}}$ which we always denote by $\widehat{\mathcal{O}}_{X,x^{(0)}}$. The main result of this theory is contained in

***Theorem 19.32** X *is a normal space at* $x^{(0)}$ *if and only if the ring* $\mathcal{O}_{X,x^{(0)}}$ *is normal, i.e. if* $\mathcal{O}_{X,x^{(0)}}$ *is algebraically closed in its quotient ring* $\mathcal{M}_{X,x^{(0)}}$ *; in symbols* :

$$\mathcal{O}_{X,x^{(0)}} = \widehat{\mathcal{O}}_{X,x^{(0)}} .$$

The set of points $x^{(0)} \in X$, where a reduced complex analytic space X is not normal, is a closed analytic subset. Since it is contained in $\operatorname{sing} X$, it is analytically thin.

In normal spaces $(X, \mathcal{O}_X)$, also *Riemann's second removable singularity theorem* remains correct:

***Theorem 19.33** *If* X *is a normal complex analytic space, and* $A \subset X$ *is a closed analytic subset of* X *of codimension at least* 2, *then the restriction map*

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(X \setminus A)$$

is an isomorphism.

The *proof* is reduced to the smooth case by a local Noether normalization $X \to U \subset \mathbb{C}^d$ and by the fact that we only have to show that functions in $\mathcal{O}(X \setminus A)$ are locally bounded near A. $\square$

Corollary 19.34 *Let A be an analytically thin subset of a normal space X. Then $f \in \mathcal{O}(X \setminus A)$ admits a holomorphic extension to X if and only if it can be continuously extended.*

We add some remarks on the integral closure $\widehat{R}$ of a reduced analytic algebra $R = \mathcal{O}_{X,x^{(0)}}$. If $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ denote the minimal prime ideals of R, then it is not difficult to convince oneself that

$$\widehat{R} = \bigoplus_{j=1}^{r} \widehat{R/\mathfrak{p}_j}\,.$$

Thus, $\widehat{R}$ is a *semilocal* algebra, having exactly as many maximal ideals as X has irreducible components at $x^{(0)}$. From this fact, one can conclude:

***Theorem 19.35** *For a reduced space $(X, \mathcal{O}_X)$, the sheaf $\widehat{\mathcal{O}}_X$ is a coherent $\mathcal{O}_X$–algebra.*

We introduce the *normalization* of a complex analytic space via the notion of the *analytic spectrum* of $\widehat{\mathcal{O}}_X$.

19.15 The analytic spectrum of a coherent algebra

We intend to associate to any coherent $\mathcal{O}_X$–algebra $\mathcal{A}$ on a complex analytic space $(X, \mathcal{O}_X)$ a new space $(Y, \mathcal{O}_Y)$ such that there exists a finite covering $\rho : Y \to X$ with $\rho_* \mathcal{O}_Y \cong \mathcal{A}$.

By coherence, it is easily checked that, locally, there exists an algebra epimorphism

$$\mathcal{O}_X(U)\,[\,s_1, \ldots, s_t\,] \longrightarrow \mathcal{A}\,(U)\;;$$

hence

$$\mathcal{A}\,(U) \cong \mathcal{O}_X(U)\,[\,s_1, \ldots, s_t\,]/\,(f_1, \ldots, f_r)$$

with functions $f_1, \ldots, f_r \in \mathcal{O}\,(U \times \mathbb{C}^t)$ which are polynomials in $s_1, \ldots, s_t$. Therefore, one is more or less forced to define Y locally by

$$|\,Y\,| = \{\,(x, s) \in U \times \mathbb{C}^t : f_1 = \cdots = f_r = 0\,\}$$

together with the structure of

$$\mathcal{O}_Y((U \times \mathbb{C}^t) \cap |\,Y\,|) = \mathcal{O}\,(U \times \mathbb{C}^t)/\,(f_1, \ldots, f_r)\,.$$

One can patch these local models together (in general not uniquely) to form a complex analytic space $(Y, \mathcal{O}_Y)$ together with a holomorphic map $\rho : Y \to X$ (locally induced by the projection $U \times \mathbb{C}^t \to U$) such that

$$(\rho_* \mathcal{O}_Y)_{x^{(0)}} \cong \bigoplus_{y^{(0)} \in \rho^{-1}(x^{(0)})} \mathcal{O}_{Y, y^{(0)}} \cong \mathcal{A}_{x^{(0)}}\,.$$

Theorem 19.36 *For any coherent analytic $\mathcal{O}_X$–algebra $\mathcal{A}$ on X, there exists a complex analytic space $(Y, \mathcal{O}_Y)$ together with a holomorphic map $\rho : Y \to X$ such that*

$$\rho_* \mathcal{O}_Y \cong \mathcal{A}\,.$$

By construction, $(\rho_* \mathcal{O}_Y)_{x^{(0)}} \cong \mathcal{A}_{x^{(0)}}$ is semilocal, i.e. a *finite* direct sum of local algebras. Hence, by definition, ρ is a *finite* holomorphic map which maps $|\,Y\,|$ to the support of the sheaf $\mathcal{A}$. We call Y the *analytic spectrum* of $\mathcal{A}$; in symbols:

$$Y = \operatorname{Specan} \mathcal{A}\,.$$

$\operatorname{Specan} \mathcal{A}$ is not canonically defined globally. However, if $\mathcal{A}$ has no nontrivial $\mathcal{O}_X$–morphisms, $\operatorname{Specan} \mathcal{A}$ is globally well–defined (up to global isomorphisms).

19.16 The normalization of reduced spaces

It is a simple exercise to check that the coherent $\mathcal{O}_X$–algebra $\widehat{\mathcal{O}}_X$ admits only trivial $\mathcal{O}_X$–algebra homomorphisms. Therefore, the results of the preceding Section imply:

Theorem 19.37 *For every reduced complex analytic space* $(X,\,\mathcal{O}_X)$ *there exists a uniquely determined complex analytic space* $(\widehat{X},\,\mathcal{O}_{\widehat{X}})$ *together with a finite holomorphic map* $\rho:\,\widehat{X}\to X$ *such that*

$$\left(\rho_*\mathcal{O}_{\widehat{X}}\right)_{x^{(0)}} \cong \widehat{\mathcal{O}}_{X,x^{(0)}}$$

for every point $x^{(0)}\in X$ *.*

By definition, there is a one–to–one correspondence between points $\widehat{x^{(0)}}\in\rho^{-1}(x^{(0)})$ and the irreducible components of X at $x^{(0)}$, and

$$\mathcal{O}_{\widehat{X},\widehat{x^{(0)}}} \cong \left(\mathcal{O}_{X,x^{(0)}}/\,\mathfrak{p}_j\right)\widehat{\ }$$

for precisely one minimal prime ideal $\mathfrak{p}_j\subset\mathcal{O}_{X,x^{(0)}}$. In particular, $\widehat{X}$ is a normal complex analytic space, called the *normalization* of X . The map $\rho:\,\widehat{X}\to X$ is surjective and finite, and $\rho^{-1}(A)$, A the set of nonnormal points in X , is analytically thin in $\widehat{X}$.

