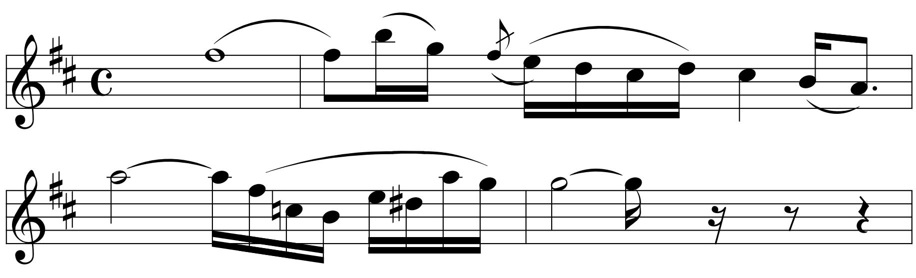
Chapter 9

So war sein Weg denn im Kreise gegangen, oder in einer Ellipse oder Spirale, oder wie immer, nur nicht geradeaus, denn das Geradlinige gehörte offenbar nur der Geometrie, nicht der Natur und dem Leben an.

(Hermann Hesse, *Das Glasperlenspiel*)



Chapter 9

Blowing down and the Grauert - Mumford criterion

After having shown that normal surface singularities can be resolved, it is a natural question to ask which manifolds M are resolutions of such singularities. The leading idea for an answer is the observation that isolated singularities (embedded in $\mathbb{C}^n$) have arbitrarily small neighborhoods with "good" boundaries such that M must as well contain such good open sets as neighborhoods of the exceptional set $E \subset M$. The resulting concept is that of *strictly pseudoconvex* open sets. The existence of such neighborhoods for one–dimensional compact analytic subsets E in a two–dimensional manifold M can be deduced from purely numerical invariants.

9.1 Strictly plurisubharmonic functions

A real–valued function φ on a complex analytic manifold M is called *(strictly) plurisubharmonic*, if φ is of class C^2 and if for each point $x^{(0)} \in M$ and any coordinate system $x_1, \ldots, x_n$ near $x^{(0)} = 0$ the *Levi form*

$$\mathfrak{L}\,(\varphi,\, x^{(0)},\, \xi) \;=\; \sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial \overline{x_k}}\,(x^{(0)})\,\xi_j\,\overline{\xi_k}\;, \quad \xi \in \mathbb{C}^n\;,$$

is positive definite. It is easily checked that this property does not depend on the analytic coordinate system.

A relatively compact open subset $V \subset\subset M$ is called *strictly pseudoconvex*, if there exists a neighborhood U of the boundary ∂V in M and a strictly plurisubharmonic function $\varphi : U \to \mathbb{R}$ such that $V \cap U = \{\, x \in U : \varphi\,(x) < 0 \,\}$. φ is then called a *defining function* for V.

It is not difficult to prove that strict pseudoconvexity is a *local property of the boundary*; i.e. $V \subset\subset M$ is strictly pseudoconvex, if and only if for all $x^{(0)} \in \partial V$ there exists a neighborhood $U = U\,(x^{(0)}) \subset M$ and a strictly plurisubharmonic function $\varphi : U \to \mathbb{R}$ such that $V \cap U = \{\, x \in U : \varphi\,(x) < 0 \,\}$. Moreover, if the boundary of V is smooth, i.e. if locally $\partial V \cap U = \{\, \varphi = 0 \,\}$, where $\varphi \in C^2(U)$ and $d\varphi \neq 0$ on U, then strict pseudoconvexity is just a property of the holomorphic tangent spaces of ∂V, that is: it is sufficient to prove

$$\mathfrak{L}\,(\varphi,\, x^{(0)},\, \xi) \;>\; 0$$

for all $x^{(0)} \in \partial V \cap U$ and all vectors $\xi = (\xi_1, \ldots, \xi_n) \neq 0$ with

$$\sum_{j=1}^{m} \frac{\partial \varphi}{\partial x_j}\,(x^{(0)}) \cdot \xi_j \;=\; 0\;.$$

For the convenience of the reader, we reproduce here the essential trick for the second statement: Replace

φ by $\psi = \varphi(e^{A\varphi})$, A sufficiently large. Then the claim follows from the formula

$$\mathfrak{L}(\psi, x^{(0)}, \xi) = (1 + A)\, e^{A\varphi(x^{(0)})} \left\{ \mathfrak{L}(\varphi, x^{(0)}, \xi) + A \left| \sum \frac{\partial\varphi}{\partial x_j}(x^{(0)})\, \xi_j \right|^2 \right\}$$

for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$.

The simplest *Example* of such a strictly plurisubharmonic function is given by $\varphi(x) = \| x \|^2$ on $\mathbb{C}^n$, since

$$\mathfrak{L}(\varphi, x^{(0)}, \xi) = \sum_{j,k=1}^{n} \delta_{jk}\, \xi_j \overline{\xi_k} = \| \xi \|^2 .$$

9.2 Grauert's criterion for exceptional sets

Restrictions of strictly plurisubharmonic functions on a complex manifold M to an analytic submanifold $N \subset M$ are again strictly plurisubharmonic. Hence, if an *isolated* singularity $(X, x^{(0)})$ is embedded in $(U, 0)$, U open in $\mathbb{C}^n$, then $\varphi_{|X\setminus\{0\}}$ is strictly plurisubharmonic, where $\varphi(x) = \| x \|^2$. Lifting φ to a resolution $\pi : M \to X$ and taking into account that, by the properness of π, the open sets $\pi^{-1}(X \cap B_\varepsilon)$, $B_\varepsilon = \{ x \in \mathbb{C}^n : \varphi(x) < \varepsilon \}$, form a basis of neighborhoods of $E = \pi^{-1}(0)$ in M, we get the following necessary condition for resolutions.

Lemma 9.1 *Let* $\pi : \widetilde{X} \to X$ *be a resolution of the isolated singularity* $(X, x^{(0)})$. *Then there exists a* C^2*–function* φ *on a neighborhood* V *of* $E = \pi^{-1}(x^{(0)})$ *in* $\widetilde{X}$, *such that :*

i) $\varphi_{|V\setminus E}$ *is strictly plurisubharmonic,*

ii) $\varphi_{|E} \equiv 0$,

iii) *the strictly pseudoconvex sets* $\widetilde{X}_\varepsilon = \{ y \in V : \varphi(y) < \varepsilon \}$, $\varepsilon > 0$ *small, form a neighborhood basis of* E.

It is a deep theorem of Grauert that the converse to Lemma 1 is true.

***Theorem 9.2** *Let* $V \subset\subset M$ *be a strictly pseudoconvex open set. Then there exists a maximal compact nowhere discrete analytic subset* E *of* V. *If* $E = \cup E^\tau$ *is the (finite) decomposition of* E *into connected components, then each* E^τ *can analytically be contracted to a point. More precisely: there exists a normal complex analytic space* X *with finitely many points* $x^{(1)}, \ldots, x^{(t)}$ *and a proper modification*

$$\pi : M \longrightarrow X$$

such that

$$\pi(E^\tau) = \{ x^{(\tau)} \}$$

and the restriction

$$\pi : M \setminus E \longrightarrow X \setminus \{ x^{(1)}, \ldots, x^{(t)} \}$$

is biholomorphic.

A compact nowhere discrete analytic subset E in a normal complex space $\widetilde{X}$ which can be contracted analytically to finitely many isolated normal singularities is called an *exceptional analytic* set. By Theorem 2 and the preceding Lemma, a compact analytic set in a manifold is exceptional, if and only if it possesses a strictly pseudoconvex neighborhood V such that it is the maximal compact nowhere discrete analytic subset in V. This is equivalent to the following statement which we will use frequently on several occasions:

A compact complex analytic subset E *in a complex manifold* M *is exceptional, if and only if it has a neighborhood basis consisting of strictly pseudoconvex open sets.*

Since, in this text, X is always supposed to be normal, $\pi_* \mathcal{O}_M = \mathcal{O}_X$ for any resolution $M \to X$. Hence, the ring of holomorphic functions on M near a (connected) exceptional set E must be isomorphic to a normal analytic algebra. Therefore, in order to be able to contract E (or - as one also says - to *blow down* E) to a point, one has to know the structure of this ring.

Before we go on, we sketch the *proof* for Theorem 2. Let $U \subset M$ be an open neighborhood of ∂V and $\varphi : U \to \mathbb{R}$ be given such that $V \cap U = \{ \varphi < 0 \}$, φ strictly plurisubharmonic. Then take $U_1 \subset\subset U$ and $\varepsilon > 0$ so small that

$$U_2 = \{ y \in U : -\varepsilon < \varphi(y) < 0 \} \cap U_1 \subset\subset U_1 \, .$$

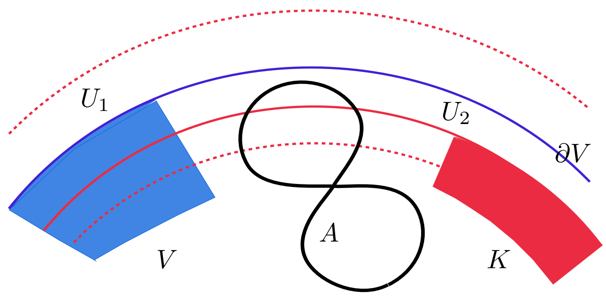


Figure 9.1

Then $K := V \setminus U_2 \subset\subset V$, and we claim that any compact nowhere discrete analytic subset A of V must be contained in K. Otherwise, there would exist a connected component of $A \cap U_2$, say B, and φ would admit a maximum on B. But restrictions of plurisubharmonic functions to subvarieties fulfill the maximum principle, hence $\varphi_{|B} = \text{const}$. which is a contradiction to the strict plurisubharmonicity of φ, since B has regular points of dimension ≥ 1.

The main point of the proof, which we omit, is to show that V is a *holomorphically convex* manifold (see the Supplement). We are then in a position to apply Remmert's *Reduction Theorem*:

***Theorem 9.3** *For any holomorphically convex manifold V, there exists a uniquely determined Stein space X_0 and a proper holomorphic map*

$$\pi : V \longrightarrow X_0$$

such that

$$\pi_* \mathcal{O}_V = \mathcal{O}_{X_0} \, .$$

We shall call X_0 the *Remmert quotient* of V. It has the additional property that all fibers $\pi^{-1}(x)$, $x \in X_0$, are connected. Moreover, X_0 is a normal space, the manifold V being normal. (Notice that the Remmert reduction exists for all holomorphically convex reduced spaces V; then X_0 is normal, if V is a normal space).

Returning to Theorem 2, denote by E the set of points $y \in V$ which are not discrete in $\pi^{-1} \circ \pi(y)$. It is not difficult to prove that E is an analytic subset of V. Since

$$E_x = E \cap \pi^{-1}(x) \, , \quad x \in X_0 \, ,$$

is compact and nowhere discrete, $E_x \subset K$ and hence

$$E = \bigcup_{x \in X_0} E_x \subset K \, ;$$

in particular, E is a compact nowhere discrete analytic subset of V.

Denote the connected components of E by $E^1, \ldots, E^t$. By Remmert's Proper Mapping Theorem, $\pi(E^\tau)$ is a (connected, compact) analytic subset of X_0. Since X_0 is a Stein space, $\pi(E^\tau)$ must be a point $x^{(\tau)}$ for all $\tau = 1, \ldots, t$. Now, $\pi_{|V \setminus E} \to X_0 \setminus \{ x^{(1)}, \ldots, x^{(t)} \}$ has discrete (connected) fibers; hence π is bijective. As a map between normal spaces, π is biholomorphic. Patching $M \setminus E$ and $X_0 \setminus \{ x^{(1)}, \ldots, x^{(t)} \}$ together along $V \setminus E$ yields X and π.

If A is an arbitrary (connected) compact analytic subset of V which is nowhere discrete, then necessarily $\pi(A) \subset \{ x^{(1)}, \ldots, x^{(t)} \}$, and $A \subset E^\tau$ for a suitable τ. □

9.3 Negative line bundles

In Chapter 4 we have seen that the zero–section of a line bundle L on a projective algebraic manifold can be analytically contracted to a point, if the dual bundle L^* is very ample. We are now going to characterize *all* line bundles with this property analytically.

Theorem 9.4 *Let L be a holomorphic line bundle on a (connected) compact complex manifold M. Then the zero–section of L is exceptional in (the total space of) L, if and only if there exist, to some trivializing covering $\mathfrak{U} = \{ U_j \}_{j \in I}$ of M with coordinate charts U_j, positive real–valued C^2–functions h_j on U_j such that*

$$h_k(x) = |f_{jk}(x)|^2 h_j(x), \quad x \in U_j \cap U_k, \tag{1}$$

and the hermitean form

$$\sum_{\nu,\mu} \frac{\partial^2 \log h_j}{\partial x_\nu \partial \overline{x_\mu}} \xi_\nu \overline{\xi_\mu} \tag{2}$$

is positive definite on U_j for all j. (Here, of course, (f_{jk}) denotes the cocyle defining L with respect to $\mathfrak{U}$ and $x_1, \ldots, x_n$ are local coordinates in U_j).

Proof. 1. We assume first the existence of the functions h_j. Denote by V_j the set of points

$$\{ (x, v) \in U_j \times \mathbb{C} : h_j(x) |v_j|^2 < 1 \} \subset U_j \times \mathbb{C} \cong L_{|U_j} =: L_j .$$

Because of (1), $V = \cup_{j \in I} V_j$ is a neighborhood of the zero–section $M \subset L$, satisfying $V \cap \pi^{-1}(U_j) = V_j$ and $V \subset\subset L$. Obviously, the boundary of V in L_j is described by $\partial V \cap L_j = \{ \varphi_j(x, v_j) := \log |v_j|^2 + \log h_j(x) < 0 \}$.

Let now $(\xi_0, \xi_1, \ldots, \xi_n) = \xi \in \mathbb{C}^{n+1}$ be a nontrivial holomorphic tangent vector on V, i.e.

$$\frac{\partial \log |v_j|^2}{\partial v_j} \xi_0 + \sum_{\nu=1}^n \frac{\partial \log h_j(x)}{\partial x_\nu} \xi_\nu = 0 .$$

Since

$$\frac{\partial \log |v_j|^2}{\partial v_j} = \frac{\partial \log v_j \overline{v_j}}{\partial v_j} = \frac{\overline{v_j}}{v_j \overline{v_j}} = \frac{1}{v_j}, \quad \frac{\partial^2 \log |v_j|^2}{\partial v_j \partial \overline{v_j}} = 0,$$

the vector $(\xi_1, \ldots, \xi_n)$ does not vanish, and we can deduce from the assumptions that

$$\mathfrak{L}(\varphi_j, (x^{(0)}, v_j), \xi) = \sum_{\nu,\mu} \frac{\partial^2 \log h_j(x^{(0)})}{\partial x_\nu \partial \overline{x_\mu}} \xi_\nu \overline{\xi_\mu} > 0 .$$

The second remark after the definition in Section 1 implies that V is indeed a strictly pseudoconvex neighborhood of M in L. By Theorem 2, the zero–section M is then contained in the exceptional set E of V.

Assume that $E \neq M$. Then there exists a point $(x^{(0)}, v_j^{(0)}) \in L_j \cap E$ with $v_j^{(0)} \neq 0$. Since L is a line bundle, one has a natural holomorphic action of $\mathbb{C}^*$ on L, where the map corresponding to $c \in \mathbb{C}^*$

is just multiplication of the fiber coordinate by c. So, cE is a compact analytic subset of L for all $c \in \mathbb{C}^*$, and, by the definition of V,

$$cE \subset V \text{ for all } |\,c\,| \le 1\,, \quad c \ne 0\,.$$

Hence,

$$\bigcup_{0<|c|\le 1} cE \subset E$$

and

$$\{\,(x^{(0)},\, v_j) : |\,v_j\,| \le |\,v_j^{(0)}\,|\,\} \subset L_j \cap E\,.$$

But then the whole fiber $L_{x^{(0)}}$ is contained in E and E cannot be compact. Contradiction!

2. Assume that M is exceptional in L. Then there exists a neighborhood V of M and a strictly plurisubharmonic C^2–function $\psi \ge 0$ on V which is zero on M and positive outside M. Without loss of generality, we may assume that V is invariant under the circle group

$$S^1 = \{\,c = e^{i\vartheta} : \vartheta \in [\,0,\, 2\pi\,]\,\}$$

with respect to the $\mathbb{C}^*$–action described under 1. Then we define

$$\varphi\,(y) = \frac{1}{2\pi}\int_0^{2\pi} \psi\,(e^{i\vartheta}y)\,d\vartheta\,, \quad y \in V\,.$$

It is easy to check that φ has the same properties as ψ, and that

$$d\varphi_x \ne 0 \text{ for } \varphi_x = \varphi_{|L_x \cap V}\,.$$

Hence, for ε sufficiently small, $\widetilde{V} = \{\,\varphi\,(y) < \varepsilon\,\} \subset V$ is a strictly pseudoconvex neighborhood of M with smooth boundary which, locally on U_j, can be described by a positive C^2–function $g_j(x)$ in the sense that

$$\partial\widetilde{V} \cap L_j = \{\,|\,v_j\,| = g_j(x)\,\} = \{\,\varphi_j(x,\, v_j) = \log\,|\,v_j\,| - \log\, g_j(x) = 0\,\}$$

and

$$\widetilde{V} \cap L_j = \{\,\varphi_j(x,\, v_j) < 0\,\}\,.$$

Since $\partial\widetilde{V}$ is globally well–defined, (1) in the statement of the Theorem is satisfied for the functions $h_j(x) := g_j^{-2}(x)$. Since $\partial\widetilde{V}$ is strictly pseudoconvex,

$$0 < \mathfrak{L}\,(\varphi_j,\, (x,\, v_j),\, \xi) = -\sum_{\nu,\mu=1}^{n} \frac{\partial^2 \log\, g_j(x)}{\partial x_\nu \partial \overline{x_\mu}}\, \xi_\nu \overline{\xi_\mu} = \frac{1}{2}\sum_{\nu,\mu=1}^{n} \frac{\partial^2 \log\, h_j(x)}{\partial x_\nu \partial \overline{x_\mu}}\, \xi_\nu \overline{\xi_\mu}$$

for all $\xi = (\xi_0,\, \xi_1, \ldots, \xi_n) \ne 0$ with

$$\frac{1}{2v_j}\,\xi_0 - \sum_{\nu=1}^{n} \frac{\partial \log\, g_j(x)}{\partial x_\nu}\,\xi_\nu = 0\,,$$

i.e. for all $(\xi_1, \ldots, \xi_n) \ne 0$, since $\xi_1 = \cdots = \xi_n = 0$ implies $\xi_0 = 0$. □

Remarks. 1. The correct interpretation of the system $\{\,h_j\,\}$ with (1) in Theorem 4 is that of a *hermitean metric* on the fibers of L which varies differentially with $x \in M$. Just set $h_x(v_j,\, w_j) = h_j(x)\, v_j \overline{w_j}$ for $(x,\, v_j)$, $(x,\, w_j) \in L_j = L_{|U_j}$. This is well defined on the whole total space of L: Recall from Section 6 of Chapter 4 that the transformation of the linear coordinates on the fibers of L is given by $v_j = f_{jk} v_k$ such that

$$h_j v_j \overline{w_j} = h_j f_{jk} v_k \overline{f_{jk} w_k} = |\,f_{jk}\,|^2\, h_j v_k \overline{w_k} = h_k v_k \overline{w_k}\,.$$

2. Condition (2) can be interpreted as a sign condition for the *curvature* of this metric. The last is related to the (global) (1, 1)–form

$$\overline{\partial}\left(\frac{1}{h}\,\partial h\right) \,=\, \overline{\partial}\partial\,\log\,h \,=\, -\sum_{\nu,\mu=1}^{n} \frac{\partial^2 \log\,h_j(x)}{\partial x_\nu \partial \overline{x_\mu}}\, dx_\nu \wedge \overline{dx_\mu}\;.$$

Due to the minus sign on the right hand side and our assumption of *positive definiteness*, the pair $(L,\,h)$ - or L for short - is under the condition (2) in Theorem 4 called a *negative* bundle. L is called *positive*, if its dual L^* is negative.

3. In this terminology, Theorem 4 can be phrased by saying that *the zero–section in a holomorphic line bundle L over a manifold is exceptional if and only if the bundle is negative*.

The following are obvious consequences from Theorem 4.

Corollary 9.5 *A holomorphic line bundle L is positive/negative, if and only if one of its powers $L^{\otimes k}$, $k > 0$, is positive/negative.*

Corollary 9.6 *Let L be a positive/negative line bundle on the complex manifold M, and let $N \subset M$ be a submanifold. Then $L_{|N}$ is positive/negative.*

Corollary 9.7 *Let L be a positive/negative line bundle on M. Then M is a Kähler manifold; i.e. there exists a hermitean metric on M such that its imaginary part Ω (which is a real alternating differential form of type $(1,\,1)$) is d–closed.*

Proof of last Corollary. Let, without loss of generality, the line bundle L be negative. Take the functions h_j as in Theorem 4 which can be chosen of class C^∞, and define on U_j the form

$$g_j \,=\, \sum_{\nu,\mu} g_{\nu\overline{\mu}}\, dx_\nu \otimes d\overline{x_\mu}$$

with

$$g_{\nu\overline{\mu}} \,=\, \frac{\partial^2 \log\,h_j}{\partial x_\nu \partial \overline{x_\mu}}\;.$$

g_j can be considered at each point $x \in U_j$ (independently of the coordinates) as a positive definite hermitean bilinear form on the tangent space $T_x \,=\, T_{M,x}$:

$$g_{j,x} : T_x \times T_x \longrightarrow \mathbb{C}\,,$$

which varies differentially with $x \in U_j$. Since, on $U_j \cap U_k$, we have

$$\log\,h_j \,=\, 2\log\,|\,f_{kj}\,| \,+\, \log\,h_k$$

and

$$\frac{\partial^2 \log\,|\,f\,|}{\partial x_\nu \partial \overline{x_\mu}} \,=\, 0$$

for holomorphic functions f, the g_j patch together to a globally defined hermitean metric g on T_M. Let Ω be the imaginary part of g. Since

$$g\,(\xi,\,\mu) \,=\, \overline{g\,(\mu,\,\xi)}\,, \quad \xi\,,\ \mu \in T_{M,x}\,,$$

it follows that

$$\Omega\,(\xi,\,\mu) \,=\, -\Omega\,(\mu,\,\xi)\;.$$

Hence, Ω is an alternating real form. It is easy to check that

$$\Omega \,=\, \frac{i}{2}\sum_{\nu,\mu} g_{\nu\overline{\mu}}\, dz_\nu \wedge d\overline{z_\mu} \,=\, \frac{i}{2}\,\partial\overline{\partial}\log\,h_j\;.$$

By the last representation of Ω, it follows that

$$(-2i)\,d\Omega \,=\, (\partial \,+\, \overline{\partial})\,\partial\overline{\partial}\log\,h_j \,=\, 0\;. \qquad \square$$

9.4 The Embedding Theorem of Kodaira

There are strong *topological* restrictions for compact complex analytic manifolds being Kähler manifolds. In fact, the class of exceptional zero–sections of holomorphic line bundles on compact manifolds is even more restricted. The following theorem was proved by Kodaira; we present it in a version due to Grauert.

Theorem 9.8 *Let $L \to M$ be a holomorphic line bundle on a compact complex manifold such that the zero–section is exceptional, i. e. assume that L is a* negative *line bundle. Then M is* projective algebraic. *More precisely, there exists a sufficiently high power $L^{-k} := (L^*)^{\otimes k}$ of the dual bundle L^* which is* very ample.

The *proof* of Grauert proceeds along the following lines:

Step 1. We first formulate the conditions for a holomorphic line bundle F on M to be *very ample* in terms that can be verified *cohomologically*. By reinterpretation of some results in Chapter 4 we conclude that the sections of a line bundle F on a complex manifold M of dimension n imbed M into some $\mathbb{P}_N$, if the following holds true:

(i) *for each point $x^{(0)} \in M$, there exists a section $s \in H^0(M, \mathcal{O}(F))$ not vanishing at $x^{(0)}$;*

(ii) *for each pair of points $x^{(1)}, x^{(2)} \in M$, $x^{(1)} \neq x^{(2)}$, there exists a section s with $s(x^{(1)}) \neq 0$, $s(x^{(2)}) = 0$;*

(iii) *for each point $x^{(0)} \in M$, there exist $n+1$ sections $s_0, s_1, \ldots, s_n \in H^0(M, \mathcal{O}(F))$ such that $s_0(x^{(0)}) \neq 0$, $s_1(x^{(0)}) = \cdots = s_n(x^{(0)}) = 0$, and*

$$df_1(x) \wedge \ldots \wedge df_n(x) \neq 0 \,, \quad f_j = \frac{s_j}{s_0} \,, \quad j = 1, \ldots, n \,,$$

for all x close to $x^{(0)}$.

Denote now by $I(x^{(1)}, x^{(2)})$ the coherent ideal sheaf in $\mathcal{O}_M$ of germs of holomorphic functions vanishing at $x^{(1)}$ and $x^{(2)}$ (for $x^{(1)} \neq x^{(2)}$), resp. vanishing at $x^{(1)} = x^{(2)}$ at least to second order. Then the quotients $\mathcal{O}(F)/I(x^{(1)}, x^{(2)})\,\mathcal{O}(F)$ are skyscraper sheaves with

$$H^0(M, \mathcal{O}(F)/I(x^{(1)}, x^{(2)})\,\mathcal{O}(F)) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}\,, & x^{(1)} \neq x^{(2)} \\ \mathbb{C}^{n+1}\,, & x^{(1)} = x^{(2)}\,. \end{cases}$$

By tensoring the exact sequence $0 \to I(x^{(1)}, x^{(2)}) \to \mathcal{O}_M \to \mathcal{O}_M/I(x^{(1)}, x^{(2)}) \to 0$ with $\mathcal{O}(F)$, we get the long exact sequence of cohomology:

$$\begin{aligned} \cdots \longrightarrow H^0(M, \mathcal{O}(F)) \xrightarrow{\;r\;} & H^0(M, \mathcal{O}(F)/I(x^{(1)}, x^{(2)})\,\mathcal{O}(F)) \\ \longrightarrow \quad & H^1(M, I(x^{(1)}, x^{(2)})\,\mathcal{O}(F)) \quad \longrightarrow \cdots \quad , \end{aligned}$$

and the assertions (i), (ii) and (iii) are just equivalent to the *surjectivity* of r for all $x^{(1)}, x^{(2)}$.

Therefore, *a holomorphic line bundle F on a compact complex manifold M is very ample, if*

$$H^1(M, I(x^{(1)}, x^{(2)})\,\mathcal{O}(F)) = 0 \text{ for all } x^{(1)}, x^{(2)} \in M\,.$$

Step 2. For any coherent sheaf S on M, there exists a number k_0 such that

$$H^\ell(M, S \otimes \mathcal{O}(L^{-k})) = 0$$

for all $\ell \geq 1$ and all $k \geq k_0$.

If $\pi : L \to M$ denotes the projection and if V is a strictly pseudoconvex neighborhood of the zero–section M in L, then one can construct an inclusion

$$\bigoplus_{k=0}^{\infty} H^\ell(M,\, S \otimes \mathcal{O}\,(L^{-k})) \hookrightarrow H^\ell(V,\, \widetilde{S})\,, \quad \ell \geq 1\,,$$

where $\widetilde{S}$ is the analytic preimage of S under $\pi_{|V}$. Denote by $\sigma : V \to Y$ the Remmert quotient of V (see the proof of Theorem 2, where we used other notations). By Grauert's Coherence Theorem (remember that Y is a Stein space),

$$H^\ell(V,\, \widetilde{S}) \,=\, H^0(Y,\, R^\ell\sigma_*\widetilde{S})\,,$$

and the vector space on the right hand side is finite dimensional, since $R^\ell\sigma_*\widetilde{S}$ is coherent and concentrated in finitely many points. Hence,

$$\dim_{\mathbb{C}} H^\ell(V,\, \widetilde{S}) \,<\, \infty\,, \quad \ell \geq 1\,,$$

(a special case of the Finiteness Theorem of Andreotti and Grauert). From $H^\ell(V,\, \widetilde{S}) \,=\, 0$ for $\ell \gg 0$, the assertion follows.

Step 3. Denoting by $I\,(x^{(1)},\, x^{(2)})$ again the ideal sheaf introduced in Step 1, Grauert deduces by a continuity argument from Step 2:

It exists a number k_0, such that for all $k \geq k_0$ and all $x^{(1)},\, x^{(2)} \in M$ we have

$$H^1(M,\, I\,(x^{(1)},\, x^{(2)}) \otimes \mathcal{O}\,(L^{-k})) \,=\, 0\,,$$

which completes the proof. □

We would like to remark here that Grauert's proof works also for exceptional *spaces* and holomorphic *vector bundles*.

Definition. A line bundle L is usually called *ample*, if some positive power $L^{\otimes k}$ is very ample. By Corollary 5 and Theorem 4, *a line bundle L is ample, if it is positive*.

To show the *converse* it is, by Corollary 6, sufficient to prove the following

Theorem 9.9 *The hyperplane bundle H on $\mathbb{P}_n$ is positive.*

Proof. Let $u_0, \ldots, u_n$ be homogeneous coordinates on $\mathbb{P}_n$. According to Chapter 4, the hyperplane bundle H is determined by the cocycle

$$f_{kj} \,:=\, \frac{u_j}{u_k}$$

with respect to the standard open covering $U_0, \ldots, U_n$, $U_j \,=\, \{\, u_j \neq 0\,\}$. So, any metric on H is a system of positive $\mathcal{C}^\infty$–functions h_j on U_j such that

$$h_k \,=\, |\, f_{jk}\,|^2 h_j \,=\, \left|\, \frac{u_k}{u_j} \,\right|^2 h_j\,.$$

A good choice is obviously

$$h_j \,:=\, \frac{|\, u_j\,|^2}{|\, u_0\,|^2 + \cdots + |\, u_n\,|^2}\,.$$

(Notice that these are indeed well-defines functions on U_j). In order to check positivity, we may and do restrict to the coordinate system $x_j \,=\, u_j/\, u_0$ on U_0 such that

$$h_0 \,=\, \frac{1}{1 \,+\, |\, x_1\,|^2 + \cdots + |\, x_n\,|^2}\,.$$

A straightforward but tedious calculation ends up with the following formula:

$$\begin{aligned}\overline{\partial}\partial \log h_0 &= \partial\overline{\partial} \log (1 + |x_1|^2 + \cdots + |x_n|^2) \\ &= \sum_{j,k} \frac{(1 + |x_1|^2 + \cdots + |x_n|^2)\delta_{jk} - \overline{x}_j x_k}{(1 + |x_1|^2 + \cdots + |x_n|^2)} \, dx_j \wedge dx_k \ .\end{aligned}$$

Introducing the standard hermitean bilinear form $\langle \cdot, \cdot \rangle$ on $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, the corresponding hermitean bilinear form can be written (up to the positive factor $(1 + \langle x, x \rangle)^{-1}$) as

$$(1 + \langle x, x \rangle) \langle \xi, \xi \rangle - |\langle x, \xi \rangle|^2$$

which by the Cauchy-Schwarz inequality $|\langle x, \xi \rangle|^2 \le |x|^2 |\xi|^2$ is always greater or equal to $|\xi|^2$ and hence positive definite. □

Remark. It is interesting to note that the standard hermitean metric h on H induces by its positive definite curvature form the *Fubini–Study metric* on $\mathbb{P}_n$ (which is automatically a Kähler metric).

9.5 Positivity of holomorphic line bundles and Chern classes

There is still another description of positive and negative line bundles using *Chern classes*. Denote by e the function

$$e(w) = \exp(2\pi i w) \ , \quad w \in \mathbb{C} \ ,$$

and look at the associated exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \xrightarrow{e} \mathcal{O}_M^* \longrightarrow 1 \ ,$$

where $\mathbb{Z}$ denotes the constant $\mathbb{Z}$–sheaf and 1 the multiplicatively written trivial sheaf (see the Supplement). Taking cohomology, we get from the long exact cohomology sequence a connecting homomorphism

$$c : H^1(M, \mathcal{O}_M^*) \longrightarrow H^2(M, \mathbb{Z}) \ .$$

Of course, denoting by $\mathcal{A}_M$ the sheaf of germs of (complex–valued) C^∞–functions on M, we have a similar map $c : H^1(M, \mathcal{A}_M^*) \to H^2(M, \mathbb{Z})$ which is always bijective, since it fits into the exact sequence

$$H^1(M, \mathcal{A}_M) \longrightarrow H^1(M, \mathcal{A}_M^*) \xrightarrow{c} H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathcal{A}_M)$$

where both groups at the ends vanish ($\mathcal{A}_M$ is a fine sheaf).

The relevance of the homomorphism c for a holomorphic (or differentiable) line bundle L rests on the following construction: One associates to L and a suitable trivializing covering $\mathfrak{U}$ of M a defining cocycle

$$(f_{jk}) \in Z^1(\mathfrak{U}, \mathcal{O}_M^*) \ (\text{resp.} \ \in Z^1(\mathfrak{U}, \mathcal{A}_M^*))$$

which generates a cohomology class in the Čech cohomology group $H^1(\mathfrak{U}, \mathcal{O}_M^*)$ and therefore also a class ξ_L in $H^1(M, \mathcal{O}_M^*)$ (resp. in $H^1(M, \mathcal{A}_M^*)$). The following is well–known (and moreover easy to prove):

***Theorem 9.10** *The map $L \mapsto \xi_L$ establishes a one–to–one correspondence between isomorphism classes of holomorphic (resp. differentiable) line bundles on M and the cohomology group*

$$H^1(M, \mathcal{O}_M^*) \ (\text{resp.} \ H^1(M, \mathcal{A}_M^*) \ ) \ .$$

Thus, defining the *(first) Chern class* $c(L)$ of a line bundle L by $c(L) = c(\xi_L) \in H^2(M, \mathbb{Z})$, we have the following properties:

a) $c(L)$ does only depend on the *differentiable* isomorphism class of L, because of the commutativity of the diagram

$$
\begin{tikzcd}
H^1(M,\, \mathcal{O}_M^*) \arrow[r] \arrow[d] & H^2(M,\, \mathbb{Z}) \arrow[d, "\mathrm{id}"] \\
H^1(M,\, \mathcal{A}_M^*) \arrow[r] & H^2(M,\, \mathbb{Z})
\end{tikzcd}
$$

b) $H^2(M,\, \mathbb{Z})$ classifies all differentiable isomorphism classes of line bundles;

c) if the map $H^2(M,\, \mathbb{Z}) \to H^2(M,\, \mathcal{O}_M)$ is zero (which, e.g., is true for Riemann surfaces C because of $H^2(C,\, \mathcal{O}_C) = 0)$, then

$$H^1(M,\, \mathcal{O}_M^*) \longrightarrow H^1(M,\, \mathcal{A}_M^*)$$

is surjective. Hence, in this case, each differentiable (even each topological) line bundle carries a complex analytic structure.

Using Čech cohomology, we can explicitly describe the *Chern homomorphism* c. It is always possible to find arbitrarily small open coverings $\mathfrak{U} = \{\, U_j \,\}_{j \in J}$ of the compact manifold M (f.i. by *triangulation*) such that

$$U_{j_0} \cap \ldots \cap U_{j_k}$$

is simply connected for all $j_0, \ldots, j_k$. Taking such a covering which trivializes L with respect to the transition functions f_{jk}, there exists always a holomorphic branch of the logarithm of $f_{jk} \in H^0(U_j \cap U_k,\, \mathcal{O}_M^*)$ on $U_{jk} = U_j \cap U_k$, i.e. a holomorphic function g_{jk} with

$$e\,(g_{jk}) = f_{jk}\ .$$

Since $f_{jk} f_{k\ell} = f_{j\ell}$ on $U_{jk\ell} = U_j \cap U_k \cap U_\ell$, we can conclude that

$$c_{jk\ell} := g_{jk} + g_{k\ell} + g_{\ell j} = \mathrm{Re}\ g_{jk} + \mathrm{Re}\ g_{k\ell} + \mathrm{Re}\ g_{\ell j}$$

is a constant function on the triple intersection $U_{jk\ell}$, hence $c_{jk\ell} \in H^0(U_{jk\ell},\, \mathbb{Z})$. Obviously, the system $(c_{jk\ell})$ is a cocycle in $Z^2(\mathfrak{U},\, \mathbb{Z})$ and, by definition, $c\,(L)$ is equal to the cohomology class

$$[\,(c_{jk\ell})\,] \in H^2(\mathfrak{U},\, \mathbb{Z}) \cong H^2(M,\, \mathbb{Z})$$

(the last isomorphism following from the fact that the covering $\mathfrak{U}$ is acyclic with respect to the sheaf $\mathbb{Z}$).

There is also a *differential–geometric* approach for defining Chern classes, since we have a canonical map

$$H^2(M,\, \mathbb{Z}) \longrightarrow H^2(M,\, \mathbb{R})$$

and (by the Poincaré Lemma) a *fine resolution* of the constant sheaf $\mathbb{R}$:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \longrightarrow \cdots \xrightarrow{d^{2n-1}} \mathcal{A}^{2n} \longrightarrow 0\ ,$$

where $\mathcal{A}^0 = \mathcal{A}_M$ and $\mathcal{A}^\nu = \mathcal{A}_M^\nu =$ sheaf of germs of real differentiable ν–forms on M, $n = \dim_{\mathbb{C}} M$. This resolution establishes the *de Rham isomorphism*

$$H^\nu_{dRh}(M,\, \mathbb{R}) := \ker d^\nu /\ \mathrm{im}\ d^{\nu-1} \cong H^\nu(M,\, \mathbb{R})\ , \quad \nu = 0,\, 1, \ldots, 2n\ .$$

We may therefore describe the image of $c\,(L)$ in $H^2(M,\, \mathbb{R})$ by a d–closed 2–form γ. The way to define γ is precisely the same as in the proof of Corollary 7: Choose a metric of L, i.e. a system of positive C^∞–functions $\{\, h_j \,\}$ on $\mathfrak{U}$ with

$$h_j = |\, f_{kj}\,|^2 h_k \ \text{ on } U_{jk}\ .$$

(Such a metric always exists, since it exists locally and can be constructed globally by patching the local metrics with the help of a C^∞–partition of unity). Then

$$\gamma = \frac{i}{2\pi}\, \overline{\partial} \partial \log\, h_j$$

is a globally defined d–closed real $(1, 1)$–form. To show that γ defines the same class in $H^2_{dRh}(M, \mathbb{R})$ as the cocycle (c_{ijk}) does, regarded as an element of $Z^2(\mathfrak{U}, \mathbb{R})$, we have to describe the isomorphism

$$H^2_{dRh}(M, \mathbb{R}) \longrightarrow H^2(M, \mathbb{R})$$

explicitly: If α is a d–closed real 2–form, then locally with respect to a covering $\mathfrak{U} = \{ U_j \}$, we find real 1–forms β_j on U_j satisfying

$$\alpha_{|U_j} = d\beta_j \ .$$

On $U_{jk} = U_j \cap U_k$, we have $d(\beta_j - \beta_k) = 0$, and therefore, there exist real–valued functions c_{jk} on U_{jk} with $dc_{jk} = \beta_j - \beta_k$. The image of (the class of) α is then the class of the cocycle of constants

$$c_{jk\ell} = c_{jk} + c_{k\ell} + c_{\ell j} \ .$$

In the case $\alpha = \gamma$, we may take

$$\beta_j = \frac{i}{4\pi} (\partial - \overline{\partial}) \log h_j \ ,$$

since

$$d\beta_j = \frac{i}{4\pi} (\partial + \overline{\partial})(\partial - \overline{\partial}) \log h_j = \frac{i}{2\pi} \overline{\partial}\partial \log h_j = \gamma_{|U_j}$$

and $\overline{\beta_j} = \beta_j$, i.e. β is a real form. Now, on $U_j \cap U_k$, we have, because of $h_j = | f_{kj} |^2 h_k$:

$$\begin{aligned}\beta_j - \beta_k &= \frac{i}{4\pi} (\partial - \overline{\partial}) \log | f_{kj} |^2 = \frac{i}{4\pi} (\overline{\partial} - \partial)(\log f_{jk} + \log \overline{f_{jk}}) = \frac{i}{4\pi} (\overline{\partial} \log \overline{f_{jk}} - \partial \log f_{jk}) \\ &= \frac{1}{4\pi i} (d \log f_{jk} - d \log \overline{f_{jk}}) \ .\end{aligned}$$

If g_{jk} are C^∞–functions with $e(g_{jk}) = e^{2\pi i g_{jk}} = f_{jk}$, then

$$d \log f_{jk} = 2\pi i \, dg_{jk} \ ,$$

and consequently,

$$\beta_j - \beta_k = d \left(\frac{1}{2} (g_{jk} + \overline{g_{jk}}) \right) = d(\mathrm{Re}\ g_{jk}) \ ,$$

i.e. $c_{jk} = \mathrm{Re}\ g_{jk}$ and

$$c_{jk\ell} = \mathrm{Re}\,(g_{jk} + g_{k\ell} + g_{\ell j}) \ .$$

Putting everything together, we have proved the "only if" part of

Theorem 9.11 *A holomorphic line bundle L on M is positive (negative), if and only if there exists a d–closed $(1, 1)$–form*

$$\gamma = \frac{i}{2} \sum_{\nu,\mu} g_{\nu\overline{\mu}} \, dx_\nu \wedge d\overline{x_\mu}$$

on M whose class in $H^2_{dRh}(M, \mathbb{R})$ is the image of $c(L)$ in $H^2(M, \mathbb{R})$ such that the hermitean form

$$\sum_{\nu,\mu} g_{\nu\overline{\mu}} \, \xi_\nu \overline{\xi_\mu}$$

is positive (negative) definite.

The "if"–part was proved by Kodaira using *Hodge theory* on the Kähler manifold M. We will not need it in the sequel.

9.6 Chern numbers and the degree of line bundles on Riemann surfaces

We first want to illustrate the results obtained so far in this Chapter by the bundles $\mathcal{O}(-k)$ on $\mathbb{P}_1$, $k = 1, 2, \ldots$. Recall the transition functions $u_0 = 1/u_1$, $v_0 = f_{01} v_1$, $f_{01} = u_1^k$, and define for arbitrary *positive* constants a_0, a_1:

$$h_0 = a_0 + a_1 \,|\, u_0 \,|^{2k} , \quad h_1 = a_1 + a_0 \,|\, u_1 \,|^{2k} .$$

(Notice that for $k = 1$ and $a_1 = a_2 = 1$, this is the "inverse" of the *standard* metric we introduced in Section 4 on the *dual* bundle $\mathcal{O}(1)$; see the proof of Theorem 9). Then, on $U_0 \cap U_1$,

$$h_0 = |\, u_0 \,|^{2k} h_1 = |\, f_{10} \,|^2 h_1 ,$$

and therefore,

$$\gamma = \frac{i}{2\pi} \, \overline{\partial} \partial \log (a_0 + a_1 \,|\, u_0 \,|^{2k}) ,$$

which can easily by computed to be equal to

$$\gamma = \frac{-i}{2\pi} \, k^2 \, \frac{a_0 \, a_1 \,|\, u_0 \,|^{2k-2}}{(a_0 + a_1 \,|\, u_0 \,|^{2k})^2} \, du_0 \wedge d\overline{u_0} \quad \text{on} \quad U_0$$

(and correspondingly on U_1). Hence,

$$\gamma = \frac{i}{2} \left\{ - \frac{k^2}{\pi} \cdot \frac{a_0 \, a_1 \,|\, u_0 \,|^{2k-2}}{(a_0 + a_1 \,|\, u_0 \,|^{2k})^2} \right\} du_0 \wedge d\overline{u_0} ,$$

such that the associated hermitean form is negative definite, forcing the line bundle $\mathcal{O}(-k)$ to be negative, as we already know.

The 2–form γ can be integrated over $\mathbb{P}_1$ (the projective line being oriented by its complex analytic structure). With $u_0 = x + iy$, we get $du_0 \wedge d\overline{u_0} = -2i \, dx \wedge dy$ and

$$\begin{aligned} \int_{\mathbb{P}_1} \gamma &= - \frac{k^2}{\pi} \int_{\mathbb{C}} \frac{a_0 \, a_1 \, (x^2 + y^2)^{k-1}}{(a_0 + a_1 \, (x^2 + y^2)^k)^2} \, dxdy \\ &= - \frac{k^2}{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{a_0 \, a_1 \, r^{2k-2}}{(a_0 + a_1 \, r^{2k})^2} \, r dr d\vartheta = - \frac{k^2 \cdot 2\pi}{\pi} \int_0^{\infty} \frac{a_0 \, a_1 \, r^{2k-1}}{(a_0 + a_1 \, r^{2k})^2} \, dr . \end{aligned}$$

Substituting $s = a_1 \, a_0^{-1} r^{2k}$, $ds = a_1 \, a_0^{-1} 2k \, r^{2k-1} dr$, this finally yields

$$\int_{\mathbb{P}_1} \gamma = -k \int_0^{\infty} \frac{ds}{(1 + s)^2} = k \left. \frac{1}{1 + s} \right|_0^{\infty} = -k .$$

The number we get by "integration of the Chern class" has different interpretations in the general situation: Since compact Riemann surfaces C are real 2–dimensional, there is by *Poincaré duality* an isomorphism

$$H^2(C, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$$

which can be normalized by associating to the *fundamental class* $[\, C \,] \in H^2(C, \mathbb{Z})$ the number $+1$. This map is then induced by the lower row map in the following diagram

$$\begin{tikzcd} H^2(C, \mathbb{Z}) \arrow[r, "\sim"] \arrow[d] & \mathbb{Z} \arrow[d] \\ H^2(C, \mathbb{R}) \arrow[r, "\sim"] & \mathbb{R} \end{tikzcd}$$

which can be described with the help of the de Rham isomorphism

$$H^2_{dRh}(C,\,\mathbb{R}) \,\cong\, H^2(C,\,\mathbb{R})$$

via

$$H^2_{dRh}(C,\,\mathbb{R}) \ni [\,\alpha\,] \longmapsto \int_C \alpha \in \mathbb{R}\,.$$

Note that this map is well–defined: If $[\,\alpha_1\,] = [\,\alpha_2\,]$, then $\alpha_1 - \alpha_2 = d\beta$, and by *Stokes' Theorem*:

$$\int_C d\beta = \int_{\partial C} \beta = 0\,, \ \text{ since } \partial C = \emptyset\,.$$

Consequently, we have the following result:

Theorem 9.12 *Let $\gamma \in \ker(d:\,\mathcal{A}^2 \to \mathcal{A}^3)$ be a representative of the Chern class $c\,(L) \in H^2(C,\,\mathbb{Z})$, L a holomorphic line bundle on the compact Riemann surface C. Then, under the natural isomorphism $H^2(C,\,\mathbb{Z}) \cong \mathbb{Z}$:*

$$c\,(L) = \int_C \gamma\,.$$

We call the image of $c\,(L)$ in $\mathbb{Z}$ the *Chern number* of L. By integration of the form γ in Theorem 11, we get the following necessary condition for positivity of a line bundle:

If a line bundle L on a Riemann surface is positive (negative), then its Chern number $c\,(L)$ is positive (negative).

Remark. We will see later, using the *Theorem of Riemann–Roch*, that the converse is also true.

The reader may have noticed that the number k for the bundle $\mathcal{O}\,(k)$, $k \in \mathbb{Z}$, can also be detected in the following way: Put $s_0 = u_0^k$, $s_1 = 1$, such that

$$s_0 = u_0^k = u_1^{-k} = f_{01}^{-1} s_1\,,$$

where $f_{01} = u_1^k$ defines $\mathcal{O}\,(k)$. Therefore, $s = (s_0,\,s_1)$ is a meromorphic section in $H^{\otimes k}$ (which is holomorphic for $k \geq 0$), whose total number of zeros counted with multiplicity (and negatively for poles) is precisely the number k (since s has a zero or a pole of order $|\,k\,|$ at $u_0 = 0$ and is holomorphic and nonvanishing everywhere else).

This again is a general fact: One can prove that for each holomorphic line bundle L on a compact Riemann surface there exists a nontrivial (i.e. not identically vanishing) meromorphic section s. Then the zeros and poles of s form a finite set, and we may count the number of zeros and poles as above. We call this number $d\,(L)$, the *degree* of L. The next theorem tells us in particular that $d\,(L)$ is in fact an invariant of L not depending on the section s. One can prove this also by remarking that the quotient $s_1/\,s_2$ of two nontrivial meromorphic sections of L is a nontrivial meromorphic function f on C. The *Residue Theorem* for compact Riemann surfaces then says that the total number of zeros of f equals the total number of poles of f. This, in turn, follows from the next theorem (by application to the trivial bundle).

Theorem 9.13 *Let $L \to C$ be a holomorphic line bundle on the compact Riemann surface C. Then the Chern number of L equals the degree of L.*

Proof. Let s be a nontrivial meromorphic section of L: $s = (s_j)_{j\in J}$, $s_j = f_{jk}s_k$, with respect to a covering $\mathfrak{U} = \{\,U_j\,\}_{j\in J}$. We may assume that the U_j are disks, that s_j has only a zero or a pole at the center of U_j, and that $\overline{U_k}$ does not contain the center of U_j for all $k \neq j$.