Normalization is *not* a functor. In general, holomorphic maps of reduced spaces cannot be lifted to the normalization. However, one can prove:

***Theorem 19.38** *If* $f\,:\,X\to Y$ *is a holomorphic mapping between reduced spaces such that the preimages of the nonnormal points in* Y *under* f *form an analytically thin subset of* X *, then there exists exactly one holomorphic mapping* $\widehat{f}:\,\widehat{X}\to\widehat{Y}$ *between the normalizations making the diagram*

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{f}"] \arrow[d] & \widehat{Y} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutative.

Corollary 19.39 *Let* X *be a reduced,* Y *be a normal complex analytic space together with a finite holomorphic map* $\rho:\,Y\to X$ *such that* $\rho^{-1}(A)$ *,* A *the nonnormal points of* X *, is analytically thin in* Y *, and* ρ *is generically (locally with respect to* X *) an isomorphism. Then* Y *is (isomorphic to) the normalization of* X *.*

19.17 The maximalization of complex structures

Normalization of a (reduced) complex space X in general changes the topological structure since it separates the irreducible components. When one wants to keep the topology, one has to be satisfied with an analytically weaker property, namely the so–called *weak* Riemann removable singularity theorem, that is:

$$(*)\qquad \mathcal{O}_{X,x^{(0)}} = \widetilde{\mathcal{O}}_{X,x^{(0)}}\cap\mathcal{C}_{X,x^{(0)}}\ .$$

The complex structure $(X,\,\mathcal{O}_X)$ is called *maximal* if $(*)$ is fulfilled. Plainly, normal spaces are maximal. It is easily shown that a maximal structure has the property that for any other reduced complex analytic structure $\mathcal{A}$ on the topological space $|\,X\,|$ that contains $\mathcal{O}_X$, i.e. $\mathcal{O}_X\subset\mathcal{A}$, one must have $\mathcal{O}_X=\mathcal{A}$. Hence the name.

The following is an easy consequence of the definition:

***Theorem 19.40** *Let* $(Y, \mathcal{O}_Y)$ *be a maximal complex analytic space, and let* $f : X \to Y$ *resp.* $g : Y \to Z$ *be a holomorphic resp. a continuous map,* X, Z *reduced spaces. Then*

i) *if* f *is a homeomorphism* $|X| \to |Y|$, *then* f *is biholomorphic,*

ii) *if* Y *is normal,* f *is injective and* $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)}$ *for all* $x \in X$, *then* f *is open and maps* X *biholomorphically onto* $f(X)$,

iii) g *is holomorphic if and only if the graph of* g *is an analytic subset of* $Y \times Z$.

In contrast to the situation for normalization, maximalization is a *functor*:

***Theorem 19.41** a) *If* $(X, \mathcal{O}_X)$ *is a reduced complex space, then* $(X, \widetilde{\mathcal{O}}_X \cap \mathcal{C}_X)$ *is a complex space with a maximal structure (the maximalization or weak normalization of* X *).*

b) *Every holomorphic map* $f : X \to Y$ *extends (uniquely) to a holomorphic map between the maximalizations.*

19.18 Preimages of coherent analytic sheaves

The σ–modifications considered earlier in this work are special cases of modifications attached to coherent ideals or even coherent sheaves of modules. In order to understand the corresponding notion of a *monoidal transformation* we first have to introduce the basic concept of analytic preimages.

The *topological* preimage of a sheaf S on X under a continuous map $f : Y \to X$ is most easily understood by regarding $S \stackrel{\pi}{\to} X$ as a fiber space with total space

$$S = \bigcup_{x \in X} S_x$$

conveniently topologized in such a way that $S(U)$, $U \subset X$, can be identified with the set of topological sections $s : U \to S$. The topological preimage is then nothing else but the topological fiber product

$$S \diamond f := S \times_X Y = \{ (s, y) \in S \times Y : \pi(s) = f(y) \}$$

endowed with the induced topology coming from the topology of the cartesian product $S \times Y$. It is not difficult to check that $S \diamond f$ is indeed a sheaf under the natural projection $S \times_X Y \to Y$ having the stalks

$$(S \diamond f)_y = S_{f(y)} ,$$

and for all U open in X there exists a canonical map

$$H^0(U, S) \longrightarrow H^0(f^{-1}(U), S \diamond f) = H^0(U, f_*(S \diamond f)) .$$

Everything here is compatible with abelian group structures. Moreover, for any sheaf S of R–modules, $S \diamond f$ is a sheaf of $(R \diamond f)$–modules.

If f is a morphism of complex spaces and if S_1 and S_2 denote analytic sheaves on Y and X, resp., we can form the abelian groups

$$\mathrm{Hom}_Y(S_2 \diamond f, S_1) \text{ and } \mathrm{Hom}_X(S_2, f_*S_1)$$

of all sheaf homomorphisms between the sheaves involved. Given

$$\varphi \in \mathrm{Hom}_Y(S_2 \diamond f, S_1)$$

there is an associated homomorphism $f_*\varphi : f_*(S_2 \diamond f) \to f_*S_1$ that can be combined with the natural homomorphism $S_2 \to f_*(S_2 \diamond f)$ (see above), leading to a group homomorphism

$$\begin{cases} \mathrm{Hom}_Y(S_2 \diamond f, S_1) \longrightarrow \mathrm{Hom}_X(S_2, f_*S_1) \\ \qquad\qquad \varphi \longmapsto \varphi^\flat . \end{cases}$$

On the other hand, the local maps

$$((f_*S_1)\diamond f)_y \xrightarrow{\sim} (f_*S_1)_{f(y)} \longrightarrow S_{1,y}$$

give rise to a natural sheaf homomorphism

$$(f_*S_1)\diamond f \longrightarrow S_1$$

which can be used similarly to define

$$\begin{cases} \mathrm{Hom}_X(S_2,\, f_*S_1) \longrightarrow \mathrm{Hom}_Y(S_2\diamond f,\, S_1) \\ \qquad\qquad \psi \longmapsto \psi^\sharp \end{cases}$$

via the composition

$$S_2\diamond f \xrightarrow{\psi\diamond f} (f_*S_1)\diamond f \longrightarrow S_1 \;.$$

It is an easy exercise to show that $\flat$ and $\sharp$ are inverses of each other:

$$\mathrm{Hom}_Y(S_2\diamond f,\, S_1) \xrightarrow{\sim} \mathrm{Hom}_X(S_2,\, f_*S_1)\;.$$

In particular, the morphism $\widehat{f} \in \mathrm{Hom}_X(\mathcal{O}_X,\, f_*\mathcal{O}_Y)$ can always be replaced by the morphism $\widehat{f}^\sharp \in \mathrm{Hom}_Y(\mathcal{O}_X\diamond f,\, \mathcal{O}_Y)$ and vice versa. (We will also denote $\widehat{f}^\sharp$ by $\widehat{f}$ in the following).

Of course, we would like to associate to an analytic sheaf S on X an $\mathcal{O}_Y$–module sheaf on Y. This is now easily done with the help of the topological preimage by putting

$$f^*S \;=\; (S\diamond f)\otimes_{\mathcal{O}_X\diamond f}\mathcal{O}_Y$$

where $\mathcal{O}_Y$ is regarded as an $(\mathcal{O}_X\diamond f)$–module via the map $\widehat{f}:\, \mathcal{O}_X\diamond f \to \mathcal{O}_Y$. Since

$$(f^*S)_y \;=\; S_{f(y)}\otimes_{\mathcal{O}_{X,f(y)}}\mathcal{O}_{Y,y}\;,\quad y\in Y\;,$$

we sometimes write shortly

$$f^*S \;=\; S\otimes_{\mathcal{O}_X}\mathcal{O}_Y\;.$$

In contrast to the direct image, the *analytic preimage* f^*S is always coherent when regarded as an $\mathcal{O}_Y$–module provided that S was coherent on X. This is a consequence of f^* being in fact a right exact functor with $f^*\mathcal{O}_X \cong \mathcal{O}_Y$.

A slight modification of our arguments above produces another isomorphism of $\mathbb{C}$–vector spaces

$$\mathrm{Hom}_Y(f^*S_2,\, S_1) \xrightarrow{\sim} \mathrm{Hom}_X(S_2,\, f_*S_1)\;.$$

In particular, when taking $S_1 = f^*S_2$ or $S_2 = f_*S_1$ and the identity on one of the two sides, we can establish the existence of natural sheaf homomorphisms

$$S_2 \longrightarrow f_*f^*S_2$$

and

$$f^*f_*S_1 \longrightarrow S_1\;.$$

19.19 Čech cohomology

It is now time to say a few words about *cohomology theory*. We know that to have a sheaf epimorphism $\psi:\, S\to S_2$ does not mean surjectivity of the homomorphisms

$$\psi_U:\, S(U) \longrightarrow S_2(U)\;,\quad U\subset X \text{ open}\;.$$

E.g., it is well-known that for the epimorphism $\exp : \mathcal{O} \to \mathcal{O}^*$ on $\mathbb{C}$, surjectivity of the morphism

$$\mathcal{O}(U) \xrightarrow{\exp} \mathcal{O}^*(U)$$

holds on an open connected subset $U \subset \mathbb{C}$ if and only if U is simply–connected.

Let us examine the necessary and sufficient conditions under which an element $s_2 \in S_2(U)$ has a preimage in $S(U)$. Since we can regard the restrictions of the sheaves S and S_2 to the subset U, we restrict ourselves to the case $U = X$. Surjectivity of ψ then implies that we may find an open covering $\mathfrak{U} = \{ U_\iota \}_{\iota \in I}$ of X and sections $s_\iota \in S(U_\iota)$ such that $\psi_{U_\iota}(s_\iota) = s_{2|U_\iota}$. Of course, we would like to have

$$s_{\iota\kappa} := s_{\kappa|U_{\iota\kappa}} - s_{\iota|U_{\iota\kappa}} = 0 , \quad U_{\iota\kappa} = U_\iota \cap U_\kappa$$

for all ι, κ. However, we only know that

$$\begin{aligned} \psi_{U_{\iota\kappa}}(s_{\iota\kappa}) &= \psi_{U_{\iota\kappa}}(s_{\kappa|U_{\iota\kappa}}) - \psi_{U_{\iota\kappa}}(s_{\iota|U_{\iota\kappa}}) \\ &= \psi_{U_\kappa}(s_\kappa)_{|U_{\iota\kappa}} - \psi_{U_\iota}(s_\iota)_{|U_{\iota\kappa}} \\ &= s_{2|U_{\iota\kappa}} - s_{2|U_{\iota\kappa}} = 0 , \end{aligned}$$

i.e.

$$s_{\iota\kappa} \in S_1(U_{\iota\kappa}) , \quad \iota, \kappa \in I ,$$

where

$$S_1 = \ker \psi .$$

By the very definition of the system

$$\{ s_{\iota\kappa} \in S_1(U_{\iota\kappa}) : (\iota, \kappa) \in I \times I \} ,$$

it satisfies obviously the conditions for a *1–cocycle* in the sense of Chapter 2(??); i.e. for all $\iota, \kappa, \lambda \in I$, we have

$$(*) \qquad s_{\iota\kappa|U_{\iota\kappa\lambda}} + s_{\kappa\lambda|U_{\iota\kappa\lambda}} + s_{\lambda\iota|U_{\iota\kappa\lambda}} = 0 , \quad U_{\iota\kappa\lambda} = U_\iota \cap U_\kappa \cap U_\lambda .$$

In particular, $s_{\iota\kappa} = -s_{\kappa\iota}$ and $s_{\iota\iota} = 0$ for all $\iota, \kappa \in I$.

Recall, that we associated to any holomorphic linebundle L on a complex manifold M a 1–cocycle in the sheaf $\mathcal{O}^*$ with respect to a trivializing covering $\mathfrak{U}$ of L, and vice versa. The group of such 1–cocycles has been denoted by $Z^1(\mathfrak{U}, \mathcal{O}^*)$. Hence, we are now forced to contemplate on such 1–cocycles in general sheaves S_1, the group of which we consequently will denote by

$$Z^1(\mathfrak{U}, S_1) = \{ (s_{\iota\kappa})_{(\iota,\kappa)\in I\times I} : s_{\iota\kappa} \in S_1(U_{\iota\kappa}) \text{ satisfying } (*) \} .$$

We shall call the system $(s_{\iota\kappa})$ in our construction above the *1–cocycle associated to local liftings of the section* $s_2 \in S_2(X)$.

Now suppose that $s_2 \in S_2(X)$ has a global lifting $\widetilde{s} \in S(X)$. Then, as above,

$$\widetilde{s}_\iota := s_\iota - \widetilde{s}_{|U_\iota} \in S_1(U_\iota) \text{ for all } \iota \in I ,$$

and, on $U_{\iota\kappa}$:

$$(**) \qquad \widetilde{s}_\kappa - \widetilde{s}_\iota = (s_\kappa - \widetilde{s}) - (s_\iota - \widetilde{s}) = s_{\iota\kappa} .$$

We call an element $(\widetilde{s}_\iota)_{\iota\in I} \in \prod_{\iota\in I} S_1(U_\iota)$ a *0–cochain* in the sheaf S_1. A 1–cocycle $(s_{\iota\kappa}) \in Z^1(\mathfrak{U}, S_1)$ is called a 1–*coboundary*, if $(**)$ holds for a 0–cochain in S_1. Hence, we have proved the „only if"–part of the following

Lemma 19.42 *A section $s_2 \in S_2(X)$ has a global lifting $s \in S(X)$ with respect to the sheaf epimorphism $\psi : S \to S_2$, if and only if each 1–cocycle associated to local liftings is a 1–coboundary with respect to $S_1 = \ker \psi$.*