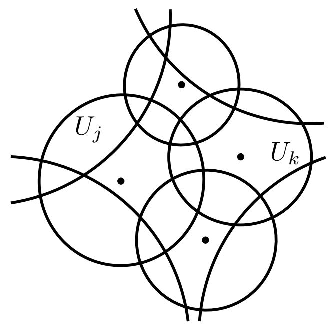


Figure 9.2

Then, by shrinking $\mathfrak{U}$ to a covering $\mathfrak{V} = \{ V_j \}$ of C with somewhat smaller disks V_j centered at the same points as the U_j, and by using a partition of unity, we find positive real functions h_j and U_j with

$$h_j = | f_{kj} |^2 h_k \text{ on } U_j \cap U_k$$

and

$$h_j = | s_j |^{-2} \text{ in a neighborhood of } \partial V_j .$$

Let J_0 be the set of elements $j \in J$ such that s_j has a zero or a pole at the center of U_j. Then, of course, we can use $h_j = | s_j |^{-2}$ for all $j \in J \setminus J_0$. Denote by γ the form associated to the metric $h = (h_j)$; then

$$\gamma_j = \gamma_{|U_j} = \frac{i}{2\pi} \overline{\partial} \partial \log h_j = 0$$

for all $j \in J \setminus J_0$, and therefore,

$$c(L) = \int_C \gamma = \int_{C_0} \gamma , \quad C_0 = \bigcup_{j \in J_0} V_j .$$

Since, without loss of generality, $V_j \cap V_k = \emptyset$ for all $j \neq k$ in J_0, we have

$$c(L) = \sum_{j \in J_0} \int_{V_j} \gamma .$$

It remains to prove that the integral

$$\int_{V_j} \gamma$$

is equal to the number of zeros or to the negative of the number of poles of s_j at the origin of V_j. Let $n_j \in \mathbb{Z}$ denote this number; then we may assume that V_j is so small that, without loss of generality,

$$s_j = z^{n_j}$$

(where z is a coordinate on V_j vanishing at the origin), and we conclude easily from Stokes' Theorem and the classical *Residue Theorem* that

$$\begin{aligned} \int_{V_j} \gamma &= \frac{i}{2\pi} \int_{V_j} d\partial \log h_j = \frac{i}{2\pi} \int_{\partial V_j} \partial \log | z |^{-2n_j} \\ &= -n_j \frac{i}{2\pi} \int_{\partial V_j} \partial \log | z |^2 = -n_j \frac{i}{2\pi} \int_{\partial V_j} \frac{dz}{z} = -n_j \frac{i}{2\pi} 2\pi i = n_j . \end{aligned}$$

9.7 Point bundles on Riemann surfaces

For the compact Riemann surfaces C, we know - as we mentioned before - that the cohomology group $H^2(C, \mathcal{O}_C)$ vanishes. Consequently, the Chern map

$$c: H^1(C, \mathcal{O}_C^*) \longrightarrow H^2(C, \mathbb{Z}) \cong \mathbb{Z}$$

is surjective. The purpose of the present Section is the construction of some line bundles on C with prescribed Chern number. For this, it is sufficient to find holomorphic line bundles $L \to C$ with $c(L) = 1$, since, from Theorem 13, one can easily deduce that the Chern map is a group homomorphism with respect to the tensor product of line bundles:

$$c(L_1 \otimes L_2) = c(L_1) + c(L_2) .$$

(Of course, this follows from the very definition for compact complex manifolds M of arbitrary dimension).

The construction is a special example of assigning a holomorphic line bundle to a *divisor* on a manifold (see Chapter 5.10). In the case of a Riemann surface C, such a divisor consists of (finitely many) points counted with multiplicity. Now, for all $x^{(0)} \in C$, there exists a distinguished sheaf on C, namely the *maximal ideal sheaf* $I_{\{x^{(0)}\}}$ associated to $\{x_0\}$ (i.e. the sheaf of germs of holomorphic functions on C which vanish at $x^{(0)}$). In the sequel, we will also use the symbol $I(x^{(0)})$ when referring to the maximal ideal sheaf. Notice that the ideal sheaf $I(x^{(1)}, x^{(2)})$ in the proof of Theorem 8 is the product $I(x^{(1)})\, I(x^{(2)})$. Clearly, $I(x^{(0)})$ is a (locally) principal ideal in $\mathcal{O}_C$, generated in a neighborhood U_0 of $x^{(0)}$ by a coordinate function z - a special feature of dimension one. Moreover, there exists a canonical isomorphism

$$I(x^{(0)})_{|U_1} \cong \mathcal{O}_{U_1} ,$$

where $U_1 = C \setminus \{x^{(0)}\}$.

The reader may amuse himself by checking that the cocycle $\{f_{01}, f_{10}\}$ given by

$$f_{10} = z_{|U_0 \cap U_1}$$

with respect to the covering $\mathfrak{U} = \{U_0, U_1\}$ defines a line bundle $L = L(x^{(0)})$ such that the sheaf of holomorphic sections $\mathcal{O}_C(L^*)$ in the dual bundle L^* is canonically isomorphic to $I(x^{(0)})$. Since $s = (s_0, s_1)$, $s_0 = z$, $s_1 = 1$, is a global holomorphic section in L, the so–called *point bundle* $L(x^{(0)})$ has Chern class

$$c(L(x^{(0)})) = 1 .$$

Notice that in the language of Chapter 5.10 the bundle $L(x^{(0)})$ is exactly the line bundle associated to the divisor $1 \cdot x^{(0)}$ on C and $I(x^{(0)}) \cong \mathcal{O}_C([-1 \cdot x^{(0)}])$.

In particular, on every compact Riemann surface C there exist negative line bundles (if we take the characterization of negative line bundles as those with negative Chern number for granted). By blowing down the zero–section, we get a normal two–dimensional singularity which is regular, as we will see, only if $C \cong \mathbb{P}_1$ and $L \to \mathbb{P}_1$ is the tautological bundle (i.e. the dual of the point bundle $L(x^{(0)})$ for an arbitrarily chosen point $x^{(0)} \in \mathbb{P}_1$). - So, we may state:

Each compact Riemann surface may be realized as the exceptional set in a resolution of a (nontrivial) normal surface singularity.

We shall call a singularity that is obtained by blowing down the zero–section in a holomorphic line bundle L with *ample* dual L^* a *generalized cone* (since *very* ample bundles L^* lead to *cones* as explained in Chapter 5). We will have to say more about such cones in Chapter 10, since they represent examples and, in fact, fundamental building objects for *quasihomogeneous* singularities such that we also refer to them as *quasi–cones*.

9.8 Statement of the Grauert - Mumford criterion

It is time to formulate the numerical criterion of Grauert and Mumford. We start with a fixed *connected reduced compact curve* A in M. Regarded as a divisor, one can associate an *intersection matrix* to A (see Chapter 5). The result whose proof we are going to attack is the famous

Theorem 9.14 (Grauert - Mumford) *A is exceptional in M if and only if the intersection matrix of A is negative–definite.*

Since the intersection theory of *smooth* embedded curves is much easier to handle than that of singular ones, we will use the fact that we have embedded resolutions of singularities of the curve A at our disposal.

So, after an appropriate sequence of blow ups with center in A we find modifications $\sigma : \widetilde{M} \to M$ such that the total transform $E = \sigma^{-1}(A)$ is a normal crossing divisor, i.e. the components E_j of E are smooth and intersect transversally. Fix such a modification. After blowing up all intersection points, we find another modification $\overline{M} \to \widetilde{M} \to M$ with total transform $\overline{E}$ of E.

Then, we have the following more precise statement.

Theorem 9.15 *The following are equivalent*:

i) *A is exceptional in M*;

ii) *the intersection matrix of A is negative definite* ;

iii) *E is exceptional in $\widetilde{M}$*;

iv) *the intersection matrix of E is negative definite* ;

v) *there exist* positive integers m_j *such that all restrictions* $L_j = L_{|E_j}$ *of* $L := [\sum_{k=1}^{r} m_k E_k]$ *are negative* ;

vi) *denoting any line bundle on $\overline{E}$ satisfying the properties in* v) *by $\overline{L}$, then $\overline{L}$ carries a metric in a neighborhood of $\overline{E}$ such that each restriction of $\overline{L}$ to a component of $\overline{E}$ is negative with respect to this metric* ;

vii) *$\overline{E}$ is exceptional in $\overline{M}$.*

In each case, the exceptional sets A resp. E resp. $\overline{E}$ contract to the same singularity.

The prize we have to pay is the need for a deeper understanding of the theory of compact Riemann surfaces and their holomorphic line bundles (see Sections 3, 5, 6, 7, 9, 10, 11 and 12).

The *proof* of Theorem 15 which will occupy the greater part of this Chapter goes along the scheme:

$$\begin{array}{ccccc}
\text{i)} & \Longleftrightarrow & \text{iii)} & \Longleftrightarrow & \text{vii)} \\
 & & \Downarrow & & \\
\text{ii)} & \Longleftrightarrow & \text{iv)} & & \Uparrow \\
 & & \Downarrow & & \\
 & & \text{v)} & \Longrightarrow & \text{vi)}
\end{array}$$

Remark that we already discussed ii) $\Leftrightarrow$ iv) in Chapter 5 (Theorem 9). The equivalences i) $\Leftrightarrow$ iii) $\Leftrightarrow$ vii) and the final statement will be a consequence of Theorem 31, and iii) $\Rightarrow$ iv) can be found in Section 19. Finally, we prove the implications iv) $\Rightarrow$ v) $\Rightarrow$ vi) $\Rightarrow$ vii) $\Rightarrow$ iii) in Sections 21, 22 and 23.

9.9 The Picard group of Riemann surfaces

By the previous considerations it is clear that we should be interested in knowing the complete structure of $H^1(C, \mathcal{O}_C^*)$ for a compact Riemann surface C. Under the operation of the tensor product, this is an abelian group which is usually called the *Picard group* of C, denoted by $\operatorname{Pic} C$. The subgroup

$$\operatorname{Pic}^0 C = \{ \xi \in \operatorname{Pic} C : c(\xi) = 0 \}$$

of holomorphic line bundles which are topologically trivial contains already all information about the variety of holomorphic line bundles on C, since for every point bundle ξ_0 there exists an isomorphism

$$\begin{cases} \operatorname{Pic}^0 C \times \mathbb{Z} \longrightarrow \operatorname{Pic} C \\ \qquad (\xi, \nu) \longmapsto \xi \otimes \xi_0^\nu . \end{cases}$$

By definition, $\operatorname{Pic}^0 C$ is the kernel of the Chern morphism $c : H^1(C, \mathcal{O}_C^*) \to H^2(C, \mathbb{Z})$, such that we may deduce from the long exact cohomology sequence associated to the exponential sequence that

$$\operatorname{Pic}^0 C \cong H^1(C, \mathcal{O}_C) / j\,(H^1(C, \mathbb{Z})) ,$$

where j denotes the canonical map

$$H^1(C, \mathbb{Z}) \longrightarrow H^1(C, \mathcal{O}_C) .$$

It is well–known that compact Riemann surfaces are topologically classified by an integer $g \geq 0$, called its *genus*: As a topological space, it looks like a sphere S^2 with g *handles* attached:

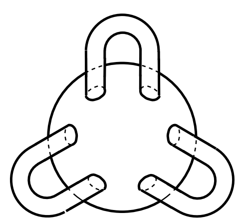


Figure 9.3

Clearly, in the case $g = 0$ we have the sphere itself:

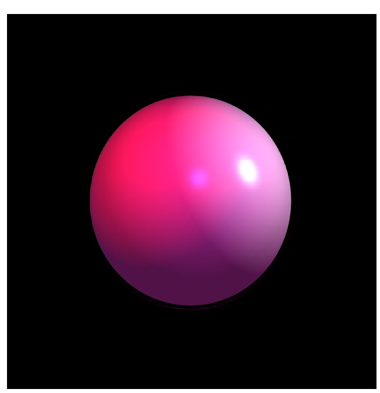


Figure 9.4

There should be no problem to the reader realizing that a genus 1 Riemann surface is topologically a *torus*:

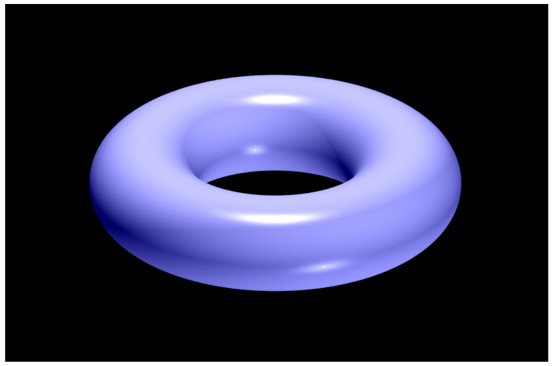


Figure 9.5

Here are pictures of surfaces of genus 2 and 3:

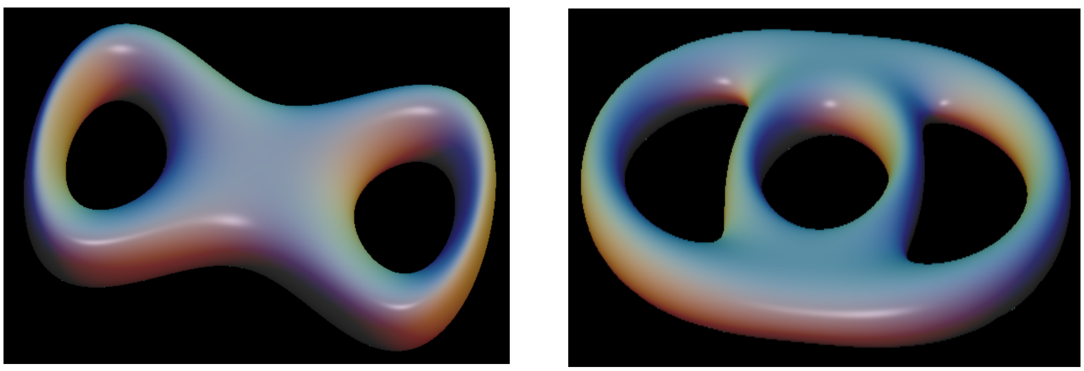


Figure 9.6

the last one also in an incarnation created by MAX BILL (standing near the "Außenalster" in Hamburg).

Figure 9.7

On such a *handle–body* (of genus g) there are obviously $2g$ mutually nonhomologous cycles:

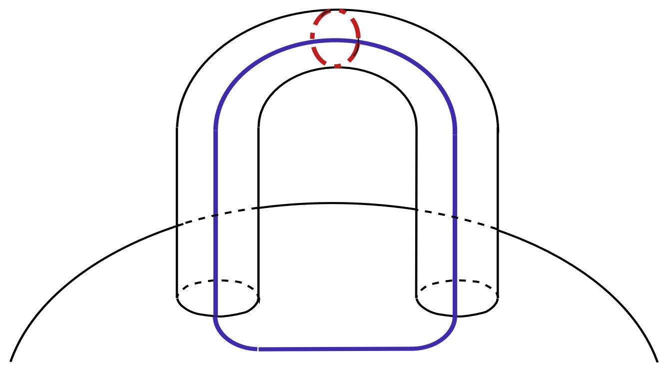


Figure 9.8

One can show, in fact, that $H_1(C, \mathbb{Z})$ is free of rank $2g$. Hence (by duality)

$$\begin{aligned} H^1(C, \mathbb{Z}) &\cong \mathbb{Z}^{2g} , \\ H^1(C, \mathbb{R}) &\cong H^1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{2g} , \\ H^1(C, \mathbb{C}) &\cong \mathbb{C}^{2g} . \end{aligned}$$

Since the map $\mathbb{C} \cong H^0(C, \mathcal{O}_C) \xrightarrow{e} H^0(C, \mathcal{O}_C^*) \cong \mathbb{C}^*$ is surjective, the map j above is necessarily injective. Moreover, $j\,(H^1(C, \mathbb{Z}))$ is a *lattice* of rank $2g$ in the complex vector space $H^1(C, \mathcal{O}_C)$, which is of dimension g (see Section 10). This implies:

The group $\operatorname{Pic}^0 C$ *is isomorphic to a g–dimensional torus* $\mathbb{C}^g / \mathbb{Z}^{2g}$.

Conceptually, this torus is the *Jacobi variety* $\operatorname{Jac} C$ of C. - For more details, we refer to Chapter 15.8 and 9, where we will have a closer look at $\operatorname{Pic}^0 C$ in order to understand the Picard group of surface singularities.

For *Example*, if $g = g\,(C) = 0$, i.e. if $C \cong \mathbb{P}_1$, then $\operatorname{Pic}^0 C = 0$ and $\operatorname{Pic} C \cong \mathbb{Z}$: The holomorphic line bundles on $\mathbb{P}_1$ are precisely the (positive and negative) tensor powers of the hyperplane bundle $H \cong L\,(x^{(0)})$, $x^{(0)} \in \mathbb{P}_1$ arbitrary.

For genus one, i.e. for $C \cong \mathbb{C}/\,\Gamma$, Γ a lattice of rank 2 in $\mathbb{C}$, $\operatorname{Pic}^0 C$ is isomorphic to the group C itself. This isomorphism is constructed in the following manner: Choose a fixed point $x^{(0)} \in C$ as the neutral element in C, then the map

$$C \longrightarrow \operatorname{Pic}^0 C$$

is given by

$$x \longmapsto L\,(x) \otimes L\,(x^{(0)})^* .$$

9.10 The Theorem of Riemann and Roch

The precise answer to the question: "How many sections does a holomorphic line bundle L on a compact Riemann surface C have?" is given by the classical *Riemann–Roch Theorem*:

***Theorem 9.16** *For all holomorphic line bundles L on a given compact Riemann surface C, the number*

$$\dim H^0(C, \mathcal{O}_C(L)) - \dim H^1(C, \mathcal{O}_C(L)) - d\,(L)$$

is equal to $1 - g$, *where* $g = g\,(C)$ *denotes the genus of* C.

In fact, one has much more information: the pairs (d, γ) for which there exists a line bundle L on a Riemann surface of genus g with $d = d(L)$ and $\gamma = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_C(L))$ lie in the *Riemann–Roch diamond*

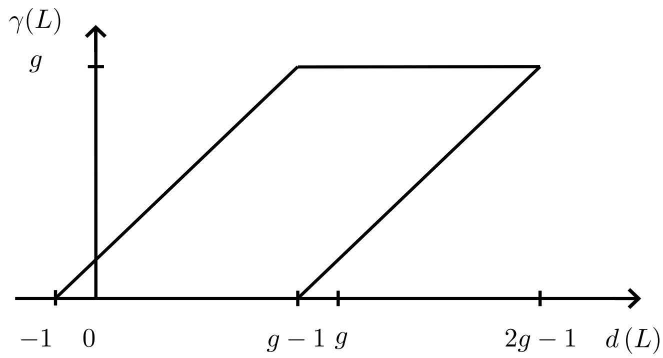


Figure 9.9

"Most" bundles lie on the lower border of the diamond. The others are called *special*. There exists an intensive study of these special line bundles (or equivalently: of special divisors). E.g., for $g \geq 1$, the trivial bundle $(d = 0,\ \gamma = 1)$ is always special. In case of a torus $\mathbb{C}/\Gamma$, all bundles $L(x) \otimes (L(x^{(0)}))^*$, $x \neq x^{(0)}$, are nonspecial of degree 0.

The classical contribution of Roch interprets the defect $\dim H^1(C, \mathcal{O}_C(L))$ (Riemann's Theorem was the *inequality*

$$\dim H^0(C, \mathcal{O}_C(L)) - d(L) \leq 1 - g\ )$$

in terms of sections in another line bundle as

$$\dim H^0(C, \mathcal{O}_C(L^* \otimes K_C)) ,$$

where K_C denotes the *canonical* line bundle on C (i.e. the cotangent bundle whose sections are the holomorphic 1–forms on C). The equality

$$\dim H^1(C, \mathcal{O}_C(L)) = \dim H^0(C, \mathcal{O}_C(L^* \otimes K_C))$$

is a consequence of *Serre duality* (see Section 12).

Modern proofs of Theorem 16 proceed by induction on $d(L)$, showing that the expression

$$\dim H^0(C, \mathcal{O}_C(L)) - \dim H^1(C, \mathcal{O}_C(L)) - d(L)$$

is constant on a given compact Riemann surface. Now, for the trivial bundle $\mathcal{O}_C$, one has $H^0(C, \mathcal{O}_C) \cong \mathbb{C}$, $d(\mathcal{O}_C) = 0$. Thus, it remains to prove that

$$\dim H^1(C, \mathcal{O}_C) = g = g(C) .$$

Again, by Serre duality, this equality is equivalent to

$$\dim H^0(C, \mathcal{O}_C(K_C)) = g ,$$

or in other words:

On a compact Riemann surface of genus g, there exist precisely g linearly independent holomorphic 1–forms.

To prove this classical result, we start with the standard resolution of the constant $\mathbb{C}$–sheaf

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_C \xrightarrow{d} \Omega^1_C \longrightarrow 0 ,$$

Ω^1_C denoting the sheaf of germs of holomorphic 1–forms on C, i.e. $\Omega^1_C = \mathcal{O}(K_C)$. Taking the long exact cohomology sequence, we get the exact sequence (writing $H^j(\cdot)$ instead of $H^j(C,\cdot)$):

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{C}) &\longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\Omega^1_C) \\ \longrightarrow H^1(\mathbb{C}) &\longrightarrow H^1(\mathcal{O}_C) \longrightarrow H^1(\Omega^1_C) \\ \longrightarrow H^2(\mathbb{C}) &\longrightarrow H^2(\mathcal{O}_C) \quad . \end{aligned}$$

Now, $H^0(\mathbb{C}) \cong \mathbb{C} \cong H^0(\mathcal{O}_C)$, such that $H^0(\mathbb{C}) \hookrightarrow H^0(\mathcal{O}_C)$ is an isomorphism. Further, $H^2(\mathcal{O}_C) = 0$ and, by Poincaré duality, $H^2(\mathbb{C}) = H_0(\mathbb{C}) \cong \mathbb{C}$. Applying Serre duality once more, we have

$$\dim H^1(\Omega^1_C) = \dim H^0(\mathcal{O}_C(T_C) \otimes \Omega^1_C) = \dim H^0(\mathcal{O}_C) = 1 ,$$

since the tangent bundle T_C is dual to the canonical bundle K_C. Therefore, the epimorphism $H^1(\Omega^1_C) \to H^2(\mathbb{C})$ must be invertible, and we deduce the exactness of the sequence

$$0 \longrightarrow H^0(\Omega^1_C) \longrightarrow H^1(\mathbb{C}) \longrightarrow H^1(\mathcal{O}_C) \longrightarrow 0 .$$

Finally, $\dim H^0(\Omega^1_C) = \dim H^1(\mathcal{O}_C)$, and $\dim H^1(\mathbb{C}) = 2g$.

9.11 Characterizations of positivity for line bundles on compact Riemann surfaces

We are now ready for a more detailed analysis of positivity (and negativity).

Theorem 9.17 *Let L be a holomorphic line bundle on a (smooth) compact Riemann surface C. Then, the following are equivalent (L_1 denotes any other holomorphic line bundle):*

i) *L is positive* ;

ii) *the degree $d(L)$ (and, equivalently, the Chern number $c(L)$) is positive* ;

iii) $\dim H^1(C, \mathcal{O}_C(L^{\otimes k} \otimes L_1)) = 0$ *for all* $k \geq k_1(L_1) > 0$;

iv) $\dim H^0(C, \mathcal{O}_C(L^{\otimes k} \otimes L_1)) = k\, d(L) + d(L_1) - 1 + g$ *for all* $k \geq k_1(L_1) > 0$;

v) *$L^{\otimes k}$ is very ample for all $k \geq k_0(L)$* ;

vi) *L is ample.*

Proof. i) $\Longrightarrow$ ii) has been seen already in Section 6. The equivalence of iii) and iv) is a direct consequence of the Riemann–Roch Theorem. v) and vi) are equivalent by definition. If v) is satisfied, the holomorphic sections of $L^{\otimes k}$ embed C into some $\mathbb{P}_N$, and $L^{\otimes k}$ is the pullback of the positive hyperplane bundle on $\mathbb{P}_N$ which gives i). So, it remains to show that ii) $\Longrightarrow$ iii) $\Longrightarrow$ v).

ii) $\Longrightarrow$ iii). By Serre duality, $\dim H^1(C, \mathcal{O}_C(L^{\otimes k} \otimes L_1))$ is the same as the dimension of $H^0(C, \mathcal{O}_C((L^{\otimes k} \otimes L_1)^* \otimes K_C))$. The degree of this line bundle is

$$-k\, d(L) - d(L_1) + d(K_C)$$

and hence negative, if $k \geq k_1$ where k_1 depends on L_1 (and C). Therefore, this bundle has no nontrivial holomorphic sections.

iii) $\Longrightarrow$ v). Following Grauert's arguments (see Section 4) we have to show that for large k the bundle $L^{\otimes k}$ satisfies $H^1(C, I(x^{(1)}, x^{(2)})\, \mathcal{O}(L^{\otimes k})) = 0$ for all $x^{(1)}, x^{(2)} \in C$ which is clear from iii) since $I(x^{(1)}, x^{(2)})$ is the sheaf of holomorphic sections in a product of two point bundles. $\square$

Remark. The constants k_0 and k_1 can be made more precise by the remark that $d(K_C) = 2g - 2$ which follows from the information the Riemann–Roch formula produces for $L = K_C$.