Proof of the "if"–part: Let $s_\iota \in S(U_\iota)$ be local liftings, let $s_{\iota\kappa} = (s_\kappa - s_\iota)_{|U_{\iota\kappa}}$ be the associated 1–cocycle in S_1, and suppose that we have

$$s_{\iota\kappa} = \widetilde{s}_\kappa - \widetilde{s}_\iota \text{ on } U_{\iota\kappa}$$

for a 0–cochain $(\widetilde{s}_\iota) \in \prod S_1(U_\iota)$. Then regard $(s_\iota - \widetilde{s}_\iota)_{\iota \in I}$ as a 0–cochain in S. Since, on $U_\iota \cap U_\kappa$,

$$(s_\iota - \widetilde{s}_\iota) - (s_\kappa - \widetilde{s}_\kappa) = (\widetilde{s}_\kappa - \widetilde{s}_\iota) - (s_\kappa - s_\iota) = s_{\iota\kappa} - s_{\iota\kappa} = 0 ,$$

the system defines a global section in $S(X)$ which maps onto s_2 under ψ. □

It is, by the way, not difficult to convince oneself that two 1-cocycles in $Z^1(\mathfrak{U}, \mathcal{O}^*)$ define the same holomorphic linebundle L on X up to isomorphism, if their difference is a 1-coboundary in $\mathcal{O}^*$.

Hence, we have seen up to now two instances where it seems to make sense to introduce the group

$$H^1(\mathfrak{U}, S) = Z^1(\mathfrak{U}, S)/ B^1(\mathfrak{U}, S) ,$$

$B^1(\mathfrak{U}, S)$ denoting the subgroup of 1-coboundaries, which we call the *first Čech cohomology group* with values in the sheaf S (with respect to the open covering $\mathfrak{U}$ of X). Two elements in $Z^1(\mathfrak{U}, S)$ are also called *cohomologuous* to eachother, if they define the same class in $H^1(\mathfrak{U}, S)$.

To get rid of the particular coverings, we observe that for a finer covering $\mathfrak{V} < \mathfrak{U}$, i.e. for a covering $\mathfrak{V} = \{ V_\lambda \}_{\lambda \in \Lambda}$ together with a map $\tau : \Lambda \to I$ such that $V_\lambda \subset U_{\tau(\lambda)}$ for all λ, there are canonical compatible restriction maps

$$Z^1(\mathfrak{U}, S) \longrightarrow Z^1(\mathfrak{V}, S) , \quad B^1(\mathfrak{U}, S) \longrightarrow B^1(\mathfrak{V}, S) ,$$

inducing maps

$$r^1_{\mathfrak{U},\mathfrak{V}} : H^1(\mathfrak{U}, S) \longrightarrow H^1(\mathfrak{V}, S) ,$$

which, in fact, do not depend on the specific map τ, and satisfying all axioms for an inductively ordered system. Hence, we form the inductive limit

$$H^1(X, S) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, S)$$

and call it the *first Čech cohomology group* of X with values in S. Clearly, we have for all open coverings $\mathfrak{U}$ canonical maps

$$r^1_{\mathfrak{U},X} : H^1(\mathfrak{U}, S) \longrightarrow H^1(X, S) ,$$

which can be shown to be injective (since, by a simple exercise, the maps $r^1_{\mathfrak{U},\mathfrak{V}}$ are monomorphisms for all $\mathfrak{V} < \mathfrak{U}$).

As a first application of this notion, we state the easily proven

***Theorem 19.43** *There is a natural 1–1 correspondence between the set of isomorphism classes of holomorphic linebundles on a complex analytic manifold X and the Čech cohomology group $H^1(X, \mathcal{O}^*)$. Moreover, this correspondence is a group isomorphism, when the group operation on linebundles is taken as the tensor product.*

If we go through the construction at the beginning of the present Section again, we easily recognize that we have associated to a short exact sequence

$$0 \longrightarrow S_1 \xrightarrow{\varphi} S \xrightarrow{\psi} S_2 \longrightarrow 0$$

of sheaves on X a group homomorphism

$$\delta : S_2(X) \longrightarrow H^1(X, S_1) ,$$

such that $s_2 \in \operatorname{im} \psi_X$, if and only if $\delta(s_2) = 0$. In other words: The sequence

$$0 \longrightarrow S_1(X) \longrightarrow S(X) \longrightarrow S_2(X) \longrightarrow H^1(X, S_1)$$

is exact.

Our next task is to prolong this sequence. In order to do so, we generalize all concepts developed so far. Define first the groups of q–cochains in the sheaf S relative to a covering $\mathfrak{U}$ by

$$C^q(\mathfrak{U}, S) = \prod_{(\iota_0,\ldots,\iota_q)\in I^{q+1}} S(U_{\iota_0\ldots\iota_q}) ,$$

where $U_{\iota_0\ldots\iota_q} = U_{\iota_0} \cap \ldots \cap U_{\iota_q}$. There is a canonical map

$$\delta^q : C^q(\mathfrak{U}, S) \longrightarrow C^{q+1}(\mathfrak{U}, S) ,$$

defined by

$$(\delta^q(s_{\iota_0\ldots\iota_q}))_{\kappa_0\ldots\kappa_{q+1}} = \sum_{\lambda=0}^{q+1} (-1)^\lambda s_{\kappa_0\ldots\widehat{\kappa_\lambda}\ldots\kappa_{q+1}|U_{\kappa_0\ldots\kappa_{q+1}}} ,$$

which commutes with the restriction maps

$$C^j(\mathfrak{U}, S) \longrightarrow C^j(\mathfrak{V}, S) , \quad j = q, q+1$$

for any refinement $\mathfrak{V} < \mathfrak{U}$. Hence, denoting by

$$Z^q(\mathfrak{U}, S) = \ker \delta^q ,$$
$$B^{q+1}(\mathfrak{U}, S) = \operatorname{im} \delta^q , \quad B^0(\mathfrak{U}, S) = 0$$

the groups of q–cocycles and q–coboundaries, resp., and remarking that $\delta^{q+1} \circ \delta^q = 0$, we can define the *Čech cohomology groups*

$$H^q(\mathfrak{U}, S) = Z^q(\mathfrak{U}, S)/B^q(\mathfrak{U}, S) \text{ and } H^q(X, S) = \varinjlim_{\mathfrak{U}} H^q(\mathfrak{U}, S)$$

for all $q \geq 0$ together with canonical homomorphisms

$$r^q_{\mathfrak{U},X} : H^q(\mathfrak{U}, S) \longrightarrow H^q(X, S) .$$

It is immediately clear from the definition that

$$H^0(X, S) = H^0(\mathfrak{U}, S) = S(X)$$

for all open coverings $\mathfrak{U}$, justifying our notions in use since Chapter 2 (???).

Given a morphism $\varphi : S \to S'$ of sheaves, we find associated group homomorphisms

$$\begin{cases} C^q(\mathfrak{U}, S) \longrightarrow & C^q(\mathfrak{U}, S') \\ (s_{\iota_0\ldots\iota_q}) \longmapsto & (\varphi_{U_{\iota_0\ldots\iota_q}}(s_{\iota_0\ldots\iota_q})) \end{cases} ,$$

which obviously give rise to group homomorphisms

$$\varphi^q : H^q(\mathfrak{U}, S) \longrightarrow H^q(\mathfrak{U}, S')$$

and

$$\varphi^q : H^q(X, S) \longrightarrow H^q(X, S') ,$$

such that

$$\varphi^0 = \varphi_X : H^0(X, S) = S(X) \longrightarrow S'(X) = H^0(X, S') .$$

Modifying the construction of the morphism $\delta : H^0(X, S_2) \to H^1(X, S_1)$ for a short exact sequence

$$0 \longrightarrow S_1 \xrightarrow{\varphi} S \xrightarrow{\psi} S_2 \longrightarrow 0 ,$$

we can also construct *connecting homomorphisms*

$$\delta^q : H^q(X, S_2) \longrightarrow H^{q+1}(X, S_1)$$

for all $q \geq 0$, provided that the topological space X is *paracompact*, i.e. if each open covering $\mathfrak{U}$ allows for a locally finite refinement $\mathfrak{V} < \mathfrak{U}$. (Then we can represent cohomology classes by cycles in $Z^q(\mathfrak{V}, S)$ with locally finite $\mathfrak{V}$). - The main theorem for Čech cohomology now states:

***Theorem 19.44** *For any short exact sequence*

$$0 \longrightarrow S_1 \xrightarrow{\varphi} S \xrightarrow{\psi} S_2 \longrightarrow 0$$

on a paracompact space X, the associated long cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, S_1) &\xrightarrow{\varphi^0} H^0(X, S) \xrightarrow{\psi^0} H^0(X, S_2) \xrightarrow{\delta^0} \\ \longrightarrow H^1(X, S_1) &\xrightarrow{\varphi^1} H^1(X, S) \xrightarrow{\psi^1} H^1(X, S_2) \xrightarrow{\delta^1} \\ \longrightarrow H^2(X, S_1) &\longrightarrow \quad \cdots \end{aligned}$$

is exact. This long exact cohomology sequence depends functorially on the given short exact sequence.

19.20 Acyclic sheaves and resolutions

A sheaf $\mathcal{A}$ is called *acyclic* on X, if all higher cohomology groups $H^q(X, \mathcal{A})$, $q \geq 1$, vanish. Assume that we have an exact sequence

$$(+) \qquad \mathcal{A}^{\cdot} : \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \longrightarrow \cdots$$

of acyclic sheaves such that $\ker d^0 \cong S$. Then we call $\mathcal{A}^{\cdot}$ an *acyclic resolution* of the sheaf S. We shall see later that each sheaf admits such resolutions (even in a canonical fashion). The important feature of acyclic resolutions consists in the fact that we are able to compute the cohomology of S by means of the sequence of sections in $\mathcal{A}^{\cdot}$. To be precise, regard the sequence of groups

$$\mathcal{A}^{\cdot}(X) : 0 \longrightarrow \mathcal{A}^0(X) \xrightarrow{d_X^0} \mathcal{A}^1(X) \xrightarrow{d_X^1} \mathcal{A}^2(X) \longrightarrow \cdots .$$

Since $d_X^{q+1} \circ d_X^q = 0$ for all $q \geq 0$ (i.e. $\mathcal{A}^{\cdot}(X)$ is a *complex*), we can form the cohomology groups

$$H^q(\mathcal{A}^{\cdot}(X)) = \ker d_X^q / \operatorname{im} d_X^{q-1} , \quad d_X^{-1} := 0 .$$

Clearly, $H^0(\mathcal{A}^{\cdot}(X)) = \ker d_X^0 = S(X) = H^0(X, S)$. That this is true for *all* cohomology groups, is the content of the *abstract de Rham Theorem*:

Theorem 19.45 *Let $\mathcal{A}^{\cdot}$ be an acyclic resolution of the sheaf S on a paracompact space X. Then there are canonical isomorphisms*

$$H^q(X, S) \cong H^q(\mathcal{A}^{\cdot}(X)) .$$

Proof. Denote by K^p the kernel of the sheaf homomorphism $d^p : \mathcal{A}^p \to \mathcal{A}^{p+1}$. Then we have

$$K^0 = S \text{ and } K^p = \ker d^p = \operatorname{im} d^{p-1} , \quad p \geq 1 .$$

Hence, the resolution (+) decomposes into short exact sequences

$$0 \longrightarrow K^p \longrightarrow \mathcal{A}^p \xrightarrow{d^p} K^{p+1} \longrightarrow 0 , \quad p \geq 0 .$$

Due to the long exact cohomology sequence, we conclude that

$$0 = H^q(X, \mathcal{A}^p) \longrightarrow H^q(X, K^{p+1}) \longrightarrow H^{q+1}(X, K^p) \longrightarrow H^{q+1}(X, \mathcal{A}^p) = 0$$

is exact for $q \geq 1$. Therefore, by induction,

$$H^q(X, S) = H^q(X, K^0) \cong H^{q-1}(X, K^1) \cong \cdots \cong H^1(X, K^{q-1})$$

for all $q \geq 1$. The initial part of the cohomology sequence contains the exact segment

$$H^0(X, \mathcal{A}^p) \xrightarrow{d_X^p} H^0(X, K^{p+1}) \longrightarrow H^1(X, K^p) \longrightarrow H^1(X, \mathcal{A}^p) = 0$$

such that

$$H^q(X, S) \cong H^1(X, K^{q-1}) \cong H^0(X, K^q)/\operatorname{im} d_X^{q-1} .$$

Applying the left–exact functor H^0 to the exact sequence

$$0 \longrightarrow K^q \longrightarrow \mathcal{A}^q \xrightarrow{d^q} \mathcal{A}^{q+1}$$

finally yields

$$H^0(X, K^q) = \ker d_X^q . \qquad \square$$

There are some important classes of acyclic sheaves on paracompact spaces X. Generalizing the concept of a partition of unity from functions to sheaves, we call a sheaf S *fine*, if for all locally finite coverings $\mathfrak{U} = \{U_\iota\}_{\iota \in I}$ there are sheaf homomorphisms $h_\iota : S \to S$ with $\operatorname{supp} h_\iota = \{x \in X : h_\iota(S_x) \neq 0_x\} \subset U_\iota$ and

$$\sum_{\iota \in I} h_\iota = \mathrm{id}_S .$$

Notice that each $\mathcal{C}_X$–module sheaf on a paracompact topological manifold X and also each $\mathcal{C}_X^\infty$–module sheaf on a paracompact $\mathcal{C}^\infty$–manifold X is fine.