9.12 Serre duality

At several other occasions, we will be led to use Serre duality which we therefore explain here in a more general set–up for a compact complex manifold M and a holomorphic vector bundle F on M. Recall that, by *Dolbeault's Lemma*, we have an exact sequence

$$(*)\qquad 0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{A}_M^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_M^{0,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}_M^{0,n} \longrightarrow 0\ , \quad n = \dim_{\mathbb{C}} M\ ,$$

where $\mathcal{A}_M^{p,q}$ denotes the (fine) sheaf of differential forms of type (p, q) on M and the differential operator $\overline{\partial}$ is defined on functions by

$$\overline{\partial} f := \sum_{\nu=1}^{n} \frac{\partial f}{\partial \overline{z_\nu}}\, d\overline{z_\nu}$$

and on forms by

$$\begin{aligned}&\overline{\partial}\left(\sum a_{\nu_1\ldots\nu_p\overline{\mu_1}\ldots\overline{\mu_q}}\, dz_{\nu_1}\wedge\ldots\wedge dz_{\nu_p}\wedge d\overline{z_{\mu_1}}\wedge\ldots\wedge d\overline{z_{\mu_q}}\right)\\ &\quad := \sum \overline{\partial} a_{\nu_1\ldots\nu_p\overline{\mu_1}\ldots\overline{\mu_q}}\wedge dz_{\nu_1}\wedge\ldots\wedge d\overline{z_{\mu_1}}\wedge\ldots\quad .\end{aligned}$$

If F is a holomorphic vector bundle on M, we denote by $\mathcal{A}_M(F)$ the sheaf of germs of C^∞–sections in F, and by

$$\mathcal{A}_M^{p,q}(F) = \mathcal{A}_M^{p,q} \otimes_{\mathcal{A}_M} \mathcal{A}_M(F)$$

the sheaf of *germs of differential forms of type* (p, q) *on* M *with values in* F. In the following, we drop the index M.

The Dolbeault complex $(*)$ gives rise to a fine resolution of the sheaf $\mathcal{O}(F)$ of holomorphic sections in F:

$$0 \longrightarrow \mathcal{O}(F) \longrightarrow \mathcal{A}^{0,0}(F) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{A}^{0,n}(F) \longrightarrow 0\ .$$

Hence,

$$H^j(M, \mathcal{O}(F)) \cong H^j_{\mathrm{Dolb}}(M, \mathcal{O}(F))\ ,$$

where

$$H^j_{\mathrm{Dolb}}(M, \mathcal{O}(F)) = \frac{\ker(\overline{\partial}:\ H^0(M, \mathcal{A}^{0,j}(F)) \longrightarrow H^0(M, \mathcal{A}^{0,j+1}(F)))}{\mathrm{im}\,(\overline{\partial}:\ H^0(M, \mathcal{A}^{0,j-1}(F)) \longrightarrow H^0(M, \mathcal{A}^{0,j}(F)))}\ .$$

Thus, each pair

$$(\zeta, \mu) \in H^j(M, \mathcal{O}(F)) \times H^{n-j}(M, \mathcal{O}(F^* \otimes K_M))$$

(K_M the *canonical bundle* of M, i.e. the n–th exterior power of the cotangent bundle) can be represented by a pair of $\overline{\partial}$–closed forms

$$\alpha \in H^0(M, \mathcal{A}^{0,j}(F))\ , \quad \beta \in H^0(M, \mathcal{A}^{0,n-j}(F^* \otimes K_M))\ .$$

Since forms in $K = K_M$ are of type $(n, 0)$, the exterior product $\alpha\wedge\beta$ is a global (n, n)–form in the usual sense:

$$\alpha\wedge\beta \in H^0(M, \mathcal{A}^{n,n})$$

which, therefore, can be integrated over M. To show that this integral is independent of the choice of α and β, assume, for instance, that $\alpha = \overline{\partial}\gamma$ with a form γ of type $(0, j-1)$. Since $\overline{\partial}\beta = 0$, we have

$$\overline{\partial}(\gamma\wedge\beta) = \overline{\partial}\gamma\wedge\beta = \alpha\wedge\beta\ .$$

Moreover,

$$\partial(\gamma\wedge\beta) = 0\ ,$$

$\gamma\wedge\beta$ being of type $(n, n-1)$. By Stokes' Theorem, it follows that

$$\int_M \alpha\wedge\beta = \int_M d(\gamma\wedge\beta) = \int_{\partial M} \gamma\wedge\beta = 0\ .$$

Serre duality can now be phrased as follows:

***Theorem 9.18** *The bilinear pairing*

$$H^j(M, \mathcal{O}(F)) \times H^{n-j}(M, \mathcal{O}(F^* \otimes K_M)) \longrightarrow \mathbb{C} ,$$

defined by

$$([\alpha], [\beta]) \longmapsto \int_M \alpha \wedge \beta ,$$

is perfect, i.e. it realizes $H^{n-j}(M, \mathcal{O}(F^* \otimes K_M))$ *as the dual space of* $H^j(M, \mathcal{O}(F))$, $j = 0, \ldots, n$.

If M is not compact, we can study cohomology with compact support (denoted by H_c^j) which can be realized by $\overline{\partial}$–closed forms with compact support. As above, one constructs a bilinear pairing

$$H^j(M, \mathcal{O}(F)) \times H_c^{n-j}(M, \mathcal{O}(F^* \otimes K)) \longrightarrow \mathbb{C}$$

which is still perfect, if one of the cohomology groups involved is finite dimensional. Moreover, $\mathcal{O}(F)$ may be replaced by any coherent analytic sheaf (and $\mathcal{O}(F^*)$ by its dual).

9.13 The adjunction formula

For a smooth curve C in a two–dimensional manifold M there is a canonical isomorphism

$$\Omega^2_{M|C} \cong \Omega^1_C \otimes N^*_{C|M} .$$

This *adjunction formula* relates the *canonical bundle* of C to the *canonical bundle* of M via the *normal bundle* of C in M. Taking into account that $\mathcal{O}(N_{C|M}) \cong \mathcal{O}_M(C)_{|C}$ (see Section 15, Theorem 22, for more details) we can rewrite this in the form

$$\Omega^1_C \cong \Omega^2_{M|C} \otimes \mathcal{O}_C(C) .$$

It is the aim of the present Section to prove this result as a very special case of elementary *multilinear algebra*. First of all, we generalize the situation by regarding a smooth hypersurface H in an n–dimensional complex analytic manifold M. For the *holomorphic tangent bundles* we have a natural inclusion

$$T_H \hookrightarrow T_{M|H}$$

which gives rise to an exact sequence of vector bundles

$$0 \longrightarrow T_H \longrightarrow T_{M|H} \longrightarrow N \longrightarrow 0 ,$$

in which the *line bundle* $N = T_{M|H}/T_H$ on H is called the *normal bundle* of H in M and denoted by $N_{H|M}$. Dualizing the sequence gives a new exact sequence

$$0 \longrightarrow N^* \longrightarrow T^*_{M|H} \longrightarrow T^*_H \longrightarrow 0 ,$$

and what we finally claim is that there is a natural isomorphism

$$\big((\wedge^n T^*_M)_{|H} \cong\big) \wedge^n T^*_{M|H} \cong N^* \otimes \wedge^{n-1} T^*_H .$$

Generalizing further, we may expect the following.

Theorem 9.19 *Let*

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

be an exact sequence of vector bundles. Then, there exists a natural isomorphism

$$\wedge^n F^* \cong \wedge^{n_1} F_1^* \otimes \wedge^{n_2} F_2^*$$

where $n = \operatorname{rang} F$, $n_1 = \operatorname{rang} F_1$, $n_2 = \operatorname{rang} F_2$.

Proof. As a general principle, any *canonical* isomorphism of vector spaces leads to such isomorphisms of vector bundles. So, e.g., the natural isomorphism $V \cong V^{**}$ of finite dimensional vector spaces induces a natural isomorphism $F \cong F^{**}$ of vector bundles. Therefore, we can replace the vector bundles in the theorem by finite dimensional vector spaces and claim: For every exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow \overline{V} \longrightarrow 0$$

of vector spaces of dimension $m, n, s = n - m$, resp., there exists a canonical isomorphism

$$\wedge^n V^* \cong \wedge^m U^* \otimes \wedge^s \overline{V}^* .$$

But the vector spaces on both sides are canonical isomorphic to the spaces

$$\mathrm{Alt}^n V \quad \text{and} \quad \mathrm{Alt}^m(U, \mathrm{Alt}^s \overline{V}) , \text{ resp.,}$$

where Alt denotes vector spaces of *alternating* multilinear forms, and there is a canonical mapping

$$\mathrm{Alt}^n V \longrightarrow \mathrm{Alt}^m(U, \mathrm{Alt}^s V)$$

by associating to an alternating multilinear form $\mu = \mu(v_1, \ldots, v_n)$ the alternating form which associates to each m–tupel $(u_1, \ldots, u_m)$ of vectors in U the alternating form μ_u on V defined by

$$\mu_u(v_{m+1}, \ldots, v_n) := \mu(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n) .$$

Since $m = \dim U$, we have automatically

$$\mu_u(v_{m+1} + u_{m+1}, \ldots, v_n + u_n) = \mu_u(v_{m+1}, \ldots, v_n)$$

for all $u_{m+1}, \ldots, u_n \in U$, such that the given map has its image in $\mathrm{Alt}^m(U, \mathrm{Alt}^s \overline{V})$. Since the map is not zero and each space on both sides is of dimension one, it is an isomorphism. □

9.14 Rudiments of intersection theory

Before we embark into the proof of the Grauert–Mumford criterion, we would like to investigate the notion of *self–intersection numbers* more intensively. In particular, we have to explain why these numbers can be negative. For that, we need a rudimentary form of *general intersection theory*.

If M is any oriented differentiable manifold of real dimension $2n$, one has a bilinear pairing

$$\langle \, . \, , \, . \, \rangle : H_n(M, \mathbb{Z}) \times H_n(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

which counts the number of intersection points in the following sense: Denote by A and B two oriented differentiable submanifolds of M of dimension n. Then the fundamental classes $[A]$ and $[B]$ of A resp. B are elements in $H_n(A, \mathbb{Z})$ resp. in $H_n(B, \mathbb{Z})$. The inclusions $A \hookrightarrow M$ and $B \hookrightarrow M$ generate canonical maps

$$H_n(A, \mathbb{Z}) \longrightarrow H_n(M, \mathbb{Z}) , \quad H_n(B, \mathbb{Z}) \longrightarrow H_n(M, \mathbb{Z}) ,$$

such that the product

$$\langle [A], [B] \rangle \in \mathbb{Z}$$

is defined. Now, if $A \cap B$ is a finite set $\{ x^{(1)}, \ldots, x^{(t)} \}$ such that, at each point $x^{(\tau)}$, the sets A and B intersect *transversely*, i.e. if for the differentiable tangent spaces

$$T_{A,x^{(\tau)}} \oplus T_{B,x^{(\tau)}} = T_{M,x^{(\tau)}} ,$$

then

$$\langle [A], [B] \rangle = \sum_{\tau=1}^{t} \varepsilon_\tau$$

where $\varepsilon_\tau = +1$, if the orientations of A and B at $x^{(\tau)}$ induce the orientation of M at $x^{(\tau)}$, or $\varepsilon_\tau = -1$, if the opposite orientation is induced.

Recall that we defined an intersection number between complex analytic curves in smooth surfaces and holomorphic line bundles and obtained a good notion of intersection numbers for curves and divisors by associating a line bundle to the divisor in Chapter 5. Thus, we restrict ourselves from now on to the case of a two–dimensional complex analytic manifold M (endowed with its canonical orientation) and a (connected) compact one–dimensional submanifold $C \subset M$. We first want to interpret the intersection number of C with a holomorphic line bundle L on M as an appropriate integral on C. The Chern class of L is, as an element of

$$H^2_{dRh}(M, \mathbb{R}) \, ,$$

given by a real d–closed 2–form γ on M. Then we define for the moment

$$[\,C,\, L\,] := [\,C,\, [\,\gamma\,]\,] = \int_C \gamma \, .$$

(Note that this definition depends only on the isomorphism class of L in $H^1(M, \mathcal{O}^*_M)$). Of course, γ restricted to C is a closed real 2–form on C whose class in

$$H^2_{dRh}(C, \mathbb{R})$$

is the Chern class of the restriction $L_{|C}$. Hence, by Theorem 12 and 13,

$$(*) \qquad [\,C,\, L\,] = c\,(L_{|C}) = d\,(L_{|C}) \in \mathbb{Z} \, ,$$

and therefore, our definition of the intersection number $(C,\, L)$ in Chapter 5 coincides in this case with the number $[\,C,\, L\,]$. Recall that we defined the *intersection number* of C with an arbitrary *divisor* D on M via the correspondence between *divisors* and line bundles by

$$(C \cdot D) := (C,\, [\,D\,]) \, .$$

Notice finally that we have $[\,D_1 + D_2\,] \cong [\,D_1\,] \otimes [\,D_2\,]$ for all divisors D_1, D_2 such that the *intersection map*

$$(C \cdot D) : \operatorname{Div} M \longrightarrow \mathbb{Z} \, , \quad C \subset M \ \text{fixed} \, ,$$

is a group homomorphism.

9.15 Self - intersection numbers

In the present Section, we want to give an interpretation of the *self–intersection numbers*

$$(E_j \cdot E_j)$$

by invoking the *adjunction formula*. Without loss of generality, we write $C = E_j$. In order to compute the line bundle $[\,C\,]_{|C}$, we first investigate the case where C is the zero–section in a holomorphic line bundle L: If v_ι is a local fiber coordinate, then

$$v_\iota = f_{\iota\kappa} v_\kappa$$

and (locally) $C = \{\, v_\iota = 0 \,\}$. Therefore, $[\,C\,]_{|C}$ is given by the cocycle

$$g_{\iota\kappa} = \frac{v_\iota}{v_\kappa} = f_{\iota\kappa} \, , \ \text{ i.e. } [\,C\,]_{|C} \cong L \, .$$

This implies

Theorem 9.20 *For the zero–section C of a line bundle L one has the identity*

$$(C \cdot C) = d\,(L) \, .$$

And, if we use again the fact that the negative line bundles are those with negative Chern number:

Corollary 9.21 *The zero–section C of a line bundle L is exceptional in L, if and only if*

$$(C \cdot C) < 0 .$$

Remark. One can easily interpret Theorem 20 *geometrically* via general intersection theory as explained in the preceding Section. If there exists a *holomorphic* section s in L then $C' := s(C)$ is a holomorphic curve in L which is homologous (and even homotopic) to C and C' intersects C in exactly $d(L) \geq 0$ points (counted with multiplicities). In the general case one has to modify a *meromorphic* section s near the poles by a *differentiable* one $\widetilde{s}$ such that the differentiable submanifold $\widetilde{s}(C)$ intersects C in the poles *negatively* with the correct pole order. To be more precise: Let the meromorphic section locally in suitable local coordinates (x, v) be described by $v = v(x) = x^{-k}$, $|x| < 2\varepsilon$. Then define

$$\widetilde{s}(x) = \begin{cases} \dfrac{1}{x^k} = \dfrac{\overline{x}^k}{|x|^{2k}} , & |x| \geq \varepsilon , \\[2ex] \dfrac{\overline{x}^k}{\varepsilon^{2k}} , & |x| \leq \varepsilon . \end{cases}$$

After some smoothing along the circle $|x| = \varepsilon$ we end up with a differentiable section $\widetilde{s}$ such that $\widetilde{s}(C)$ intersects C negatively of order k at the given pole.

Coming back to the general case of a smooth curve $C \subset M$, we can find a covering $\mathfrak{U} = \{U_\iota\}$ of C in M with coordinates (u_ι, v_ι) in U_ι such that

$$C \cap U_\iota = \{(u_\iota, v_\iota) : v_\iota = 0\} .$$

The bundle $[C]_{|C}$ is then defined by the cocycle

$$g_{\iota\kappa} = g_{\iota\kappa}(u_\kappa) = \left.\frac{v_\iota(u_\kappa, v_\kappa)}{v_\kappa}\right|_{v_\kappa=0} .$$

In the local expansion

$$v_\iota = v_\iota(u_\kappa, v_\kappa) = \sum_{\nu,\mu} a_{\nu\mu} u_\kappa^\nu v_\kappa^\mu ,$$

all coefficients $a_{\nu 0}$ are 0, since $v_\kappa = 0$. Hence,

$$g_{\iota\kappa} = \sum_\nu a_{\nu 1} u_\kappa^\nu = \left.\frac{\partial v_\iota}{\partial v_\kappa}\right|_{v_\kappa=0} .$$

By the same reasoning, we get

$$\left.\frac{\partial v_\iota}{\partial u_\kappa}\right|_{v_\kappa=0} = 0 .$$

The tangent bundle T_M of M is (locally near C) given by the cocycle

$$\Theta_{\iota\kappa} = \begin{pmatrix} \dfrac{\partial u_\iota}{\partial u_\kappa} & \dfrac{\partial u_\iota}{\partial v_\kappa} \\[2ex] \dfrac{\partial v_\iota}{\partial u_\kappa} & \dfrac{\partial v_\iota}{\partial v_\kappa} \end{pmatrix} .$$

Its restriction to C (denoted as usual by $T_{M|C}$) is then defined by

$$\Theta_{\iota\kappa|C} = \Theta_{\iota\kappa}\big|_{v_\kappa=0} = \begin{pmatrix} \left.\dfrac{\partial u_\iota}{\partial u_\kappa}\right|_{v_\kappa=0} & * \\[2ex] 0 & g_{\iota\kappa} \end{pmatrix}$$

where $(*)$ stands for an arbitrary entry. Due to the zeros in the left lower corner, the tangent bundle T_C of C defined by

$$\left.\frac{\partial u_\iota}{\partial u_\kappa}\right|_{v_\kappa = 0}$$

is in a canonical way a subbundle of $T_{M|C}$, and the quotient bundle $N = N_{C|M}$ will be given by the cocycle $(g_{\iota\kappa})$. Thus, we have

Theorem 9.22 *If C is a smooth compact Riemann surface embedded in a two–dimensional complex analytic manifold M, then*

$$[\,C\,]_{|C} \cong N_{C|M} \, .$$

In particular, the self–intersection number $(C{\cdot}C)$ equals the Chern number of the normal bundle $N_{C|M}$ of C in M.

Remark. Since *topologically* the differentiable manifold M looks near C like a neighborhood of the zero–section in the normal bundle $N_{C|M}$, the self–intersection number of C in M coincides with the sum of the intersections (counted with multiplicity and the correct sign according to orientation) of C with any differentiable submanifold $C' \subset M$ which is homologous to C in M and intersects C transversally (in the differentiable sense) in finitely many points.

We still need another description of $[\,C\,]_{|C}$, or - to be more precise - of its *dual* bundle $[\,-C\,]_{|C}$. To this end, we give another description for $[\,-C\,]$: Let, for the moment, D be any divisor on M. Assume, moreover, that

$$D = \sum n_j E_j$$

is *positive* (also called *effective*), i.e. that all $n_j \geq 0$, or - what amounts to the same - that D is locally (on U_ι) the divisor of a *holomorphic* function h_ι. Then, if f is a holomorphic function which vanishes on each E_j at least to order n_j, i.e. if div $f - D$ is positive, we have locally

$$f = s_\iota \cdot h_\iota \, , \quad s_\iota \in H^0(U_\iota, \mathcal{O}_M) \, ,$$

such that the system $\{\, s_\iota \,\}$ defines a holomorphic section in $[\,-D\,]$. If, on the other hand, we have a holomorphic section $\{\, s_\iota \,\}$ in $[\,-D\,]$, then

$$s_\iota = \frac{h_\kappa}{h_\iota}\, s_\kappa \ ,$$

and $f = s_\iota h_\iota$ defines a global holomorphic function with div $f - D$ positive. Since all these considerations are also true locally, we proved:

Theorem 9.23 *If $D = \sum n_j E_j$ is a positive divisor, then the sheaf $\mathcal{O}_M(-D)$ of germs of holomorphic sections in the line bundle $[\,-D\,]$ can be identified with the ideal sheaf $I_D \subset \mathcal{O}_M$ of germs of holomorphic functions f with* div $f \geq D$.

In the case $D = C \subset M$, which we studied before, I_C is nothing else but the ideal sheaf of C in M, and analytic restriction to C gives

$$\mathcal{O}_C([\,-C\,]_{|C}) \cong I_C \otimes_{\mathcal{O}_M} \mathcal{O}_C \cong I_C \otimes_{\mathcal{O}_M} (\mathcal{O}_M/\,I_C) \cong I_C/\,I_C^2 \, .$$

Therefore, we have proved:

Corollary 9.24 *Under the assumption of* Theorem 22, *the sheaf $\mathcal{O}\,(N_C^*)$ is isomorphic to $I_C/\,I_C^2$.*

9.16 Divisors supported on E

In the following sections, we restrict our considerations to divisors $D = \sum_{j=1}^{r} n_j E_j$ supported on a fixed compact 1–dimensional complex analytic set $E \subset M$ whose irreducible components E_j are always assumed to be *smooth*. We denote the abelian group of such divisors by

$$\mathrm{Div}_E\, M\ .$$

By linear extension of the intersection pairing, we get a bilinear map

$$(\,.\,,\,.\,) : \mathrm{Div}_E\, M \times \mathrm{Div}_E\, M \longrightarrow \mathbb{Z}\ .$$

Of course, this map is completely determined by the matrix of the intersection numbers $(E_j,\, E_k)$ which we also denote by

$$(E_j \cdot E_k)\ .$$

We first deal with the case $j \neq k$ and claim that for components intersecting transversely our definition is correct. (For the notion of normal crossing divisors, see Chapter 5.9).

Theorem 9.25 *For a normal crossing divisor E, the number $(E_j \cdot E_k)$, $j \neq k$, equals the number of points of $E_j \cap E_k$. In particular, the intersection matrix*

$$((E_j \cdot E_k))_{j,k}$$

is symmetric and has nonnegative entries outside the main diagonal.

Proof. In a small neighborhood of E_j (which only counts for the computation of $(E_j \cdot E_k)$), the divisor E_k decomposes into ℓ small disks D_λ, all intersecting E_j transversely, where ℓ is the cardinality of the finite set $E_j \cap E_k$.

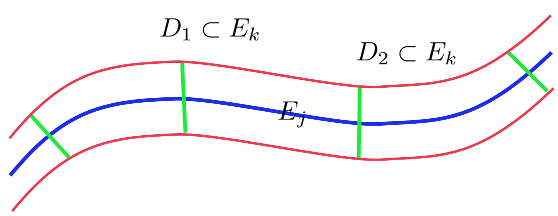


Figure 9.10

So, near E_j, the line bundle $[\,E_k\,]$ has a holomorphic section which vanishes exactly on the D_λ (precisely to first order). Hence,

$$[\,E_k\,]_{|E_j}$$

has a holomorphic section with precisely ℓ zeros such that

$$(E_j \cdot E_k) \,=\, d\,([\,E_k\,]_{|E_j}) \,=\, \ell\ . \qquad \square$$

9.17 Intersection numbers and blowing up

The present Section is devoted to the question what happens to intersection numbers under σ–processes (recovering and generalizing the results of Chapter 5.12). We consider our standard situation of a smooth Riemann surface $C \subset M$ and blow up a point $x^{(0)} \in C$:

$$\begin{tikzcd} & \widetilde{M} \arrow[d, "\sigma"] \\ C \arrow[r] & M \end{tikzcd}$$

Let $\overline{C}$ be the strict transform of C under σ. Using suitable coordinates on C near $x^{(0)}$, it is very easy to show that the injection $\overline{C} \hookrightarrow \widetilde{M}$, followed by σ, defines a biholomorphic map $\tau : \overline{C} \xrightarrow{\sim} C$. Hence, if L is any holomorphic line bundle on M, and if $\widetilde{L}$ denotes the lifted bundle $\sigma^* L$, then by the commutativity of the diagram

$$\begin{tikzcd} \overline{C} \arrow[r] \arrow[d, "\tau"'] & \widetilde{M} \arrow[d, "\sigma"] \\ C \arrow[r] & M \end{tikzcd}$$

$\widetilde{L}_{|\overline{C}} = \tau^*(L_{|C})$ which implies the important formula

$$(C, L) = (\overline{C}, \sigma^* L) .$$

If $L = [D]$, with $D = \sum_{j=1}^{r} n_j E_j$ a divisor with smooth components E_j, it is easily checked that

$$\widetilde{L} = \sigma^* L = [\sigma^* D]$$

with

$$\sigma^* D = n_0 \overline{E}_0 + \sum_{j=1}^{r} n_j \overline{E}_j ,$$

where $\overline{E}_0 = \sigma^{-1}(x^{(0)}) \cong \mathbb{P}_1$, $\overline{E}_j =$ strict transform of E_j in $\widetilde{M}$, and $n_0 = \sum_{j \in J_0} n_j$, where $J_0 = \{ j = 1, \ldots, r : x^{(0)} \in E_j , \ n_j \neq 0 \}$.