Theorem 19.46 *Every fine sheaf S on a paracompact space X is acyclic.*

Proof. Since X is paracompact, it suffices to show that

$$H^q(\mathfrak{U}, S) = 0 , \quad q \geq 1$$

for all locally finite coverings $\mathfrak{U}$ of X. Now, for a fixed partition $(h_\iota)_{\iota \in I}$ of id_S, define homomorphisms

$$k^q : C^q(\mathfrak{U}, S) \longrightarrow C^{q-1}(\mathfrak{U}, S) , \quad q \geq 1 ,$$

by

$$(k^q(s_{\iota_0 \ldots \iota_q}))_{\kappa_0 \ldots \kappa_{q-1}} := \sum_{\iota \in I} h_\iota(s_{\iota \kappa_0 \ldots \kappa_{q-1}}) ,$$

where $h_\iota(s_{\iota \kappa_0 \ldots \kappa_{q-1}})$ denotes the trivial extension of this element from $U_{\kappa_0 \ldots \kappa_{q-1}} \setminus U_\iota$ to $U_{\kappa_0 \ldots \kappa_{q-1}}$. It is easily checked that $\delta^{q-1} \circ k^q + k^{q+1} \circ \delta^q = \mathrm{id}_{C^q(\mathfrak{U}, S)}$, such that $Z^q(\mathfrak{U}, S) = \mathrm{id}_{C^q(\mathfrak{U}, S)} Z^q(\mathfrak{U}, S) \subset B^q(\mathfrak{U}, S)$. $\square$

Other examples of acyclic sheaves are the following, as we shall see in a moment: A sheaf F is called *flabby*, if for all open subsets $U \subset X$ the restriction homomorphism

$$H^0(X, F) \longrightarrow H^0(U, F)$$

is surjective. Every sheaf S can be embedded into a flabby one (its "flabbyfication"), called $F(S)$, whose groups of sections consists of *all* (i.e. not necessarily continuous) sections in S; in other words:

$$F(S)(U) = \{ s : U \to S : \pi \circ s = \mathrm{id}_U \} .$$

Each section in $F(S)(U)$ can trivially be extended to the whole space X by putting $s(x) = 0_x$, $x \in X \setminus U$.

Moreover, we can build up a canonical *flabby resolution* of a given sheaf S by setting $F^0 := F(S)$, $F^1 := F(F^0/S)$ etc.:

$$0 \longrightarrow S \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots .$$

In order to prove that flabby sheaves are acyclic, we need the following lemmata.

Lemma 19.47 *If* $0 \to F_1 \to F \to F_2 \to 0$ *is a short exact sequence with a flabby sheaf* F_1 *, then the sequence*

$$0 \longrightarrow H^0(X,\,F_1) \longrightarrow H^0(X,\,F) \longrightarrow H^0(X,\,F_2) \longrightarrow 0$$

is exact.

Proof. Given a section $f_2 \in F_2(X)$, we choose a lifting $f \in F\,(U)$ with a maximal open set $U \subset X$ and claim that $U = X$. Suppose, to the contrary, that $U \neq X$. Then there exists a nonvoid open set V, not contained in U, and a lifting $\widetilde{f}$ of f_2 over V. $f - \widetilde{f}$ can be extended from $U \cap V$ to a global section f_1 in F_1, and the pair $(f,\, f_1 + \widetilde{f})$ defines a lifting of f_2 to $U \cup V$. Contradiction ! □

Lemma 19.48 *For a short exact sequence* $0 \to F_1 \to F \to F_2 \to 0$ *with flabby sheaves* F_1 *and* F*, the cokernel* F_2 *is also flabby.*

Proof. Trivial.

Theorem 19.49 *Flabby sheaves are acyclic.*

Proof. Choose a flabby resolution $0 \to F \to F^0 \to F^1 \to \cdots$ of F, and define the exact sequences

$$0 \longrightarrow K^p \longrightarrow F^p \longrightarrow K^{p+1} \longrightarrow 0\,, \quad p \geq 0\,,$$

as in the proof of Theorem ?. By induction on p, we may conclude that all sheaves K^p are flabby. Using Lemma ? above, we find

$$\ker\, d_X^{p+1} = \operatorname{im}\, d_X^p\,, \quad p \geq 0\,,$$

and the conclusion follows from Theorem ?. □

It is also not difficult to show that the special flabby sheaves $F\,(S)$ are even fine.

The same reasoning shows that there exists on a paracompact space X only one “cohomology theory” normalized in such a way that flabby sheaves (or only sheaves of type $F\,(S)$) have trivial higher cohomology groups.

19.21 Leray and Stein coverings

We have remarked in the previous Section that the canonical homomorphisms $r^1_{\mathfrak{U},X} : H^1(\mathfrak{U},\,S) \to H^1(X,\,S)$ are always injective. If the covering $\mathfrak{U}$ is “cohomologically good”, these maps and the corresponding ones in arbitrary dimension q are bijective.

An open covering $\mathfrak{U}$ of X has the *Leray property* for the sheaf S in dimension p, if $H^q(U_{\iota_1\ldots\iota_p},\,S) = 0$ for all $q = 1,\ldots,p$ and all p–fold intersections. It is called a *Leray covering* for S, if the Leray property is satisfied for all natural numbers $p \geq 1$. With this notion, one can prove *Leray’s Theorem*:

***Theorem 19.50** *For a covering* $\mathfrak{U}$ *of* X *with the Leray property for* S *in dimension* p *the canonical homomorphism* $r^p_{\mathfrak{U},X} : H^p(\mathfrak{U},\,S) \to H^p(X,\,S)$ *is bijective. This is true for all* $p \geq 0$*, if* $\mathfrak{U}$ *is a Leray covering for* S.

It is now very important to have Leray coverings simultaneously for a whole class of sheaves. This remark, for instance, applies to the class of constant sheaves, if all intersections $U_{\iota_0\ldots\iota_q}$ are (connected and) contractible (e.g. a covering of $\mathbb{R}^n$ with convex sets U_ι).

In the case of complex analytic spaces X, we would like to work with all *coherent* analytic sheaves S at the same time. Therefore, we are interested in open subsets $U \subset X$ with the property that $H^q(U,\,S) = 0$ for every $q \geq 1$ and each coherent analytic sheaf S on U. We call such sets *B–domains* in X; if X itself is a B–domain, it is called a *B–space*, which is shorthand for stating that the space X *satisfies Theorem B of Cartan.*

B–spaces have very strong properties. They satisfy, for instance, the so–called *Theorem A of Cartan*, as well:

Theorem 19.51 *Let* S *be a coherent analytic sheaf on the B–space* X *. Then, for all* $x \in X$ *, there are finitely many sections* $s_1, \ldots, s_k \in S(X)$ *such that the stalk* S_x *is generated as an* $\mathcal{O}_{X,x}$*–module by the germs* $s_{1x}, \ldots, s_{kx}$ *.*

Proof. Denote by $I = I_x$ the ideal sheaf of holomorphic function germs vanishing at the point x. From the exact sequence

$$0 \longrightarrow I \cdot S \longrightarrow S \longrightarrow S/ I \cdot S \longrightarrow 0 ,$$

we deduce exactness of

$$H^0(X, S) \longrightarrow H^0(X, S/ I \cdot S) \longrightarrow H^1(X, I \cdot S) = 0 ,$$

since $I \cdot S$ is a coherent subsheaf of S. But $(S/ I \cdot S)_y \neq 0_y$ for all $y \neq x$, whence

$$H^0(X, S/ I \cdot S) = S_x / \mathfrak{m}_{X,x} S_x ,$$

and the claim follows from Theorem 5. ??. $\square$

By the same reasoning as above for the ideal sheaf $I_{x,y}$ of germs of holomorphic functions vanishing at x and y, $x \neq y$, we also get immediately

Theorem 19.52 *Every B–space* X *is holomorphically separable, i.e. for* $x_1 \neq x_2$ *in* X *there exists a holomorphic function* $f \in \mathcal{O}(X)$ *with* $f(x_1) \neq f(x_2)$ *.*

Clearly, each of the last two Theorems implies that *a B–space is compact if and only if it consists of isolated points only*.