Theorem 9.26 *Let E_1, $E_2 \subset M$ be (eventually identical) smooth compact Riemann surfaces, and let $\overline{E}_1$, $\overline{E}_2$ be their strict transforms under a σ–process with center $x^{(0)} \in E_1 \cap E_2$. Then*

$$(\overline{E}_1 \cdot \overline{E}_2) = (E_1 \cdot E_2) - 1 .$$

Proof. We have

$$(E_1 \cdot E_2) = (E_1, [E_2]) = (\overline{E}_1, [\sigma^* E_2]) = (\overline{E}_1, \overline{E}_0 + \overline{E}_2) = 1 + (\overline{E}_1 \cdot \overline{E}_2) ,$$

since $(\overline{E}_1, \overline{E}_0) = 1$. □

From Theorem 26, we can immediately conclude that the statement of Theorem 25 remains true for divisors E with not necessarily normal crossings:

Corollary 9.27 *Let $E = \sum_{j=1}^{r} E_j$ be a divisor with smooth components E_j. Then the intersection matrix $((E_j \cdot E_k))$ is symmetric and has nonnegative entries outside the main diagonal.*

Proof. Let $E_j \cap E_k = \{ x^{(1)}, \ldots, x^{(t)} \}$, $j \neq k$, and denote by c_τ the *order of contact* between E_j and E_k at $x^{(\tau)}$ (see Chapter 5.9). As we have seen there, this number drops by one after performing a σ–process at $x^{(\tau)}$. Therefore,

$$(E_j \cdot E_k) = \sum_{\tau=1}^{t} (c_\tau + 1) .$$ □

9.18 Principal divisors

The preceding considerations may also be used to show that there is always a large subgroup of $\mathrm{Div}\, M$ in the kernel of the intersection map (for a fixed curve $C \subset M$).

Two divisors D_1 and D_2 are called *linearly equivalent*, if their difference is a *principal divisor*, i.e. the divisor of a *global* meromorphic function on M; we denote this fact by $D_1 \sim D_2$. Thus,

$$D_1 \sim D_2 \Longleftrightarrow \begin{cases} \exists \text{ a meromorphic function } h \text{ on } M \\ \text{such that } D_1 - D_2 = \mathrm{div}\, h\,. \end{cases}$$

Since for two meromorphic functions h_1, h_2 it is true that

$$\mathrm{div}\,(h_1 \cdot h_2) = \mathrm{div}\, h_1 + \mathrm{div}\, h_2\,,$$

the set $\mathrm{Div}^H M$ of *principal* divisors (*Hauptdivisoren* in German) is a subgroup of $\mathrm{Div}\, M$. The line bundle associated to a principal divisor is obviously trivial (and vice versa: if the line bundle attached to a divisor D is trivial, then D is principal). Therefore, we have:

Theorem 9.28 *For every principal divisor D, the intersection number $(C, [D])$ is zero. Hence, the intersection map may be thought of as being a group homomorphism*

$$\mathrm{Div}\, M / \mathrm{Div}^H M \longrightarrow \mathbb{Z}\,.$$

We give *another proof*. It is to show that

$$(C, \mathrm{div}\, h) = 0$$

for all meromorphic functions h on M. If we blow up a point $x^{(0)} \in M$, then

$$\sigma^* \mathrm{div}\, h = \mathrm{div}\,(h \circ \sigma)$$

and

$$(C, \mathrm{div}\, h) = (\overline{C}, \mathrm{div}\,(h \circ \sigma))\,,$$

$\overline{C}$ the strict transform of C. Hence, we may assume that

$$\mathrm{div}\, h = n\, C + \sum n_j E_j\,,$$

where the E_j are pairwise disjoint and intersect C transversely (at precisely one point). Near such an intersection point, h can be written in the form

$$h = e_j u_j^{n_j} v^n\,,$$

where e_j is a unit and u_j and v are local coordinates such that

$$C = \{ v = 0 \}\,, \quad E_j = \{ u_j = 0 \}\,.$$

Hence, the system $e_j^{-1}(u_j, 0) u_j^{-n_j}$ defines a meromorphic section in the n–th power of $[C]_{|C} \cong N_C$ such that

$$n\,(C \cdot C) = d\,(N_C^{\otimes n}) = -\sum n_j = -\sum n_j (C \cdot E_j)\,,$$

i.e.

$$(C, \mathrm{div}\, h) = (C, n\, C + \sum n_j E_j) = n\,(C \cdot C) + \sum n_j (C \cdot E_j) = 0\,. \qquad \square$$

9.19 Necessity of the Grauert - Mumford criterion

The next Sections are devoted to the proof of the Grauert–Mumford criterion in the precise form of Theorem 15. (See also Corollary 21 for the special case of the zero–section in a holomorphic line bundle).

The following result due to Mumford is exactly the implication iii) $\Longrightarrow$ iv) in Theorem 15. (Notice that we already proved ii) $\Longleftrightarrow$ iv), see Theorem 5.9).

Theorem 9.29 *Let $E \subset M$ be a connected normal crossing divisor with smooth components E_j. If E is exceptional in M, then the intersection matrix*

$$((E_j \cdot E_k))_{1 \le j,k \le r}$$

is negative definite.

Proof. Blow E down to an isolated singular point and take any nontrivial holomorphic function g which vanishes at this point. Then g must have zeros arbitrarily close to the singular point. Let f be the pull–back of g to M; then we have

$$\operatorname{div} f = \sum_{j=1}^{r} m_j E_j + D_0$$

with $m_j > 0$ for all j, and D_0 is not empty, positive and intersects the connected set $E = \cup E_j$ in a nonempty set of isolated points. For a fixed index k, we may assume (after possibly blowing up some more points and applying Theorem 5.9 once more) that all components of D_0 intersect E_k transversely (see also the proof and statement of Theorem 28) such that

$$(+) \qquad 0 = (E_k, \operatorname{div} f) = (E_k, \sum_j m_j E_j) + (E_k, D_0) \ge \sum_j m_j (E_j \cdot E_k)\,,$$

and this inequality is strict for at least one index k.

The inequalities (+) are already sufficient for the negative–definiteness of the real symmetric matrix $S = (c_{jk})$ where $c_{jk} = (E_j \cdot E_k)$ and, in particular, $c_{jk} \ge 0$ for $j \ne k$: A matrix S is evidently (negative) definite, if and only if it exists an invertible matrix Q such that $QS\,{}^tQ$ is (negative) definite. Taking $Q = (q_{jk})$ to be the diagonal matrix with the entries $q_{jj} = m_j$, we see that the problem is reduced to the statement that

$$\widetilde{S} = QS\,{}^tQ = (\widetilde{c}_{jk})\,, \quad \widetilde{c}_{jk} = m_j m_k (E_j \cdot E_k)$$

is negative definite. The real symmetric matrix $\widetilde{S}$ has now the following properties:

I) $\widetilde{c}_{jk} \ge 0\,, \quad j \ne k\,;$

II) $\displaystyle\sum_{j=1}^{r} \widetilde{c}_{jk} \le 0\,, \quad k = 1, \ldots, r\,,$
and this inequality is strict for some k (say $k = 1$);

III) it is not possible to split the index set $\{1, \ldots, r\}$ into two nonempty disjoint subsets J_1 and J_2 such that $\widetilde{c}_{jk} = 0$ for all $j \in J_1$, $k \in J_2$. (This, of course, follows from the connectedness of $E = \cup E_j$).

Lemma 9.30 *A real symmetric matrix $\widetilde{S} = (\widetilde{c}_{jk})$, satisfying the conditions* I), II) *and* III) *as before, is negative definite.*

Proof. We first show that $\widetilde{S}$ is negative semi–definite: Let $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$; then

$$\sum_{j,k} \widetilde{c}_{jk} x_j x_k = \sum_j \widetilde{c}_{jj} x_j^2 + 2 \sum_{j<k} \widetilde{c}_{jk} x_j x_k = \sum_k (\sum_j \widetilde{c}_{jk}) x_k^2 - \sum_{j<k} \widetilde{c}_{jk} (x_j - x_k)^2 \le 0\,.$$

Moreover, if the left hand side vanishes for x, then

a) $(\sum_j \widetilde{c}_{jk})x_k^2 \,=\, 0$ for all k ;

b) $\widetilde{c}_{jk}(x_j \,-\, x_k)^2 \,=\, 0$ for all $j \neq k$.

From a) and II) we then conclude that $x_1 \,=\, 0$; hence

$$J_1 \,=\, \{\, j \in \{\, 1, \ldots, r \,\} : x_j \,=\, 0 \,\} \neq \emptyset \,.$$

Suppose now that also

$$J_2 \,=\, \{\, 1, \ldots, r \,\} \setminus J_1 \,=\, \{\, k \in \{\, 1, \ldots, r \,\} : x_k \,\neq\, 0 \,\}$$

is not empty. Then, by b), $\widetilde{c}_{jk} \,=\, 0$ for all $j \in J_1$, $k \in J_2$, which is a contradiction to III). □

9.20 Sufficiency of the criterion: Reduction to $A = E$

A partial completion of the proof of Theorem 15 will be provided by showing the equivalences i) $\Longleftrightarrow$ ii) $\Longleftrightarrow$ vii). This reduction step follows from the next Theorem which is obviously more generally valid for iterated σ–processes (and therefore for any point modification $\sigma : \widetilde{M} \to M$ with center in A; see Section 29). It also implies the last statement in Theorem 15.

Theorem 9.31 *Let $\sigma : \widetilde{M} \to M$ be the σ–process at a point $x^{(0)} \in A \subset M$, A a compact connected one–dimensional complex analytic subset, and let $\widetilde{A} := \sigma^{-1}(A)$ be its total transform. Then $\widetilde{A}$ is exceptional in $\widetilde{M}$, if and only if A is exceptional in M , and if so, they blow down to the same singularity.*

Proof . Since σ is a proper mapping, the preimages of all open neighborhoods of A form a neighborhood basis for the connected compact set $\widetilde{A}$, and vice versa for the images. Clearly, A and $\widetilde{A}$ have the "same" boundary, σ being biholomorphic outside $\widetilde{A}$. This shows the first statement. Moreover, by Riemann's Extension Theorem, $\sigma_* \mathcal{O}_{\widetilde{M}} \cong \mathcal{O}_M$ such that

$$H^0(\sigma^{-1}(U), \, \mathcal{O}_{\widetilde{M}}) \,\cong\, H^0(U, \, \mathcal{O}_M)$$

for all open sets $U \subset M$. Therefore,

$$\varinjlim_{\widetilde{A} \subset V} H^0(V, \, \mathcal{O}_{\widetilde{M}}) \,\cong\, \varinjlim_{\widetilde{A} \subset \sigma^{-1}(U)} H^0(\sigma^{-1}(U), \, \mathcal{O}_{\widetilde{M}}) \,\cong\, \varinjlim_{A \subset U} H^0(U, \, \mathcal{O}_M)$$

are isomorphic analytic algebras. □

9.21 An equivalent criterion

For the next step, we need the *converse* to a result which we used implicitly in the proof of Theorem 29 (cf. Lemma 30).

Lemma 9.32 *Let $S \,=\, (c_{jk})$ be a real symmetric matrix with $c_{jk} \,\geq\, 0$ for all $j \neq k$, satisfying condition* III)*: There is no decomposition $\{\, 1, \ldots, r \,\} \,=\, J_1 \cup J_2$ with $c_{jk} \,=\, 0$ for all $j \in J_1$, $k \in J_2$.*

a) *Assume that there are positive numbers $m_1, \ldots, m_r$ such that*

$$(*) \qquad \sum_{j=1}^{r} m_j c_{jk} \,\leq\, 0 \,, \quad k \,=\, 1, \ldots, r \,.$$

Then S is negative semi–definite. If, moreover, in $(*)$ strict inequality holds for some k , then S is negative definite.

b) *If* S *is negative definite, then there exist positive* integers $m_1, \ldots, m_r$ *such that the strict inequalities in* $(*)$ *are satisfied.*

The converse b) will be a consequence of the next Theorem which is the *key lemma* in the *classification of irreducible root systems*, too.

Theorem 9.33 *Let* $S = (c_{jk})$ *be a real symmetric negative semi–definite* $r \times r$*–matrix with* $c_{jk} \geq 0$ *for all* $j \neq k$ *satisfying condition* III)*. Then there exists an invertible matrix* Q *such that* $QS\,{}^tQ$ *is a diagonal matrix of type*

$$D = \operatorname{diag}(-1, \ldots, -1)$$

or of type

$$D = \operatorname{diag}(-1, \ldots, -1, 0)$$

with eigenvalue 0 *of multiplicity* 1*.*

In the first case, S *is negative definite. In the second case, it exists a* positive *vector* $x = (x_1, \ldots, x_r) \in \mathbb{R}^r_{>0}$ *such that* $xS = 0$ *(which, of course, spans the annihilator of* S*).*

Proof. For $r = 1$, all the statements are trivially correct. So, assume that the Theorem is verified for $(r - 1) \times (r - 1)$–matrices, and take an $r \times r$–matrix $S = (c_{jk})$ with associated quadratic form

$$q(x) := xS\,{}^tx \, .$$

Substituting $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $x = (1, \ldots, 1)$, resp., yields

$$c_{jj} \leq 0 \text{ for all } j$$

and

$$\sum_{j=1}^{r} \Big(\sum_{k=1}^{r} c_{jk} \Big) \leq 0 \, ,$$

hence

$$\sum_{k=1}^{r} c_{jk} \leq 0 \text{ for at least one value of } j$$

which (after application of a suitable permutation matrix Q) we may assume to be $j = 1$. Since $c_{1k} = 0$ for all $k > 1$ is impossible, we get

$$c_{11} < 0 \, ,$$

and we assume without loss of generality that

$$c_{11} = -1 \, , \quad c_{12} > 0 \, .$$

Take now the matrix

$${}^tQ = \left(\begin{array}{c|ccc} 1 & c_{12} & \cdots & c_{1r} \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array} \right)$$

and check that

$$QS\,{}^tQ = \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & S' & \\ 0 & & & \end{array} \right)$$

where $S' = (c'_{jk})_{2 \le j,k \le r}$ with

$$c'_{jk} = c_{jk} + c_{1j}c_{1k}\ , \quad 2 \le j\ ,\ k \le r\ .$$

Obviously, $S' \le 0$ and $c'_{jk} \ge 0$ for all $j \ne k$. If there would exist a decomposition

$$\{\,2, \ldots, r\,\} = J'_1 \cup J'_2\ , \quad J'_1 \cap J'_2 = \emptyset\ ,$$

with $c'_{jk} = 0$ for all $j \in J'_1$, $k \in J'_2$, then we could assume without loss of generality that $J'_1 = \{\,2, \ldots, s\,\}$, $J'_2 = \{\,s+1, \ldots, r\,\}$, and we would get out of

$$0 = c'_{jk} = c_{jk} + c_{1j}c_{1k} \ge 0$$

that $c_{jk} = 0$ for all $j \in J'_1$, $k \in J'_2$. Taking $j = 2$ yields also $c_{1k} = 0$ for all $k \in J'_2$, which is a contradiction to the properties of S. - This implies the first statements.

The last statement shall be shown by induction, too. If S is not definite, then also S'. Therefore, we find a vector

$$y' = (y_2, \ldots, y_r)\ , \quad y_j > 0\ , \quad j = 2, \ldots, r\ ,$$

with $y'S' = 0$. By the definition of Q, the vector

$$x = yQ\ , \quad y = (0, y_2, \ldots, y_r)$$

is positive, and $xS = 0$. □

Proof of Lemma 32, part b). S has r negative eigenvalues. Let λ be the eigenvalue of smallest absolute value. Then $\widetilde{S} = S - \lambda E$ satisfies the assumptions of Theorem 33. Since $\widetilde{S}$ is degenerate, it exists a vector $x > 0$ with $x\widetilde{S} = 0$. Thus, $xS = \lambda x < 0$. By continuity, we may choose x with positive *rational* coordinates, and hence, there exist positive integers $m_1, \ldots, m_r$ such that

$$\sum_{k=1}^{r} c_{jk}m_k < 0$$

for all $j = 1, \ldots, r$. □

We are now in the position to perform the next important step in proving the sufficiency of the Grauert–Mumford criterion in Theorem 15 by showing the implication iv) $\Longrightarrow$ v).

Theorem 9.34 *Let $E = \cup E_j$ be a one–dimensional connected compact complex analytic subset of the smooth surface M with smooth components E_j and negative definite intersection matrix $S = (c_{jk})$, $c_{jk} = (E_j \cdot E_k)$. Then there exist positive integers m_k such that the line bundle $L = [\sum m_k E_k]$ has the property that $L_{|E_j}$ has negative degree for all $j = 1, \ldots, r$.*

Proof. Choose positive integers $m_1, \ldots, m_r$ according to Lemma 32, part b), put

$$E^{(m)} := \sum_{k=1}^{r} m_k E_k \in \mathrm{Div}_E\, M$$

and

$$L := [\,E^{(m)}\,] = \bigotimes_{k=1}^{r} [\,E_k\,]^{\otimes m_k}\ .$$

Then the degree of the restriction

$$L_j = L_{|E_j} = \bigotimes_{k=1}^{r} ([\,E_k\,]_{|E_j})^{\otimes m_k}$$

is equal to

$$d\,(L_j) = \sum_{k=1}^{r} m_k \cdot d\,([\,E_k\,]_{|E_j}) = \sum_{k=1}^{r} m_k (E_k \cdot E_j) = \sum_{k=1}^{r} m_k c_{jk} < 0\ . \qquad \square$$

9.22 The equivalent criterion under blowing up

In this Section, we shall prove that the existence of a bundle L satisfying the assumptions of Theorem 34 is preserved under σ–processes. This is even a little bit more than we need for the implication v) $\Longrightarrow$ vi) in Theorem 15.

Theorem 9.35 *Let $E \subset M$ be a compact divisor with smooth components E_j in a two–dimensional complex analytic manifold M such that, for some positive integers m_k, the degree*

$$d\,([\sum_{k=1}^{r} m_k E_k\,]_{|E_j})$$

is negative for all $j = 1, \ldots, r$. Let $\sigma : \widetilde{M} \to M$ be a σ–process at a point $x^{(0)} \in E$; denote by $\widetilde{E}_j$ the strict transform of E_j in $\widetilde{M}$ and by $\widetilde{E}_0$ the subvariety $\sigma^{-1}(x^{(0)}) \cong \mathbb{P}_1$ such that $\sigma^{-1}(E) = \cup_{j=0}^{r} \widetilde{E}_j$. Then the restrictions of the line bundle

$$\widetilde{L} = [\,m\widetilde{E}_0 + 2\sum_{k=1}^{r} m_k \widetilde{E}_k\,]$$

to the curves $\widetilde{E}_j$, $j = 0, \ldots, r$, have negative degree, when $m = 1 + 2\sum' m_j$, $\sum'$ denoting summation over all j with $x^{(0)} \in E_j$.

Proof. Let J_0 be the set of indices j such that $x^{(0)} \in E_j$. Then

$$\begin{aligned}
(\widetilde{E}_j \cdot \widetilde{E}_k) &= (E_j \cdot E_k) - 1\,, && j, k \in J_0\,,\\
(\widetilde{E}_j \cdot \widetilde{E}_k) &= (E_j \cdot E_k)\,, && k \in J_1 = \{\,1, \ldots, r\,\} \setminus J_0\,,\\
(\widetilde{E}_j \cdot \widetilde{E}_0) &= \begin{cases} 1\,, & j \in J_0\,,\\ 0\,, & j \in J_1\,,\\ -1\,, & j = 0\,. \end{cases}
\end{aligned}$$

This implies for $\ell \in J_0$:

$$\begin{aligned}
d\,(\widetilde{L}_{|\widetilde{E}_\ell}) &= ((m\widetilde{E}_0 + 2\sum_{k=1}^{r} m_k \widetilde{E}_k) \cdot \widetilde{E}_\ell)\\
&= m(\widetilde{E}_0 \cdot \widetilde{E}_\ell) + 2\sum_{k \in J_0} m_k (\widetilde{E}_k \cdot \widetilde{E}_\ell) + 2\sum_{k \in J_1} m_k (\widetilde{E}_k \cdot \widetilde{E}_\ell)\\
&= (1 + 2\sum_{k \in J_0} m_k) + 2\sum_{k \in J_0} m_k ((E_k \cdot E_\ell) - 1) + 2\sum_{k \in J_1} m_k (E_k \cdot E_\ell)\\
&= 1 + 2\,d\,([\sum m_k E_k\,]_{|E_\ell}) \le 1 - 2 < 0\,;
\end{aligned}$$

for $\ell \in J_1$:

$$d\,(\widetilde{L}_{|\widetilde{E}_\ell}) = d\,(L_{|E_\ell}) < 0\,;$$

and for $\ell = 0$:

$$d\,(\widetilde{L}_{|\widetilde{E}_0}) = m(\widetilde{E}_0 \cdot \widetilde{E}_0) + 2\sum_{k=1}^{r} m_k (\widetilde{E}_k \cdot \widetilde{E}_0) = -(1 + 2\sum_{k \in J_0} m_k) + 2\sum_{k \in J_0} m_k = -1 < 0\,.$$

This completes the proof of the Theorem. $\square$

Remark. The effect of a σ–process to an intersection matrix $S = (c_{jk})_{1\le j,k\le r}$ can be understood in completely a formal way. Let S be an arbitrary symmetric $r\times r$–matrix with integer entries c_{jk} such that $c_{jk} \ge 0$, $j \ne k$. Suppose that there exists a nonempty subset $J \subset \{1,\dots,r\}$ such that $c_{jk} \ge 1$ for all $j, k \in J$, $j \ne k$. Then the formal σ–process of S with respect to J is the following $(r+1)\times(r+1)$–matrix $\widetilde{S} = (\widetilde{c}_{jk})_{0\le j,k\le r}$ (assume $J = \{1,\dots,s\}$):

$$\widetilde{S} = \left(\begin{array}{c|ccc|ccc} -1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & & & \\ \vdots & & c_{jk}-1 & & & c_{jk} & \\ 1 & & & & & & \\ \hline 0 & & & & & & \\ \vdots & & c_{jk} & & & c_{jk} & \\ 0 & & & & & & \end{array}\right)$$

From this concrete form of $\widetilde{S}$ one can easily conclude the correctness of our statement in Theorem 35. Alternatively, one can argue as follows: By addition of the zeroth column to the next s columns and addition of the zeroth row to the next s rows (which is a transformation of type ${}^tQ\widetilde{S}Q$), $\widetilde{S}$ is transformed into the matrix

$$\left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & S \end{array}\right)$$

Hence, as we already know, $\widetilde{S}$ is negative definite, if and only if S is negative definite. Thus, Lemma 32 gives at least the *existence* of positive integers $\widetilde{m}_0,\dots,\widetilde{m}_r$ with the desired properties.

9.23 Sufficiency of the Grauert - Mumford criterion

We are now well–prepared for the *proof of the sufficiency* of the Grauert–Mumford criterion.