Finally, the following holds true:

Theorem 19.53 *Every B–space* X *is holomorphically convex, i.e. for each compact set* $K \subset X$ *the holomorphically convex hull*

$$\widehat{K} := \{ x \in X : |f(x)| \leq \sup |f(K)| \text{ for all } f \in \mathcal{O}(X) \}$$

is compact.

Proof. Suppose that there exists a compact set $K \subset X$ such that the convex hull $\widehat{K}$ is not compact. Then there is an infinite series $\{x_j\}_{j \in \mathbb{N}}$ in $\widehat{K}$ without accumulation points in $\widehat{K}$. Since $\widehat{K}$ is closed in X, the set $A = \{x_j : j \in \mathbb{N}\}$ has no accumulation points in X, as well. Hence, A is a discrete analytic subset in X with a coherent ideal sheaf I_A. By property B, we have exactness of the sequence

$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X / I_A) \longrightarrow H^1(X, I_A) = 0 ,$$

such that we can find a holomorphic function $f \in \mathcal{O}_X(X)$ with $\lim_{j \to \infty} f(x_j) = \infty$, contradicting

$$\sup |f(x_j)| \leq \sup |f(K)| < \infty . \qquad \square$$

Calling a complex analytic space X a *Stein space* (or *holomorphically complete*) if it is holomorphically separable and holomorphically convex, we may state that *B–spaces are Stein spaces*. Remember that we assume all topological spaces to have countable topology. However, this follows automatically (at least for each connected component) for holomorphically separable spaces. This taken for granted, one can show that one may replace the assumption of a Stein space X to be holomorphically convex by the following one:

For each infinite discrete closed set $Z \subset X$*, there exists a holomorphic function* $f \in \mathcal{O}(X)$ *such that*

$$\sup_{x \in Z} |f(x)| = \infty .$$

For Stein spaces, the following permanence results are more or less obvious:

***Theorem 19.54**

1. *Every region in* $\mathbb{C}$ *is a Stein space.*
2. *Products of Stein spaces are Stein; in particular, every polydisk* $P \subset \mathbb{C}^n$ *is a Stein space.*
3. *Closed subvarieties of Stein spaces are Stein spaces; in particular, each point* x *in a complex analytic space* X *has arbitrarily small Stein neighborhoods.*
4. *Intersections of finitely many Stein domains in a complex space are Stein spaces; in particular, every complex space* X *(with countable topology) has arbitrarily fine (locally finite) Stein coverings* $\mathfrak{U} = \{ U_\iota \}_{\iota \in I}$ *, i.e. coverings with* $U_{\iota_0} \cap \ldots \cap U_{\iota_q}$ *Stein for all* $\iota_0, \ldots, \iota_q \in I^{q+1}$ *.*

Our task concerning Leray coverings on complex spaces is now accomplished by the famous and very deep *Theorem B* of Cartan:

***Theorem 19.55** *Every Stein space is a B–space.*

Let us finish the present Section by sampling some results for Stein domains in $\mathbb{C}^n$. Obviously, every such domain G is holomorphically separable. Hence, in these cases, "holomorphically convex" is equivalent to "property B". In fact, there are many different characterizations of such Stein domains in $\mathbb{C}^n$. Among them are the characterizations as

i) domains of holomorphy;

ii) pseudoconvex domains;

iii) domains with $H^1(G, I) = 0$ for all coherent ideal sheaves $I \subset \mathcal{O}_G$ with discrete zero–set $N(I)$;

iv) domains with $H^q(G, \mathcal{O}_G) = 0$ for $q = 1, \ldots, n-1$.

19.22 Grauert's Coherence Theorem

In many applications, one is forced to study a relative situation $f : X \to Y$ together with some sheaves S on X. Similarly to the absolute case, an exact sequence

$$(*) \qquad 0 \longrightarrow S_1 \longrightarrow S \longrightarrow S_2$$

on X gives rise to an exact sequence

$$0 \longrightarrow f_* S_1 \longrightarrow f_* S \longrightarrow f_* S_2$$

of direct images. (Recall that the direct image sheaf $f_* S$ of S, say, is defined by $(f_* S)(V) = H^0(f^{-1}(V), S)$, $V \subset Y$ open, the left–exactness of the functor f_* following from Theorem ????). It is clear that, for obvious reasons, we also have to introduce similar sheaves for all cohomology groups.

Therefore, we associate to any open subset $V \subset Y$ the cohomology group $H^q(f^{-1}(V), S)$. If $\mathfrak{U} = \{ U_\iota \}_{\iota \in I}$ denotes an open covering of $f^{-1}(V)$, and if W is an open subset of V, then

$$\mathfrak{V} = \mathfrak{U} \cap f^{-1}(W) = \{ V_\iota = U_\iota \cap f^{-1}(W) \}_{\iota \in I}$$

is an open covering of $f^{-1}(W)$ with natural restriction maps

$$S(U_\iota) \longrightarrow S(V_\iota)$$

inducing homomorphisms $C^q(\mathfrak{U}, S) \longrightarrow C^q(\mathfrak{V}, S)$ commuting with δ such that there are canonical restriction homomorphisms

$$H^q(\mathfrak{U}, S) \longrightarrow H^q(\mathfrak{V}, S)$$

and

$$H^q(f^{-1}(V), S) \longrightarrow H^q(f^{-1}(W), S)$$

for all $q \geq 0$. This, in fact, defines a presheaf which, in general, is not a sheaf for $q \geq 1$. The associated sheaf is denoted by

$$R^q f_* S \text{ or } f_*^q S$$

such that $R^0 f_* S = f_*^0 S = f_* S$. $R^q f_* S$ is usually called the *q–th direct image sheaf of S under the map f*. In case that f is a morphism of ringed spaces $(X, R_X) \to (Y, R_Y)$, these sheaves carry a natural structure of R_Y–modules.