Step 1. We first show the implication v) $\Longrightarrow$ vi) in Theorem 15. We start with a connected compact normal crossing divisor E and a holomorphic line bundle L in a neighborhood of E such that the restrictions $L_j := L_{|E_j}$ are negative. A metric on $L_{|E}$ is nothing else but a collection of metrics h_j on $L_j = L_{|E_j}$ which coincide at the singular points of E. So, we start with some negative metrics on L_j according to Theorem 17. If these metrics do not fit together at an intersection point $x^{(0)} \in \cup_{j\ne k}(E_j \cap E_k)$, we perform a σ–process at such a point:

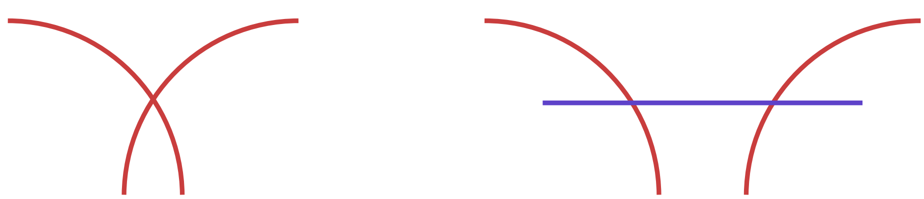


Figure 9.11

So, we are reduced to the case of a connected compact normal crossing divisor $E = \cup_{j=1}^r E_j$ with the following property:

$(*)$ *It exists $s < r$ such that no two of the E_j, $j \le s$, and no two of the E_j, $j > s$, intersect at all. The latter curves are rational and intersect exactly two curves of the first kind in 0 and ∞.*

Lemma 9.36 *Let E be a connected compact normal crossing divisor satisfying condition $(*)$ and L a holomorphic line bundle in a neighborhood of E such that the restrictions L_j are negative. Then there exists a metric h on $L_{|E}$ that induces negative metrics h_j on L_j for all j.*

Proof. Choose negative metrics on the L_j, $j \leq s$ (Theorem 17). Since for given $j > s$ we have $E_j \cong \mathbb{P}_1$, the corresponding negative line bundle L_j is isomorphic to some $\mathcal{O}(-\ell)$ such that we can choose a negative metric on L_j which coincides with the given values at 0 and ∞ according to the example at the beginning of Section 6. $\square$

Step 2. The rest of this Section is devoted to the implication vi) $\Longrightarrow$ vii) in Theorem 15. From now on, we can assume that we are in the following situation: $E = \cup E_j$ has only normal crossings, M is covered by finitely many bidisks P_ι with coordinates u_ι, v_ι such that

$$E \cap P_\iota = \{ v_\iota = 0 \} \text{ or } \{ u_\iota v_\iota = 0 \},$$

and, for suitable positive integers m_k, the line bundle $L := [\sum m_k E_k]$ has the property that its restrictions $L_j = L_{|E_j}$ are negative with respect to a metric $\{ h_{j\iota} \}$ attached to L_j with the help of the trivializing covering $\mathfrak{U}_j = \{ U_{j\iota} = E_j \cap P_\iota \}$. Furthermore, we may suppose that the covering $\mathfrak{U} = \{ P_\iota \}$ has a shrinking $\mathfrak{V} = \{ Q_\iota \}$ with $Q_\iota \subset\subset P_\iota$ having the same properties as $\mathfrak{U}$ with respect to E; i.e.: $E \subset \cup Q_\iota$ and $E \cap Q_\iota = \{ v_\iota = 0 \}$ or $= \{ u_\iota v_\iota = 0 \}$, and that the negative metrics $\{ h_{j\iota} \}_\iota$ patch together at the singular points of E. Now, if

$$E \cap P_\iota = E_j \cap P_\iota = \{ v_\iota = 0 \},$$

we extend $h_{j\iota}$ arbitrarily to a positive C^∞–function $\widetilde{H}_\iota$ on P_ι. If

$$E \cap P_\iota = (E_j \cup E_k) \cap P_\iota = \{ u_\iota v_\iota = 0 \},$$

we define

$$\widetilde{H}_\iota = \frac{1}{h_{j\iota}(0)} \cdot h_{j\iota} h_{k\iota}$$

which is positive and restricts to the functions $h_{j\iota}$ and $h_{k\iota}$ on $E_j \cap P_\iota$ and $E_k \cap P_\iota$, resp. Moreover,

$$\log \widetilde{H}_j = \log h_{j\iota} + \log h_{k\iota} - \log h_{j\iota}(0)$$

is strictly plurisubharmonic, since the Levi form of $\log \widetilde{H}_\iota$ is diagonal with positive entries along the diagonal. Next, we take a partition of unity, say $\{ \rho_\iota \}$, subordinate to the covering $\mathfrak{U}$; that is: the ρ_ι are C^∞–functions on M with $0 \leq \rho_\iota \leq 1$, supp $\rho_\iota \subset\subset P_\iota$ and

$$\sum \rho_\iota \equiv 1$$

in a neighborhood V of E in M. Let $\{ F_{\iota\kappa} \}$ denote a cocycle representing L with respect to the trivializing covering $\mathfrak{U}$. Since ρ_λ is identically zero in a neighborhood of $\partial P_\lambda \cap P_\iota$ for all λ and ι:

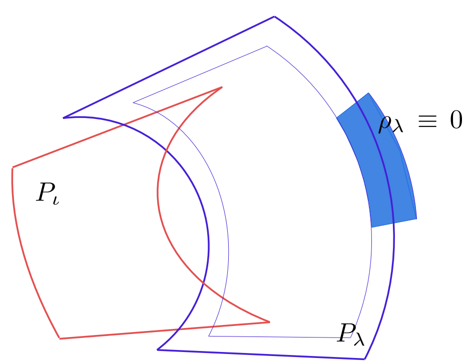


Figure 9.12

there is a well–defined C^∞–function H_ι on P_ι given by

$$H_\iota = \sum_\lambda \rho_\lambda |\, F_{\lambda\iota}\,|^2 \widetilde{H}_\lambda \,,$$

and the system $\{ H_\iota \}$ satisfies the following condition on $U_\iota \cap U_\kappa$:

$$H_\kappa = \sum_\lambda \rho_\lambda |\, F_{\lambda\kappa}\,|^2 \widetilde{H}_\lambda = |\, F_{\iota\kappa}\,|^2 \cdot \sum_\lambda \rho_\lambda |\, F_{\lambda\iota}\,|^2 \widetilde{H}_\lambda = |\, F_{\iota\kappa}\,|^2 H_\iota \,.$$

For a fixed curve E_j, denote by $\{ f_{\iota\kappa}^{(j)} \}$ the cocycle $\{ F_{\iota\kappa|E_j} \}$ which represents the restriction $L_{|E_j}$. Then

$$H_{\iota|E_j} = \sum_\lambda \rho_\lambda |\, f_{\lambda\iota}^{(j)}\,|^2 h_{j\lambda} = (\sum_\lambda \rho_\lambda) h_\iota = h_\iota \,.$$

Therefore, after eventually shrinking V, we can deduce that

$$H_{\iota|Q_\iota \cap V}$$

is a positive function, and this implies that the system $\{ H_\iota \}$ is a metric on $L_{|V}$ with respect to the covering $\{ Q_\iota \cap V \}$ which extends the given metric on $L_{|E}$.

Of course, for $E \cap P_\iota = \{ u_\iota v_\iota = 0 \}$, we see at once that $\rho_\iota \equiv 1$ near 0; thus $\log H_\iota$ is strictly plurisubharmonic around 0. In the case $E \cap P_\iota = \{ v_\iota = 0 \}$, at least the derivatives

$$\frac{\partial^2 \log H_\iota}{\partial u_\iota \partial \overline{u_\iota}}$$

must be positive (by continuity) for all $(u_\iota, v_\iota) \in Q_\iota$ with small $|\, v_\iota\,|$.

Hence, we can construct new coverings which we again denote by $\mathfrak{U} = \{ P_\iota \}$, $\mathfrak{V} = \{ Q_\iota \}$, with the following properties:

a) $Q_\iota \subset\subset P_\iota$, and P_ι, Q_ι are bidisks with respect to coordinates u_ι, v_ι,

b) $E \cap P_\iota = \{ v_\iota = 0 \}$ or $= \{ u_\iota v_\iota = 0 \}$,

c) there exist positive C^∞–functions H_ι on P_ι with

$$\frac{\partial^2 \log H_\iota}{\partial u_\iota \partial \overline{u_\iota}} > 0 \text{ on } P_\iota \text{ (in the first case)},$$

or $\log H_\iota$ is strictly plurisubharmonic on P_ι (in the second case),

d) $\{ H_\iota \}$ is a metric on the line bundle $L = [\sum m_j E_j]$, all $m_j > 0$, such that on $P_\iota \cap P_\kappa$ we have (we can always assume that at least P_ι is of the first kind):

$$H_\iota = \left| \frac{v_\kappa}{v_\iota} \right|^{2m} \cdot H_\kappa \,, \quad m = m_j \,,$$

when P_κ is of the first type, or

$$H_\iota = \frac{|\, u_\kappa\,|^{2m_1} |\, v_\kappa\,|^{2m_2}}{|\, v_\iota\,|^{2m}} H_\kappa \,,$$

$m = m_j$, $m_1 = m_j$ and $m_2 = m_k$ (or vice versa), when P_κ is of the second type.

Step 3. The construction above leads immediately to the following globally defined C^∞–function ψ:

$$\psi_{|P_\iota} = \psi_\iota = \begin{cases} H_\iota |\, v_\iota\,|^{2m} \\ H_\iota |\, u_\iota\,|^{2m_1} |\, v_\iota\,|^{2m_2} \end{cases}$$

(according to the different natures of $E \cap P_\iota$) which vanishes on E and has the property that

$$\frac{\partial^2 \log \psi}{\partial u_\iota \partial \overline{u_\iota}} > 0 \text{ on } P_\iota \setminus E \text{ (first case)}$$

and $\log \psi_{|P_\iota \setminus E}$ is strictly plurisubharmonic (in the second case). The system

$$V_\varepsilon = \{ x \in V : \varphi := \log \psi < \log \varepsilon \} ,$$

ε small, is a fundamental system of neighborhoods of E in M, and hence, it suffices to show that ∂V_ε has a defining strictly plurisubharmonic function for small ε. To prove this, we first compute the holomorphic gradient of $\varphi_\iota = \log \psi_\iota$ (in the first case):

$$\frac{\partial \varphi_\iota}{\partial u_\iota} = \frac{1}{H_\iota} \cdot \frac{\partial H_\iota}{\partial u_\iota} , \quad \frac{\partial \varphi_\iota}{\partial v_\iota} = \frac{1}{H_\iota} \cdot \frac{\partial H_\iota}{\partial v_\iota} + \frac{m}{v_\iota} .$$

By continuity of $H_\iota^{-1}(\partial H_\iota / \partial v_\iota)$, the last derivative must be different from zero on $\partial V_\varepsilon \cap Q_\iota$, i.e. ∂V_ε has a smooth boundary there, and the holomorphic tangent vectors are all complex multiples of

$$\xi = (\xi_1, \xi_2) = \left(\frac{\partial \varphi_\iota}{\partial v_\iota} , - \frac{\partial \varphi_\iota}{\partial u_\iota} \right) .$$

It is then easily checked that

$$\mathfrak{L}(\varphi_\iota, \xi) = \frac{\partial^2 \log H_\iota}{\partial u_\iota \partial \overline{u_\iota}} \| \xi_1 \|^2 + \frac{\partial^2 \log H_\iota}{\partial v_\iota \partial \overline{v_\iota}} \| \xi_2 \|^2 + 2 \operatorname{Re} \left(\frac{\partial^2 \log H_\iota}{\partial u_\iota \partial \overline{v_\iota}} \xi_1 \overline{\xi_2} \right)$$

can be written in the form

$$\mathfrak{L}(\varphi_\iota, \xi) = \frac{m^2}{|v_\iota|^2} \left\{ \frac{\partial^2 \log H_\iota}{\partial u_\iota \partial \overline{u_\iota}} + R(u_\iota, v_\iota) \right\} ,$$

where

$$| R(u_\iota, v_\iota) | \le C_0 | v_\iota | \text{ on } Q_\iota .$$

Since there is, by construction, a positive constant B such that

$$\frac{\partial^2 \log H_\iota}{\partial u_\iota \partial \overline{u_\iota}} \ge B \text{ on } Q_\iota ,$$

it follows that

$$\mathfrak{L}(\varphi_\iota, \xi) > 0$$

for all holomorphic tangent vectors $\xi \neq 0$ on $\partial V_\varepsilon \cap Q_\iota$, ε small. By the usual trick (see the remark at the end of Section 1),

$$\widetilde{\varphi} = (\varphi - \log \varepsilon) \, e^{A(\varphi - \log \varepsilon)}$$

is strictly plurisubharmonic for all sufficiently large values A in a neighborhood of $\partial V_\varepsilon \cap Q_\iota$, and there, we have locally $V_\varepsilon = \{ \widetilde{\varphi} < 0 \}$. Obviously, $\widetilde{\varphi}$ is strictly plurisubharmonic on P_ι with $P_\iota \cap E = \{ u_\iota v_\iota = 0 \}$ for all positive numbers A. Hence, $\widetilde{\varphi}$ is a defining function for V_ε. □

9.24 Subsets of exceptional sets

In the last Sections of this Chapter we draw some conclusions from the Grauert–Mumford criterion. In the present, we are going to show that connected subconfigurations of exceptional sets are again exceptional.

Let $E = \cup_{j=1}^r E_j \subset M$ be exceptional (we do *not* assume the components E_j to be smooth), and let $E' = \cup_{j=1}^s E_j$ be a connected subset. After a suitable iterated σ–modification, $\sigma^{-1}(E)$ has normal crossings such that the intersection matrix is defined. Obviously, $\sigma^{-1}(E')$ is a connected subconfiguration. Hence, in order to prove that also E' is exceptional, we can assume that we are already in the normal crossing case. Since a real symmetric matrix is positive–definite, if and only if all its main minors (i.e. all quadratic submatrices distributed symmetrically around the main diagonal) have positive determinant, the negative–definiteness of the matrix $((E_j \cdot E_k))_{1\le j,k\le r}$ implies the same property for the intersection matrix of E'.

Or, to give an alternative proof not using the result on positive definite matrices: E is exceptional, if and only if there exist positive integers m_j such that

$$\sum_{j=1}^{r} m_j (E_j \cdot E_k) \, < \, 0 \, , \quad k \, = \, 1, \ldots, r \, .$$

But then, for all $\ell \, = \, 1, \ldots, s$:

$$\sum_{j=1}^{s} m_j (E_j \cdot E_\ell) \, < \, - \sum_{j=1+s}^{r} m_j (E_j \cdot E_\ell) \, \le \, 0 \, .$$

We proved:

Theorem 9.37 *Let E be the exceptional set of a resolution of a two–dimensional normal singularity. Then each connected compact nondiscrete subvariety $E' \subset E$ is exceptional.*

In other words: One can blow down exceptional sets *partially*. In particular, each individual curve $E_j \subset E$ can be contracted (which, of course, is equivalent to the fact that its self–intersection number is negative, if E_j is smooth).

9.25 Exceptional curves of the first kind and minimal resolutions

The last remark can be applied, for instance, to so–called *exceptional curves of the first kind* or (-1)–*curves*. By definition, these are *rational* curves $E \cong \mathbb{P}_1$ with self–intersection number -1. They can be created at will for resolutions of any singularity by just blowing up a given resolution at some points. This fact gives oneself the impression that such curves are *inessential* with respect to resolutions. This is, indeed, true according to:

Theorem 9.38 *Let $\pi : \widetilde{M} \to M$ be the contraction of a (-1)–curve $E \subset \widetilde{M}$. Then M is smooth and π is (isomorphic to) the σ–process of M at the point $x^{(0)} \, = \, \pi\,(E)$.*

Remark. This result implies *Theorem 5.1 in the surface case: *Two σ–processes of smooth surfaces are canonically isomorphic*.

Proof of Theorem 38. Let $I \subset \mathcal{O}_{\widetilde{M}}$ be the ideal sheaf of E. Let further $\widetilde{M}$ be strongly pseudoconvex with exceptional set E. Then, we will show below in Section 27 that

$$(*) \qquad H^1(\widetilde{M}, I^r) \, = \, 0 \, , \quad r \, \ge \, 0 \, .$$

By assumption, $E \cong \mathbb{P}_1$ and the conormal bundle $(I/I^2)_{|E}$ is isomorphic to $\mathcal{O}_{\mathbb{P}_1}(1)$. Hence, using $(*)$ with $r \, = \, 2$ and the long exact cohomology sequence associated to the sequence

$$0 \longrightarrow I^2 \longrightarrow I \longrightarrow I/I^2 \longrightarrow 0 \, ,$$

yields the surjectivity of the homomorphism

$$H^0(\widetilde{M}, I) \longrightarrow H^0(\widetilde{M}, I/I^2) \cong H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1)) \cong \mathbb{C}^2 \, .$$

Therefore, we find two sections $f_1, f_2 \in H^0(\widetilde{M}, I)$ projecting down to a basis of $H^0(\widetilde{M}, I/I^2)$.

Let us show that the map $f := (f_1, f_2) : \widetilde{M} \to \mathbb{C}^2$ is isomorphic to a σ–process. About a point $x \in E$ choose local coordinates (u, v) such that $x = (0, 0)$ and $E = \{ v = 0 \}$. Then

$$f_1(u, v) = v\, f_{11}(u) + v^2\, f_{12}(u) + \cdots = v\, h_1(u, v)$$

$$f_2(u, v) = v\, f_{21}(u) + v^2\, f_{22}(u) + \cdots = v\, h_2(u, v)\ ,$$

where

$$h_j(u, v) = f_{j1}(u) + v\, f_{j2}(u) + \cdots\ , \quad j = 1, 2\ .$$

The image of f_j in $H^0(\widetilde{M}, I/I^2)$ will be represented by $f_{j1}(u)$. Now, f_{11} and f_{21} have no common zero because otherwise a certain linear combination would possess a zero of second order which is impossible for a holomorphic section of $\mathcal{O}_{\mathbb{P}_1}(1)$. Therefore, $|\, h_1\,|^2 + |\, h_2\,|^2 \neq 0$ on E.

Let U_ε be the connected component of the set $\{ x \in \widetilde{M} : |\, f_1\,|^2 + |\, f_2\,|^2 < \varepsilon \}$ containing E. Then $U_\varepsilon \subset\subset \widetilde{M}$, $f : U_\varepsilon \to B_\varepsilon = \{ z \in \mathbb{C}^2 : \|\, z\,\|^2 < \varepsilon \}$ is proper and holomorphic, and $E = f^{-1}(0)$.

We calculate the Jacobi determinant of f near $x = (0, 0) \in E$. Since (f_1, f_2) and $(f_1, f_2 + \alpha\, f_1)$ have the same Jacobi determinant, we may and will assume that $f_{21}(0) = 0$. Since f_1, f_2 form a basis of $H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1))$, we have $f_{11}(0) \neq 0$ and $f'_{21}(0) \neq 0$. Therefore,

$$\begin{aligned} I_f(u, v) &= \det \begin{pmatrix} v\, f'_{11}(u) + \cdots & f_{11}(u) + \cdots \\ v\, f'_{21}(u) + \cdots & f_{21}(u) + \cdots \end{pmatrix} \\ &= v\,(f'_{11}(u)\, f_{21}(u) - f_{11}(u)\, f'_{21}(u)) + v^2\,(\cdots) + \cdots\ , \end{aligned}$$

and $f'_{11}(u)\, f_{21}(u) - f_{11}(u)\, f'_{21}(u) \neq 0$ close to 0. Consequently,

$$J_f(u, v) \neq 0$$

for all $v \neq 0$ of sufficiently small modulus. Since $f : U_\varepsilon \setminus E \to B_\varepsilon \setminus \{ 0 \}$ is proper and locally a homeomorphism, and $B_\varepsilon \setminus \{ 0 \}$ is simply connected, f must be biholomorphic outside E.

Let now $\sigma : \widetilde{B}_\varepsilon \to B_\varepsilon$ be the σ–modification of B_ε at the origin. So, we have in coordinates $z = (z_1, z_2) = (\xi_0\eta_0, \eta_0) = (\eta_1, \xi_1\, \eta_1)$. Let $C \cong \mathbb{P}_1$ be the exceptional curve. Since f_1 and f_2 have no common zero, the pair $(f_1(x)), f_2(x))$, $x \in E$, is a well–defined point in E, giving rise to a biholomorphic map $E \to C$, and thus, we can extend $f : U_\varepsilon \setminus E \to B_\varepsilon \setminus \{ 0 \}$ to a map $F : U_\varepsilon \to \widetilde{B}_\varepsilon$ by an obvious definition. Locally about $x \in E$ with $f_2(x) \neq 0$, F is given by

$$\begin{aligned} (\eta, \xi) &= (z_1/\, z_2, z_2) = (f_1/\, f_2, f_2) \\ &= \left(\frac{f_{11}(u) + v\, f_{12}(u) + \cdots}{f_{21}(u) + v\, f_{22}(u) + \cdots}\ , \quad v\, f_{21}(u) + v^2\, f_{22}(u) + \cdots \right)\ . \end{aligned}$$

Here, $f_{21}(u) \neq 0$ because of $f_2(x) \neq 0$. Therefore, F is holomorphic near x. F is necessarily proper, and hence, the set–theoretical inverse map F^{-1} is bounded close to C. By the Riemann Removable Singularity Theorem, F^{-1} is holomorphic everywhere. $\square$

Remark. It is possible to shorten the proof of Theorem 38 considerably by using the theory of *rational singularities* (to be developed in Chapter 12 in full detail): By the definition of the *fundamental cycle* Z_0 of $\widetilde{M}$, we have $Z_0 = E \cong \mathbb{P}_1$ such that the *virtual genus* $p(Z_0)$ is equal to

$$1 - \chi\,(Z_0, \mathcal{O}_{Z_0}) = 1 - (\dim H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) - \dim H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1})) = 1 - (1 - 0) = 0\ .$$

By Theorem 12.24([**??**]), M has a *rational singularity* with embedding dimension (see Theorems 12.13 and 12.15)

$$e = m + 1 = -(Z_0 \cdot Z_0) + 1 = 1 + 1 = 2\,.$$

Hence, M is smooth, and π must be equal to a finite sequence of σ–processes (as we will finally prove in Section 28). Since there is only one irreducible curve in the preimage of $x^{(0)}$, the claim follows. $\square$

The next result will be (in combination with Theorems 38) one of the two main ingredients for the existence of *minimal resolutions* of surface singularities which we are going to establish in the rest of this Section.

Corollary 9.39 *Let $E = \cup_{j=1}^{r} E_j$ be an exceptional set in a two–dimensional manifold M, and let $\tau : \widetilde{M} \to M$ be a point modification. If no E_j is exceptional of the first kind, then no strict transform $\overline{E}_j$ in $\widetilde{M}$ is exceptional of the first kind.*

Proof. Let τ be the iterated sequence of σ–processes $\sigma_1, \ldots, \sigma_\ell$ (see Theorem 48). Denote by τ_λ the map $\sigma_\lambda \circ \ldots \circ \sigma_1$. Assume that the strict transform $\overline{E}_j^{(\lambda)}$ of a curve E_j under one of the maps τ_λ is exceptional of the first kind, and choose λ minimal with respect to this property. Then, since $\lambda \geq 1$ by assumption, the exceptional set $\tau_\lambda^{-1}(E)$ contains the irreducible component $\overline{E}_j = \overline{E}_j^{(\lambda)}$ and the component $\overline{E}_0$, the fiber over the center of σ_λ. Since $\overline{E}_0$ and $\overline{E}_j$ must intersect in $n \geq 1$ points (otherwise $\overline{E}_j^{(\lambda-1)} \cong \overline{E}_j^{(\lambda)}$ would have been of first kind), the intersection matrix of the connected subconfiguration $\overline{E}_j \cup \overline{E}_0 \subset \tau_\lambda^{-1}(E)$ is given by

$$\begin{pmatrix} -1 & n \\ n & -1 \end{pmatrix}$$

which is not negative definite. A contradiction to Theorem 37. $\square$

The second ingredient we need consists in the existence of resolutions *dominating* resolutions of a given singularity.

Theorem 9.40 *Let $\pi' : M' \to X$ and $\pi'' : M'' \to X$ be two resolutions of a normal surface singularity $(X, x^{(0)})$. Then, there is a resolution $\pi : M \to X$* dominating *them, i.e. it exists a commutative diagram*

$$\begin{tikzcd}
& M \arrow[dl, "\sigma'"'] \arrow[dd, "\pi"] \arrow[dr, "\sigma''"] & \\
M' \arrow[dr, "\pi'"'] & & M'' \arrow[dl, "\pi''"] \\
& X &
\end{tikzcd}$$

with modifications σ' and σ''.

Proof. The fibre product $M' \times_M M''$ has a reduced connected component that is biholomorphic to $X \setminus \{x^{(0)}\}$ outside of the fibre over $x^{(0)}$. Normalizing this component and resolving the singularities leads to M. $\square$

Definition. A resolution $\pi : M \to X$ is called *minimal*, if all other resolutions $\pi' : M' \to X$ *factorize* over π. In other words: If there exists an iterated σ–modification $\sigma : M' \to M$ such that $\pi' \circ \sigma = \pi$.

Thus, a resolution can only be minimal, if it does not contain any exceptional curve of the first kind. Therefore, it is reasonable to starting with any resolution and blowing down (-1)–curves successively. This process stops automatically after finitely many steps.

Theorem 9.41 *A resolution $M \to X$ of a normal surface singularity is minimal, if its exceptional divisor E does not contain any exceptional curve of the first kind. Minimal resolutions exist and are canonically determined by X.*

To get the last result, it is sufficient to prove the following.

Theorem 9.42 *If under the assumptions of* Theorem 40 *the exceptional sets $E' \subset M'$ and $E'' \subset M''$ do not contain exceptional curves of the first kind then there exists a uniquely determined biholomorphic map $\sigma : M' \to M''$ making the triangles commutative* :

$$\begin{tikzcd} & M \arrow[dl, "\sigma'"'] \arrow[dr, "\sigma''"] & \\ M' \arrow[rr, "\sigma"] \arrow[dr, "\pi'"'] & & M'' \arrow[dl, "\pi''"] \\ & X & \end{tikzcd}$$

Proof. If σ' and σ'' are isomorphisms, there is nothing to prove. If exactly one of the modifications σ', σ'' is an isomorphism, say σ'', then M'' must contain an exceptional curve of the first kind. Contradiction! So, we may suppose that σ' and σ'' are not isomorphic such that we have in both cases a *first* exceptional curve E_1' resp. E_1'' that will be contracted by σ' resp. σ''. Due to our considerations above we have $E_1' \cap E_1'' = \emptyset$ or $E_1' = E_1''$. Blowing these curves resp. this curve down leads to a similar diagram with less σ–processes on both sides, finally giving an isomorphism at least on one side. $\square$

9.26 Good and minimal good resolutions

Unfortunately, *minimal* resolutions may not have the property that its exceptional sets are normal crossing divisors. As an example, we look at the following configuration:

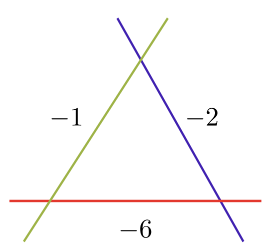


Figure 9.13

which can be realized with any three Riemann surfaces by *plumbing* (see Section 30 below). Blowing the intersection matrix formally down yields the the negative–definite intersection matrix

$$\begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}$$

that belongs to the next situation:

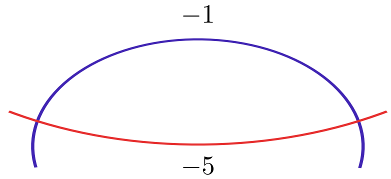


Figure 9.14

This is still a resolution with exceptional normal crossing divisor, but not a *minimal* resolution. Obviously, by blowing down the (-1)–curve we end up with just one exceptional curve having an *ordinary node*.

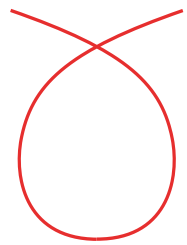


Figure 9.15

We now give the following

Definition. A resolution of a normal surface singularity $(X, x^{(0)})$ with exceptional set $E = \bigcup E_j$ is called a *good resolution*, if the following are satisfied:

i) all E_j are smooth;

ii) for all $j \neq k$ one has $(E_j \cdot E_k) = 1$ or $= 0$;

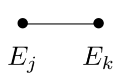
iii) no three (different) curves E_j, E_k, E_ℓ pass through one point.

In other words: For a good resolution the exceptional divisor $E = \bigcup E_j$ is a normal crossing divisor such that $(E_j \cdot E_k) = 1$ if $E_j \cap E_k \neq \emptyset, E_j \neq E_k$.

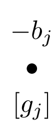
Theorem 9.43 *There exist good resolutions.*

Proof. Clear by Theorem 5.5. □

Remark. We can simply attach a *dual resolution graph* $\check{\Gamma} = \check{\Gamma}_M$ - or a *dual graph* for short - to a good resolution M by assigning to each curve E_j a vertex and connecting two vertices belonging to E_j and E_k, $j \neq k$, if and only if $(E_j \cdot E_k) = 1$:



We call $\check{\Gamma}$ a *weighted* dual graph, if the vertices are decorated with symbols

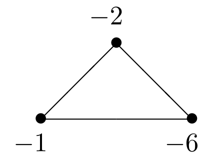


where $-b_j = (E_j \cdot E_j)$ and g_j is the genus of the Riemann surface E_j. In particular, it makes sense to speak of the *intersection matrix* of a weighted dual graph.

Starting with any good resolution we can perform the process of blowing down (-1)–curves step by step only in case that the result is again a *good* resolution. This procedure stops, as one may convince oneself, at a *uniquely determined* resolution.

***Theorem 9.44** *To each normal surface singularity there exists a uniquely determined* minimal good *resolution.*

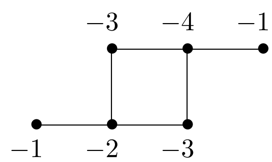
Example. In the example at the beginning of this Section, the dual resolution graph of the minimal good resolution is the following:



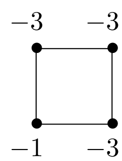
Warning. In general, the resolution method of Jung (see Chapter 7) does not automatically lead to the minimal resolution of a surface singularity. For example, Jung's procedure gives for the hypersurface singularity

$$z^2 = (x + y^2)(x^2 + y^7)$$

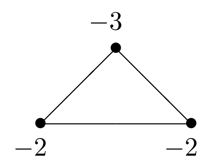
a resolution with the following dual graph (see Chapter 7, Appendix A):



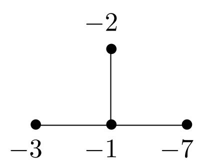
After blowing down the exceptional curves of first kind we find



and



We close this Section with another *Example*, the hypersurface singularity $z^2 = x^3 + y^7$. With Jung's method one finds the following (dual) resolution graph:



This represents already the *minimal good* resolution since, after blowing down the (-1)–curve, we get a "triple intersection point":

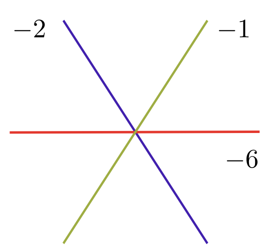


Figure 9.16

This is exactly the curve configuration we find by blowing up a *cusp singularity* two times. Hence, the exceptional divisor of this surface singularity consists in one curve only which possesses a cusp singularity.

9.27 A vanishing result for resolutions of surface singularities

We first state the main result which often may be used to replace GRAUERT's Comparison Theorem.

Theorem 9.45 *Let* $\pi : \widetilde{X} \to X$ *be a strongly pseudoconvex resolution of an isolated surface singularity* $(X, x^{(0)})$ *with exceptional set* $E = \pi^{-1}(x^{(0)})$, *and let* $I = \mathcal{O}_{\widetilde{X}}(-E)$ *by the ideal sheaf of* E. *Then there exists an integer* k_0 *such that, for all coherent sheaves* S *on* $\widetilde{X}$, *the canonical map*

$$H^j(\widetilde{X}, I^k S) \longrightarrow H^j(\widetilde{X}, S)$$

is zero for all $j \geq 1$ *and* $k \geq k_0$.

We first deduce $(*)$ in Section 25 from Theorem 45. In other words, we are going to prove

Lemma 9.46 *Let* $\widetilde{M} \to M$ *and* I *be as in* Theorem 38. *Then we have*

$$H^1(\widetilde{M}, I^r) = 0 \text{ for all } r \geq 0 .$$

Proof. We use the exact sequences

$$0 \longrightarrow I^{r+1} \longrightarrow I^r \longrightarrow I^r / I^{r+1} \longrightarrow 0$$

and the isomorphism $I^r / I^{r+1} \cong (I/I^2)^{\otimes r}$. By assumption,

$$H^1(\widetilde{M}, I^r / I^{r+1}) \cong H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(r)) ,$$

and it is easy to check by Laurent expansion that $H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(r)) = 0$ for $r \geq -1$ (use Serre duality and the remark at the end of Section 4.9). Hence, the homomorphisms $H^1(\widetilde{M}, I^{r+1}) \to H^1(\widetilde{M}, I^r)$ are surjective for all $r \geq 0$. Iterating, we get epimorphisms

$$H^1(\widetilde{M}, I^s) \longrightarrow H^1(\widetilde{M}, I^t)$$

for all $s \geq t$. Invoking Theorem 45 gives the claim. □

Proof of Theorem 45. We use the simple fact that $H^j(\widetilde{X}, S) = 0$ for all $j \geq j_0$, independently of S. (Since π is proper, X can be covered by finitely many Stein open sets). Hence, we may proceed by descending induction on j. Moreover, by GRAUERT's Coherence Theorem (or by the Finiteness Theorem of ANDREOTTI–GRAUERT), the modules $H^j(\widetilde{X}, S)$ are finite dimensional complex vector spaces.

Choose holomorphic functions $f_1, f_2 \in H^0(X, \mathcal{O}_X)$ having only the point $x^{(0)}$ as common zero, and denote by J the ideal sheaf on $\widetilde{X}$ generated by $f_1 \cdot \pi$ and $f_2 \cdot \pi$. Then there exists an integer $r \geq 1$ with

$$I^r \subset J \subset I . \tag{+}$$

We regard the sheaf homomorphism

$$\begin{cases} S \oplus S \longrightarrow & J\,S \\ (s_1, s_2) \longmapsto & (f_1 \circ \pi)\, s_1 + (f_2 \circ \pi)\, s_2 . \end{cases}$$

The exact sequence of sheaf homomorphisms

$$0 \longrightarrow R \longrightarrow S \oplus S \longrightarrow J\,S \longrightarrow 0$$

implies exactness of

$$0 \longrightarrow I^s R \longrightarrow I^s S \oplus I^s S \longrightarrow J\,I^s\,S \longrightarrow 0 .$$

(I is locally free). For any exponent $k < s$, we have a commutative diagram with exact rows

$$\begin{array}{ccccc} H^j(\widetilde{X},\, I^s\, S \oplus I^s\, S) & \longrightarrow & H^j(\widetilde{X},\, I^s\, J\, S) & \longrightarrow & H^{j+1}(\widetilde{X},\, I^s\, R) \\ \big\downarrow & & \big\downarrow \beta & & \big\downarrow \gamma \\ H^j(\widetilde{X},\, I^k\, S \oplus I^k\, S) & \xrightarrow[\Phi]{} & H^j(\widetilde{X},\, I^k\, J\, S) & \longrightarrow & H^{j+1}(\widetilde{X},\, I^k\, R) \end{array}$$

By induction assumption, $\gamma = 0$ for $s > k$ sufficiently large. Thus, $\operatorname{im} \beta \subset \operatorname{im} \Phi$.

For simplicity, we write $\operatorname{im} H^j(\mathfrak{a}S) = \operatorname{im}(H^j(\widetilde{X}, \mathfrak{a}\, S) \to H^j(\widetilde{X}, S))$ for any coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_{\widetilde{X}}$. Since

$$\operatorname{im} H^j(I^\ell S) \subset \operatorname{im} H^j(I^k S) \text{ for } \ell > k\,,$$

the sequence of submodules $\operatorname{im} H^j(I^\ell S)$ will stabilize for large ℓ. Moreover,

$$\operatorname{im} H^j(JI^s S) = \operatorname{im} H^j(JI^k S) = \operatorname{im} H^j(I^k S)$$

for $s \ge k \ge k_0$ because of (+).

Let $\xi_1, \ldots, \xi_m \in \operatorname{im} H^j(JI^s S)$ be elements whose images in $H^j(\widetilde{X}, S)$ form a $\mathbb{C}$–basis of the submodule $\operatorname{im} H^j(I^s S)$. Further, let $\eta_1, \ldots, \eta_n$ form a $\mathbb{C}$–basis of the kernel of

$$H^j(\widetilde{X},\, I^{k_0} S) \longrightarrow H^j(\widetilde{X},\, S)\,.$$

The relation $\operatorname{im} \beta \subset \operatorname{im} \Phi$ yields an equation

$$\xi_\mu = f_1(\sum_\lambda u^{(1)}_{\lambda\,\mu}\, \xi_\lambda + \sum_\nu v^{(1)}_{\nu\mu}\, \eta_\nu) + f_2(\sum_\mu u^{(2)}_{\lambda\,\mu}\, \xi_\lambda + \sum_\nu v^{(2)}_{\nu\mu}\, \eta_\nu)\,.$$

By construction, the η_ν go to zero in $H^j(\widetilde{X}, S)$; hence,

$$\operatorname{im} H^j(JI^{k_0} S) \subset (f_1,\, f_2) \operatorname{im} H^j(JI^{k_0} S)\,.$$

Now, the modules $\operatorname{im} H^j(JI^{k_0} S)$ are finitely generated over the local ring $\mathcal{O}_{X,x^{(0)}}$ by the Coherence Theorem. Applying Nakayama's Lemma implies

$$H^j(JI^{k_0} S) = 0\,. \qquad \square$$

An easy consequence of Theorem 45 is the following

Corollary 9.47 *Let* $\pi : \widetilde{X} \to X$ *be given as in* Theorem 45*, and let* S *be a coherent analytic sheaf on* $\widetilde{X}$*. Then*

$$R^j \pi_* S = 0 \text{ and } H^j(\widetilde{X},\, S) = 0 \text{ for } j > 1\,.$$

Proof. Let I denote the ideal sheaf of E. From Theorem 45 and the definition of the direct image sheaves, it follows that for $j \ge 1$ and $k \gg 0$ the sheaf homomorphisms

$$R^j \pi_* I^k S \longrightarrow R^j \pi_* S$$

are zero. Hence, the exact sequence $0 \to I^k S \to S \to S/\, I^k S \to 0$ implies exactness of

$$R^j \pi_* I^k S \xrightarrow{0} R^j \pi_* S \longrightarrow R^j \pi_*(S/\, I^k S)$$

for $j \ge 1$. Since E is one–dimensional and compact, we have

$$(R^j \pi_*(S/\, I^k S))_{x^{(0)}} \cong H^j(E,\, S/\, I^k S) = 0\,, \quad j \ge 2\,.$$

Therefore, $(R^j \pi_* S)_{x^{(0)}} \subset (R^j \pi_*(S/\, I^k S))_{x^{(0)}} = 0$, $j \ge 2$. Outside $x^{(0)}$, the higher direct image sheaves vanish by trivial reasons. Finally, since X is Stein,

$$H^j(\widetilde{X},\, S) \cong H^0(X,\, R^j \pi_* S) = 0\,, \quad j \ge 2\,. \qquad \square$$

In a more general situation, we will deduce a similar result from GRAUERT's Comparison Theorem (see Lemma 12.1).

9.28 The structure of proper modifications of smooth surfaces

The goal of the present Section consists in providing the reader with some arguments for the following central result.

Theorem 9.48 *Any proper modification $\pi : N \to M$ of connected smooth surfaces is a (with respect to M) locally finite iterated σ–process.*

For a rather elementary *proof* we need four steps.

a) The fibers of π are connected.

b) There exists a minimal locally finite set $S \subset M$ such that the restriction $\pi : N \setminus \pi^{-1}(S) \to M \setminus S$ is biholomorphic.

c) Locally at a point $y^{(0)} \in \pi^{-1}(S)$, π is given by two functions f_1, f_2 such that

$$(f_1(y) : f_2(y)) , \quad y \notin \pi^{-1}(S) ,$$

converges to a point of $\mathbb{P}_1$ if $y \to y^{(0)}$.

d) π factorizes over the σ–transformation at any point $x^{(0)} \in S$.

ad a). If $E = \pi^{-1}(x^{(0)})$ is not connected, we have disjoint open sets $V_1, V_2 \subset N$ with $E \subset V_1 \cup V_2$, but $E \not\subset V_1, E \not\subset V_2$. Since π is proper and surjective, there exists a connected open neighborhood U of $x^{(0)}$ such that $U \subset \pi(V_1) \cup \pi(V_2)$. $U \setminus \{x^{(0)}\}$ is connected and $\pi : \pi^{-1}(U) \setminus E \to U \setminus \{x^{(0)}\}$ is biholomorphic. Consequently, $(V_1 \setminus E) \cup (V_2 \setminus E)$ is connected close to E. Contradiction to $V_1 \cap V_2 = \emptyset$!

ad b). For general reasons on modifications, the set $A \subset M$ of *base points* of π is of codimension 2, i.e. A consists of isolated points. If the fiber $\pi^{-1}(x^{(0)})$, $x^{(0)} \in A$, is 0–dimensional, it consists in exactly one point $y^{(0)}$ by part a), and π is locally biholomorphic near $y^{(0)}$ due to Riemann's removable singularity theorem. Thus, we get S just by removing all such points $x^{(0)}$ from A, and for all $x^{(0)} \in S$, the fibers $E = \pi^{-1}(x^{(0)})$ are connected curves.

ad c). Let $y = (y_1, y_2)$ be local coordinates for N at a point $y^{(0)} \in \pi^{-1}(S)$, and $x = (x_1, x_2)$ be local coordinates for M at $x^{(0)} = \pi(y^{(0)})$ such that $x^{(0)} = 0$, $y^{(0)} = 0$ and π is locally at $y^{(0)}$ given by $x_1 = f_1(y_1, y_2)$, $x_2 = f_2(y_1, y_2)$. We claim that

$$\lim_{\substack{y \to 0 \\ y \notin E}} (f_1(y), f_2(y))$$

exists as a point in $\mathbb{P}_1$. To see this, we write (as germs at 0) $f_1 = q\, g_1$, $f_2 = q\, g_2$ with the greatest common divisor q of f_1 and f_2, and we have to show that $g_1(0) = g_2(0) = 0$ is impossible. Since g_1 and g_2 are relatively prime, this holds also for $r_1 g_1 + g_2$ and $r_2 g_1 + g_2$ for different constants $r_1, r_2 \in \mathbb{C}$. Let now $h = 0$ be a local equation of E near $y^{(0)}$. Each prime factor of h can divide $r\, g_1 + g_2$ for at most one value r. Consequently, there exist elements $r \in \mathbb{C}$ such that h and $g := r\, g_1 + g_2$ are relatively prime. So, after choosing representatives of the given germs in a sufficiently small neighborhood V of $y^{(0)}$, we find a curve $C = N(g) \subset V$ which intersects $E \cap V = N(h)$ in exactly this point. If we change the coordinates (x_1, x_2) to $(\xi_1, \xi_2) := (x_1, r\, x_1 + x_2)$, then π will be described by

$$(\xi_1, \xi_2) = q\,(y_1, y_2)\,(g_1(y_1, y_2), g\,(y_1, y_2))$$

such that $\pi(C) \subset C_0 := \{\xi_2 = 0\}$ and $\pi(N(q)) = x^{(0)}$, i.e. $N(q) \subset E$. Let now $\rho : \widetilde{C} \longrightarrow C$ be an embedded resolution. Then, $\pi \circ \rho : \widetilde{C} \longrightarrow C_0$ is bijective outside $\rho^{-1}(x^{(0)})$. Hence, $\widetilde{C}$ has only one irreducible component, and $\pi \circ \rho$ is biholomorphic. This implies that we have a holomorphic inverse $\pi^{-1} = \rho \circ (\pi \circ \rho)^{-1} : C_0 \to C$. Denote by D a small disc around the origin in $\mathbb{C}$, and write ι for the embedding $D \hookrightarrow C_0$ defined by $\iota(\zeta) := (\zeta, 0)$. Then, $\pi^{-1} \circ \iota$ will be given by a pair of functions (p_1, p_2) in the variable ζ, and by the definition of π we have the identity

$$(+) \qquad \zeta \,=\, f_1(p_1(\zeta),\, p_2(\zeta)) \,=\, Q\,(\zeta)\, G_1(\zeta)$$

with $Q\,(\zeta) \,=\, q\,(p_1(\zeta),\, p_2(\zeta)),\, G_1(\zeta) \,:=\, g_1(p_1(\zeta),\, p_2(\zeta))$. Suppose finally that $g_1(0,\,0) \,=\, 0$. Then $G_1(0) \,=\, 0$, and by differentiating $(+)$ we find

$$Q\,(0)\, \frac{\partial G_1}{\partial \zeta}\bigg|_{\zeta=0} \,=\, G_1(0)\, \frac{\partial Q}{\partial \zeta}\bigg|_{\zeta=0} + Q\,(0)\, \frac{\partial G_1}{\partial \zeta}\bigg|_{\zeta=0} \,=\, \frac{\partial \zeta}{\partial \zeta}\bigg|_{\zeta=0} \,=\, 1\,.$$

Hence, $q\,(0,\,0) \,=\, Q\,(0) \,\neq\, 0$. But this contradicts the fact that f_1 and f_2 vanish on E and thus have a common nontrivial factor.

ad d). Suppose without loss of generality that $g_2(0,\,0) \,\neq\, 0$. If $(\xi_1,\,\xi_2) \,=\, (\xi\eta,\,\eta) \,=\, (\xi',\,\xi'\eta')$ denotes the σ–process $\widetilde{\pi} :\, \widetilde{M} \longrightarrow M$ at $x^{(0)}$ with exceptional curve E_0 , then the association

$$(y_1,\,y_2) \longmapsto \left(\frac{f_1(y_1,\,y_2)}{f_2(y_1,\,y_2)} \,=\, \frac{g_1(y_1,\,y_2)}{g_2(y_1,\,y_2)}\,,\; f_2(y_1,\,y_2) \right)$$

defines a holomorphic mapping $\phi :\, N \longrightarrow \widetilde{M} \,=:\, M^{(1)}$ which is an extension of the mapping $\widetilde{\pi}^{-1} \circ \pi :\, N \setminus E \longrightarrow \widetilde{M} \setminus E_0$. Therefore, $\pi \,=\, \widetilde{\pi} \circ \phi$.

Continuing in the same vein yields a decomposition of π into a chain of modifications

$$N \longrightarrow M^{(r)} \longrightarrow \cdots \longrightarrow M^{(1)} \longrightarrow M$$

in which all mappings are quadratic transformations besides the first one that is a modification with discrete fibers, i.e. an isomorphism. □

Remark. The modification $\pi \,:\, N \,\to\, M$ may locally be regarded as a *resolution* of the "regular" singularity $x^{(0)} \,\in\, M$ which is *rational* by definition. As we will show in Chapter 12 in a general context, the exceptional curves of any resolution of a rational singularity are always smooth rational curves. Therefore, $E \,=\, \cup_{j=1}^{\ell} E_j$ and all $E_j \,\cong\, \mathbb{P}_1$. The canonical sheaf Ω^2_N has in arbitrarily small neighborhoods of E nontrivial holomorphic sections (see Chapter 15, Section 8) and is therefore defined by an effective divisor on N which we call a *canonical divisor* K_N . Moreover, Ω^2_N is trivial in $V \setminus E$, $V \,:=\, \pi^{-1}(U)$, for a sufficiently small neighborhood U of $x^{(0)}$ in which Ω^2_M is trivial. Hence, $K \,:=\, K_N$ is supported on E :

$$K \,=\, \sum_{j=1}^{\ell} a_j\, E_j\,, \quad a_j \in \mathbb{N}_{>0}\,.$$

By the Grauert–Mumford criterion, we have $(K \cdot K) \,<\, 0$. Hence, $(K \cdot E_j) \,<\, 0$ for at least one curve E_j , and $(E_j \cdot E_j) \,<\, 0$. By "reinterpretation" of the *adjunction formula* and Riemann–Roch, we obtain

$$(K \cdot E_j) \,+\, (E_j \cdot E_j) \,=\, ((K \,+\, E_j) \cdot E_j) \,=\, d\,(\Omega^1_{E_j}) \,=\, 2\,g\,(E_j) \,-\, 2 \,=\, -\,2\,,$$

which is only possible if

$$(E_j \cdot E_j) \,=\, (K \cdot E_j) \,=\, -1\,.$$

It is simply realized as before that $\pi \,:\, N \,\to\, M$ factorizes over the σ–process contracting the (-1)–curve E_j . □

9.29 Neighborhoods of smooth components of the exceptional set

The main result of Section 25 can obviously be formulated in the following way:

Theorem 9.49 *For any (-1)–curve $E \subset M$ there exists a neighborhood of E which is biholomorphic to a neighborhood of the zero section in the total space of the line bundle $L \,\cong\, N_{E|M}$ on $E \,\cong\, \mathbb{P}_1$ with Chern number -1 .*

One can ask more generally when a similar statement is correct for an arbitrary *smooth* compact curve $E \subset M$. Grauert proved that for two such *exceptional* curves $E \subset M$, $E' \subset M'$ one has an actual biholomorphic map between neighborhoods of E and E' if there exists a *formal isomorphism* between M and M' along E resp. E'. As an application in the situation where E' is the zero section in the total space of the (negative) line bundle $N := N_{E|M}$ he comes up with the following important result.

***Theorem 9.50** *If the smooth exceptional curve $E \subset M$ satisfies*

$$H^1(E, T_E \otimes (N^*)^{\otimes \ell}) = H^1(E, (N^*)^{\otimes \ell}) = 0\,, \quad \ell \geq 1\,,$$

then a neighborhood of E in M is biholomorphic to a neighborhood of the zero section in the total space of the line bundle $N := N_{E|M}$.