It should also be self–evident how to construct canonical morphisms $R^q \varphi_* : R^q f_* S \to R^q f_* S'$ associated to sheaf homomorphisms $\varphi : S \to S'$ on X and connecting homomorphisms $\delta^q : R^q f_* S_2 \to R^{q+1} f_* S_1$ for exact sequences $0 \to S_1 \to S \to S_2 \to 0$ on X.

***Theorem 19.56** *For any short exact sequence $0 \to S_1 \to S \to S_2 \to 0$ of $\mathcal{O}_X$–modules on a complex analytic space X and any holomorphic map $f : X \to Y$, one has a canonical long exact sequence of $\mathcal{O}_Y$–modules:*

$$\begin{aligned} 0 \longrightarrow R^0 f_* S_1 &\longrightarrow R^0 f_* S \longrightarrow R^0 f_* S_2 \longrightarrow \\ R^1 f_* S_1 &\longrightarrow R^1 f_* S \longrightarrow R^1 f_* S_2 \longrightarrow \\ R^2 f_* S_1 &\longrightarrow \quad \cdots \quad . \end{aligned}$$

Grauert's Coherence Theorem now states:

***Theorem 19.57** *For a coherent analytic sheaf S on X and all proper holomorphic maps $f : X \to Y$, the direct image sheaves $R^q f_* S$, $q \geq 0$, are coherent. Moreover, if $V \subset Y$ is a Stein domain, then the natural morphisms*

$$H^q(f^{-1}(V), S) \longrightarrow H^0(V, R^q f_* S)\,, \quad q \geq 0\,,$$

are bijective.

Every compact complex space X carries a (proper) constant holomorphic map. Hence, Grauert's Coherence Theorem implies:

***Theorem 19.58** *The cohomology groups $H^q(X, S)$, $q \geq 0$, on a compact complex space X with values in a coherent $\mathcal{O}_X$–module sheaf S are finite dimensional complex vector spaces.*

We now want to compare the stalks $(R^q f_* S)_y$, $y \in Y$ fixed, with some canonical cohomology groups on the fibers $X_y := f^{-1}(y)$ carrying the usual complex analytic structure with respect to the proper holomorphic map f. We denote by S_y the analytic restriction of the sheaf S to the subvariety X_y, i.e.

$$S_y = S \otimes_{\mathcal{O}_X} \mathcal{O}_{X_y} = (S / f^*(I_y)S)_{|f^{-1}(y)}\,,$$

$I_y \subset \mathcal{O}_Y$ the ideal sheaf of the point y. Since any section in S over an open set U defines a section of S_y on $U \cap f^{-1}(y)$, we have a canonical homomorphism

$$(R^q f_* S)_y \longrightarrow H^q(X_y, S_y)$$

whose kernel contains $\mathfrak{m}_y (R^q f_* S)_y$ where $\mathfrak{m}_y$ denotes the stalk of I_y at $y \in Y$, i.e. the maximal ideal of $\mathcal{O}_{Y,y}$. Hence, we find a canonical homomorphism of finite dimensional vector spaces

$$r_S^q(y) : (R^q f_* S)_y \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y} / \mathfrak{m}_y) \longrightarrow H^q(X_y, S_y)$$

which, in general, is neither injective nor surjective. However, defining higher infinitesimal neighborhoods of $f^{-1}(y)$, namely the spaces

$$X_y^{(n)} \text{ with structure sheaf } \mathcal{O}_Y / f^*(I_y^n) \mathcal{O}_Y\,,$$

all having the same underlying topological space $f^{-1}(y)$, and correspondingly forming the analytic restrictions

$$S_y^{(n)} = S \otimes_{\mathcal{O}_Y} \mathcal{O}_{X_y^{(n)}} \text{ of } S \text{ to } X_y^{(n)}$$

for all n, we get a projective system of homomorphisms

$$(R^q f_* S)_y \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y} / \mathfrak{m}_y^n) \longrightarrow H^q(X_y, S_y^{(n)})$$

giving rise to a homomorphism from the $\mathfrak{m}_y$–adic completion

$$(R^q f_* S)\widehat{\ }_y = \varprojlim_n (R^q f_* S)_y / \mathfrak{m}_y^n (R^q f_* S)_y$$

of the finite $\mathcal{O}_{Y,y}$–module $(R^q f_* S)_y$ into the projective limit

$$\varprojlim_n H^q(X_y, S_y^{(n)}) .$$

Grauert's Comparison Theorem (called the *Theorem of Formal Functions* in Algebraic Geometry) says that this is a bijection:

***Theorem 19.59** *Let $f : X \to Y$ be a proper holomorphic map of complex spaces, and let S be a coherent analytic sheaf on X. Then, for all points $y \in Y$ and all $q \geq 0$, the canonical homomorphisms*

$$(R^q f_* S)\widehat{\ }_y \longrightarrow \varprojlim_n H^q(X_y, S_y^{(n)})$$

are isomorphisms.

We finish this Section by formulating *Grauert's Semicontinuity Theorems*. Here, we are dealing with the functions

$$d_S^q(y) := \dim_{\mathbb{C}} H^q(X_y, S_y) , \quad y \in Y ,$$

for proper holomorphic maps $f : X \to Y$ and coherent analytic sheaves S that are *flat* over Y. Under these assumptions, we have

***Theorem 19.60**

1. *The maps d_S^q are upper semicontinuous on Y. More precisely, for all $k \in \mathbb{N}$, the sets*

$$\{ y \in Y : d_S^q(y) \geq k \}$$

   are closed analytic subsets of Y.

2. *If d_S^q is a locally constant function, and if Y is reduced, then $R^q f_* S$ is a locally free sheaf, and the canonical homomorphisms*

$$r_S^q(y) : (R^q f_* S)_y / \mathfrak{m}_y (R^q f_* S)_y \longrightarrow H^q(X_y, S_y)$$

   are bijections for all $y \in Y$.

3. *The Euler–Poincaré characteristic*

$$\chi_S(y) = \sum_{q=0}^{\infty} (-1)^q d_S^q(y)$$

   is (defined and) locally constant on Y.

We will have the opportunity to use the following Corollary to the Semicontinuity Theorem:

Corollary 19.61 *Under the assumptions of the preceding Theorem, if* $H^q(X_y, S_y)$ *vanishes for some* $y = y^{(0)} \in Y$ *and some* $q \in \mathbb{N}$, *then*

$$H^q(X_y, S_y) = 0$$

for all y *in a neighborhood* V *of* $y^{(0)}$, *and* $(R^q f_* S)_{|V} = 0$.

This is a direct consequence of Theorem ? provided Y is reduced at $y^{(0)}$. In the general case, it follows from the next result, called the *Base Change Theorem*:

***Theorem 19.62** *If the canonical homomorphism* r_S^q *is surjective at one point* $y^{(0)} \in Y$, *then it is an isomorphism in a whole neighborhood* V *of* $y^{(0)}$. *Moreover, under this assumption,* $R^q f_* S$ *is locally free near* $y^{(0)}$ *if and only if* $r_S^{q-1}(y^{(0)})$ *is surjective.*

Notes and References

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To be continued.