In particular, blowing down such curves amounts to the same as to blow down the zero section in negative line bundles. Hence, the resulting singularities are *generalized cones* (see Section 7 and Chapter 10.4). By Serre duality, these assumptions are equivalent to

$$H^0(E, \Omega^1_E \otimes N^{\otimes \ell}) = H^0(E, N^{\otimes \ell}) = 0\,, \quad \ell \geq 1\,,$$

Since the canonical divisor has degree $2g - 2$, $g = g(E)$ the genus of the curve, these two cohomology groups vanish if

$$\ell\, d(N) < 2 - 2g\,.$$

As a consequence, we may state:

Corollary 9.51 *The conclusion of the Theorem above is always valid for exceptional rational and for exceptional elliptic curves. For an arbitrary smooth curve this is true if the degree of the normal bundle is smaller than* $\min(0, 2 - 2g)$, $g = g(E)$.

So, by blowing down rational curves $E \cong \mathbb{P}_1$, we get exactly the cones over the rational normal curves (see Chapter 4.12). If E is an *elliptic* curve the corresponding singularities are called *simple elliptic* (see Chapter 10.5).

9.30 The plumbing construction

The rest of the present Chapter will be concerned with a construction which associates to any weighted dual graph with negative definite intersection matrix a good resolution of a surface singularity.

Let M be a two–dimensional complex analytic manifold containing a connected one–dimensional compact analytic subset E with smooth irreducible components $E_1, \ldots, E_r$. We say that M is *plumbed along* E, if (eventually after shrinking M with respect to E) the following is true: There exist line bundles L_j on E_j, neighborhoods U_j, V_j of E_j in M resp. of the zero–section E_j in L_j and biholomorphic maps

$$\lambda_j : V_j \longrightarrow U_j$$

such that

(i) $\displaystyle\bigcup_{j=1}^{r} U_j = M$,

(ii) $\lambda_{j|E_j}$ is the identity on E_j,

(iii) $E_j \cap E_k = \emptyset \implies U_j \cap U_k = \emptyset$,

(iv) if $E_j \cap E_k \neq \emptyset$, then $E_j \cap E_k = \{ x^{(0)} \}$ and $U_{jk} = U_j \cap U_k$ is connected; more precisely: $\lambda_j^{-1}(U_{jk})$ resp. $\lambda_k^{-1}(U_{jk})$ are of the form $\{ \, | \, u_j \, | < \varepsilon , \, | \, v_j \, | < \varepsilon \}$ resp. $\{ \, | \, u_k \, | < \varepsilon , \, | \, v_k \, | < \varepsilon \}$, where u_j and u_k are local parameters on E_j and E_k, and v_j and v_k are fiber coordinates of L_j and L_k near $x^{(0)}$, resp., such that the bijection

$$\lambda_j^{-1}(U_{jk}) \stackrel{\lambda_k^{-1} \circ \lambda_j}{\longrightarrow} \lambda_k^{-1}(U_{jk})$$

is given by $(u_k, v_k) = (v_j, u_j)$.

In particular, the curves E_j, E_k meet transversely with $(E_j \cdot E_k) = 0$ or 1, and no three curves go through one point. Naively, the picture of a plumbed manifold looks as follows:

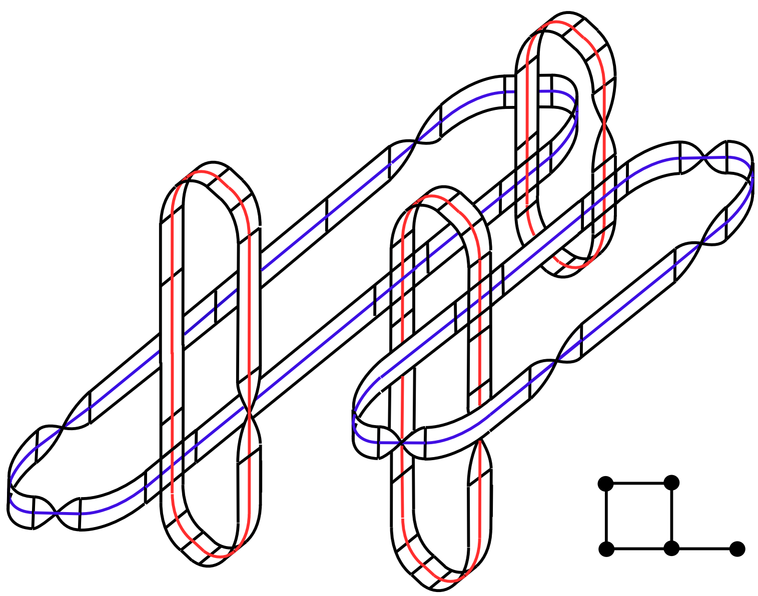
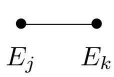
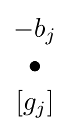


Figure 9.17

As in the case of resolutions we can simply attach a *dual graph* $\check{\Gamma} = \check{\Gamma}_M$ to a plumbed manifold M by assigning a vertex to each curve E_j and connecting two vertices belonging to E_j and E_k, if and only if $(E_j \cdot E_k) = 1$:



or, more precisely, a *weighted* dual graph, if the vertices are replaced by symbols



where $-b_j = (E_j \cdot E_j)$ and g_j is the genus of the Riemann surface E_j.

Now, given any connected weighted dual graph $\check{\Gamma}$, it is an easy task to construct a complex analytic manifold plumbed in such a way that its associated dual weighted graph is the given one: Just notice that compact Riemann surfaces carry holomorphic line bundles of all degrees and patch suitably chosen neighborhoods of the zero–section in such line bundles together.

In the C^∞–category, a plumbed manifold M is completely determined by $\check{\Gamma}_M$. The main reason for this fact lies in the *Tabular Neighborhood Theorem* of differential geometry saying that there exist arbitrarily small neighborhoods of E_j in M which are diffeomorphic to a neighborhood of E_j in the normal bundle of E_j in M. But, in our case, the normal bundle is already differentiably determined by its Chern number. It is also easily seen that there are no C^∞–obstructions against the plumbing construction.

For the determination of the complex analytic structure, the knowledge of $\check{\Gamma}$ is not sufficient. In fact, more is needed:

(1) *The analytic structure of the curves* E_j *and of the bundles* L_j, $j = 1, \ldots, r$,

(2) *the sets* $E_j \setminus \bigcup_{k \neq j} E_k$, $j = 1, \ldots, r$ *up to analytic automorphisms of the Riemann surfaces* E_j.

For a rational curve $\mathbb{P}_1$, however, each line bundle L is isomorphic to a bundle $\mathcal{O}(b)$, and each set of three points can be moved by an analytic automorphism to $0, 1, \infty$, say. Consequently, we can state:

Theorem 9.52 *To each set of data* (E_j, b_j), $E_j \cong \mathbb{P}_1$, $b_j \in \mathbb{Z}$, $j = 1, \ldots, r$, *with* $E_j \cap E_k$ *empty or consisting of one point,* $j \neq k$, *and* $E_j \cap \cup_{k \neq j} E_k$ *having not more than three points for all* j, *there exists one (and up to analytic isomorphism only one) plumbed manifold having these data.*

9.31 Existence of normal surface singularities associated to abstract graphs

Using the plumbing construction in combination with the Grauert–Mumford criterion immediately yields

Theorem 9.53 *Given any weighted dual graph* $\check{\Gamma}$ *with negative definite intersection matrix, there exists a normal surface singularity having* $\check{\Gamma}$ *as dual resolution graph.*

As we remarked before, the plumbing construction will lead in general to many nonisomorphic singularities with the *same* weighted dual graph. Moreover, although we encountered up to now only singularities having a plumbed resolution, one should be aware of the fact that there might exist still more nonisomorphic singularities having the same data for a good resolution which, however, *cannot be constructed by plumbing*. The reason for this phenomenon does lie in the existence of obstacles against the construction of *biholomorphic* maps between neighborhoods of a curve $E_j \subset M$ and neighborhoods of the zero–section in the normal bundle N_j of E_j in M; and even if these obstacles vanish it might be impossible to patch two such neighborhoods together according to the rules of plumbing (see also the Appendix A). Note however, that the first difficulty is not essential since by blowing up sufficiently many points on curves of higher genus we may force their normal bundles to have small enough degrees.

We finally want to show in this Section that the philosophy of Theorem 53 remains true, if we drop the implicit assumption for the resolution $\widetilde{X}$ to be *good*. Of course, in this situation the weighted dual graph is not enough information. But we can start with weighted "graphs" of the following type (see next page), to which we can associate an intersection matrix $C = (c_{jk})$ with $c_{jj} = -b_j$ and c_{jk} counting the number of intersection points of the j–th and k–th "curve" (with multiplicity, if we attach to each intersection point numbers according to the prescribed contact numbers). The general assumption again is the negative–definiteness of C.

As in the case of abstract intersection matrices, we can perform in a completely formal manner blow–ups of such graphs. The associated intersection matrix will remain negative definite such that after a finite number of steps we end up with a "good graph" for which we can construct by plumbing an associated complex analytic manifold M. But the "curves" inserted by the formal σ–processes are now concrete submanifolds of M isomorphic to $\mathbb{P}_1$ which can be actually contracted by Theorem 38 (in the opposite order of their formal creation). So, finally, we find a complex manifold together with an embedded system of smooth Riemann surfaces E_j having the correct genus and self–intersection number and intersecting each other combinatorially in the prescribed manner.

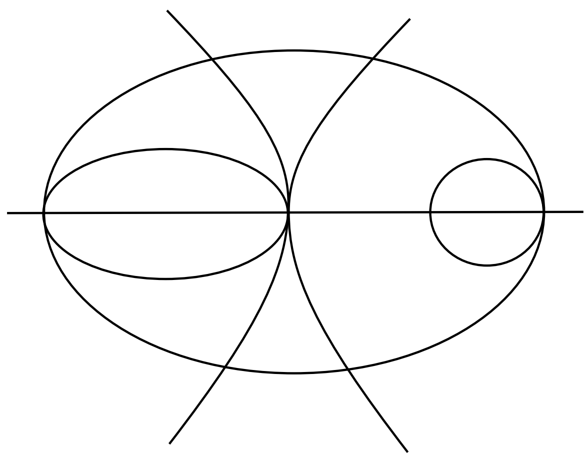
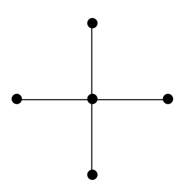


Figure 9.18

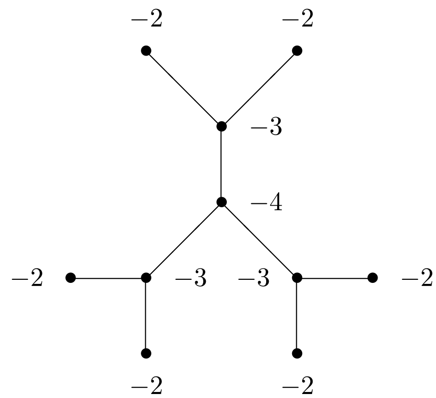
9.A Appendix A: Taut singularities

A normal two–dimensional singularity X is called *taut* by Henry B. Laufer, if all other normal two–dimensional singularities X' with the same (weighted) dual resolution graph as X are biholomorphically equivalent to X. In other words: A taut singularity is analytically determined by the topological information inherent in its dual graph.

It is clear that already the plumbing construction yields many different analytic structures associated to a given graph, if there is a curve of genus ≥ 1. So, necessarily, all curves must be rational for a taut singularity. Moreover, since the cross ratio is an analytic invariant on $\mathbb{P}_1$, no curve can be cut by more than three other curves. Hence, a graph of type



(with suitable self–intersection numbers to make the intersection matrix negative–definite) does not belong to a taut singularity. But also the rational graph of Example 11.2 is not taut (due to Tjurina).



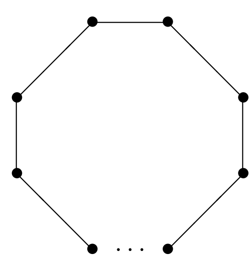
The results on quotient surface singularities imply that they are taut; in particular, all rational double points are taut. Tjurina proved that rational triple points have the same property. A complete list of all taut graphs was finally given by Laufer. He shows that if a dual graph with given weights belongs to a taut singularity, so does the same graph with more negative weights. With this in mind, the "minimal" graphs of taut singularities are those of

– rational double points

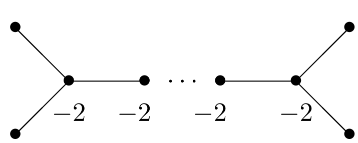
– rational triple points

– most of the singularities whose dual weighted graph may be obtained by joining the longest arms of two of the rational double or triple points

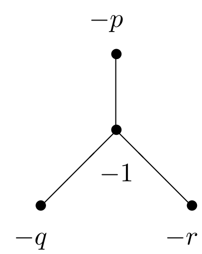
– cusp singularities (which will be studied in more detail in Chapter [**??**]). Their dual graph is a cycle



Of course, cusp singularities are not rational (they are minimally elliptic; see Chapter [**??**], [**??**]). Most of the other taut singularities are rational; the only exceptions are in the third class those with negative–definite dual graph



Laufer also classified all *pseudotaut* singularities which have by definition the property that there are only countably many analytically different normal singularities with the same resolution graph. It turns out that to a pseudotaut graph there correspond only *finitely* many singularities. One of these graphs is the following (which we will see again in Chapter [**??**]):



9.B Appendix B: Riemann-Roch and duality on embedded singular curves

Up to now it has been our strategy to avoid *singular* curves in the exceptional divisor of resolutions of surface singularities. In the present Appendix, we sample a few general facts about duality on such curves and the correct version of the Riemann–Roch Theorem that shall be needed in Chapter 12.

9.B.1 Interpretation of the adjunction formula

We have seen in Section 12 that for a smooth curve C in a two–dimensional manifold M there is a canonical isomorphism

$$\Omega^2_{M|C} \cong \Omega^1_C \otimes N^*_{C|M} .$$

Taking into account that $\mathcal{O}\,(N_{C|M}) \cong \mathcal{O}_M(C)_{|C}$ we can rewrite this in the form

$$\Omega^1_C \cong \Omega^2_M \otimes \mathcal{O}_C(C) .$$

We now define for an *arbitrary* curve $C \subset M$ the sheaf ω_C by the same formula:

$$\omega_C := \Omega^2_M \otimes \mathcal{O}_C(C) .$$

It is easy to conclude that ω_C is the *dualizing sheaf* of C in the sense of Chapter 14; in other words:

Lemma 9.54 $$\omega_C := \mathrm{Ext}^1(\mathcal{O}_C,\, \Omega^2_M) .$$

Proof. From the exact sequence

$$(*) \qquad 0 \longrightarrow I = \mathcal{O}_M(-C) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_C \longrightarrow 0 ,$$

which is a free resolution of $\mathcal{O}_C$ over $\mathcal{O}_M$, we deduce the exactness of

$$\mathrm{Hom}\,(\mathcal{O}_C,\, \Omega^2_M) \longrightarrow \mathrm{Hom}\,(\mathcal{O}_M,\, \Omega^2_M) \longrightarrow \mathrm{Hom}\,(I,\, \Omega^2_M) \longrightarrow$$

$$\mathrm{Ext}^1(\mathcal{O}_C,\, \Omega^2_M) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_M,\, \Omega^2_M) \;=\; 0 .$$

Now, $\mathrm{Hom}\,(\mathcal{O}_M,\, \Omega^2_M) \cong \Omega^2_M$ and $\mathrm{Hom}\,(I,\, \Omega^2_M) \cong \Omega^2_M(C)$. On the other hand, tensoring $(*)$ by $\Omega^2_M \otimes \mathcal{O}_M(C)$ yields the exact sequence

$$(**) \qquad 0 \longrightarrow \Omega^2_M \longrightarrow \Omega^2_M(C) \longrightarrow \omega_C \longrightarrow 0\,,$$

whence the claim. □

Remark. In addition, the preceding proof implies the *vanishing* of the group $\mathrm{Hom}\,(\mathcal{O}_C,\, \Omega^2_M)$. This, however, is a general fact which we state in a "local" version as follows.

Lemma 9.55 *Let R be a local noetherian ring, M an R–module and $\mathfrak{a}$ an ideal in R. Then,*

$$\mathrm{Hom}_R(R/\mathfrak{a},\, M) \;\cong\; \{\, m \in M : a\,m \,=\, 0 \text{ for all } a \in \mathfrak{a}\,\}\,.$$

In particular, if for all $m \in M$, $m \neq 0$, there exists an element $a \in \mathfrak{a}$ with $a\,m \neq 0$, then

$$\mathrm{Hom}_R(R/\mathfrak{a},\, M) \;=\; 0\,.$$

Proof. Left–exactness of the functor $\mathrm{Hom}\,(\_\,,\, M)$ gives the injectivity of the homomorphism $\mathrm{Hom}_R(R/\mathfrak{a},\, M) \longrightarrow \mathrm{Hom}_R(R,\, M)$, and it is immediate that

$$\mathrm{Hom}_R(R/\mathfrak{a},\, M) \;=\; \{\, \varphi \in \mathrm{Hom}_R(R,\, M) : \varphi|\mathfrak{a} \,=\, 0\,\}\,.$$

Now, as we already used before, $M \;\cong\; \mathrm{Hom}_R(R,\, M)$, the isomorphism being given by $m \longmapsto \varphi_m$ where $\varphi_m(a) := a\,m$. □

9.B.2 Duality on embedded singular curves

Suppose now that $C \subset M$ is a (not necessarily reduced) *compact* curve in a two–dimensional complex analytic manifold M. Notice that $\mathcal{O}_C$ is a Cohen–Macaulay ring as the local quotient of $\mathcal{O}_M$ by an equation f of C and even a *complete intersection*. Therefore, the dualizing sheaf ω_C is locally free. Moreover, C is *projective algebraic* (see Chapter 4). Hence, the general duality theorem in Chapter 14 gives the following special result.

***Theorem 9.56** *In the situation described as before, there is a canonical isomorphism*

$$H^0(C,\, \mathcal{F}^* \otimes \omega_C) \xrightarrow{\sim} H^1(C,\, \mathcal{F})^*$$

for arbitrary locally free sheaves $\mathcal{F}$ on C.

Proof. The left hand side is canonically isomorphic to $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F},\, \omega_C)$ since

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F},\, \omega_C) \;=\; H^0(C,\, \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F},\, \omega_C))$$

and $\mathcal{H}om_{\mathcal{O}_C}(\mathcal{F},\, \omega_C) \;\cong\; \mathcal{F}^* \otimes \omega_C$, ω_C being locally free. □

Remark. In order to describe the isomorphism in the duality theorem, we have to associate to any homomorphism $\varphi : \mathcal{F} \to \omega_C$ a linear form on the vector space $H^1(C,\, \mathcal{F})$. Now, φ induces a homomorphism

$$H^1(\varphi) : H^1(C,\, \mathcal{F}) \longrightarrow H^1(C,\, \omega_C)$$

and it suffices to find a (nontrivial) canonical homomorphism $H^1(C,\, \omega_C) \to \mathbb{C}$. Using $(**)$ in Section B.1, there is a homomorphism

$$H^1(C,\, \omega_C) \longrightarrow H^2_c(M,\, \Omega^2_M)\,,$$

the subscript c denoting cohomology with compact support. By Dolbeault's lemma, cohomology classes in $H^2_c(M,\, \Omega^2_M)$ can be represented by (automatically $\overline{\partial}$–closed) $(2,\, 2)$–forms α with compact support on M, and Stokes theorem implies that

$$\begin{cases} H^2_c(M,\, \Omega^2_M) \longrightarrow \mathbb{C} \\[1ex] \qquad \overline{\alpha} \qquad\quad \longmapsto \displaystyle\int_M \alpha \end{cases}$$

does not depend on the representative α of the class $\overline{\alpha}$.

9.B.3 Riemann-Roch on embedded singular curves

It is one of the ironies in mathematics that one has to develop the theory of embedded curves in full generality (taking into account the possibility of nonsmooth reduced components, nontransversality of intersections and so on) in order to show that all these pathologies cannot occur for the exceptional set E in an arbitrary resolution of, e.g., a *rational* singularity X.

Before we can begin our brief summary about the generalized Riemann–Roch Theorem, we would like to rephrase the classical version (Theorem 16). Recall that the *Euler–Poincaré characteristic* for a coherent analytic sheaf S on a compact complex projective algebraic manifold M of dimension n is defined by

$$\chi(M, S) = \sum_{j=0}^{n} (-1)^j \dim H^j(M, S) .$$

Remark. From the long exact cohomology sequence and the simple observation that for any long exact sequence

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_N \longrightarrow 0$$

of finite dimensional complex vector spaces V_j we have

$$\sum_{j=0}^{N} (-1)^j \dim_{\mathbb{C}} V_j = 0 ,$$

we easily conclude that the functor χ is *additive*.

Lemma 9.57 *If* $0 \longrightarrow S_1 \longrightarrow S \longrightarrow S_2 \longrightarrow 0$ *is a short exact sequence of coherent sheaves on* M *, then*

$$\chi(M, S) = \chi(M, S_1) + \chi(M, S_2) .$$

We always put

$$\chi(M) = \chi(M, \mathcal{O}_M) ,$$

such that for a smooth compact Riemann surface C of genus g we have

$$\chi(C) = 1 - g ,$$

and therefore, given any holomorphic line bundle $L \to C$ of degree $d(L)$, the Riemann–Roch Theorem asserts that

$$(+) \qquad \chi(C, \mathcal{O}_C(L)) = d(L) + \chi(C) .$$

We consider now the case of an arbitrary compact (not necessarily reduced) analytic curve C in a two–dimensional complex manifold M, defined by the (necessarily locally principal) ideal $I \subset \mathcal{O}_M$. If

$$C = \bigcup_{j=1}^{r} C_j$$

denotes the decomposition of C into irreducible components C_j and if I_j is the ideal of all holomorphic functions vanishing on C_j, then we can find positive integers n_j such that

$$(++) \qquad I = I_1^{n_1} \cdot \ldots \cdot I_r^{n_r} .$$

We associate to C the positive divisor $\sum n_j C_j$. On the other hand, each positive divisor defines a curve via $(++)$. Thus, the concepts of curves embedded in M and of positive divisors are the same.

To generalize the Theorem of Riemann and Roch to holomorphic line bundles L on such curves C, we need the correct definition for their degree $d(L)$. We will see below that the following is a good choice:

$$d(L) = \sum_{j=1}^{r} n_j\, d(\widetilde{L}_j)\ ,$$

where $\nu_j : \widetilde{C}_j \to C_j$ denotes the *normalization* of the reduced curve C_j and $\widetilde{L}_j$ is the pull–back of L_j under ν_j. In particular, if all C_j are smooth, then

$$d(L) = \sum_{j=1}^{r} n_j\, d(L_{|C_j})\ .$$

Now, if C is reduced, then the normalization $\nu : \widetilde{C} \to C$ has $\widetilde{C}_1, \ldots, \widetilde{C}_r$ as connected components. Moreover, the higher direct images $R^j \nu_* S$ vanish for coherent analytic sheaves S on $\widetilde{C}$, since preimages of suitably small open sets in C are disjoint unions of open sets in $\mathbb{C}$, hence Stein. The *Leray spectral sequence* then implies

$$\sum_{j=1}^{r} \chi(\widetilde{C}_j, \mathcal{O}_{\widetilde{C}_j}(\widetilde{L}_j)) = \chi(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(\widetilde{L})) = \chi(C, \nu_* \mathcal{O}_{\widetilde{C}}(\widetilde{L}))\ .$$

The canonical map $\mathcal{O}_C(L) \to \nu_* \nu^* \mathcal{O}_C(L)$ is injective (the kernel having isolated support), and its cokernel is isomorphic to that of $\mathcal{O}_C \to \nu_* \nu^* \mathcal{O}_C$ because of the *projection formula* $\nu_* \nu^* \mathcal{F} \cong \mathcal{F} \otimes \nu_* \mathcal{O}_{\widetilde{C}}$. Hence, we get finally from the classical Riemann–Roch Theorem and the additivity of χ on short exact sequences the formula

$$\chi(C, \mathcal{O}_C(L)) - \chi(C) = \chi(C, \nu_* \mathcal{O}_{\widetilde{C}}(\widetilde{L})) - \chi(C, \nu_* \mathcal{O}_{\widetilde{C}})$$

$$= \sum_{j=1}^{r} (\chi(\widetilde{C}_j, \mathcal{O}_{\widetilde{C}_j}(\widetilde{L}_j)) - \chi(\widetilde{C}_j)) = \sum_{j=1}^{r} d(\widetilde{L}_j)\ .$$

If C is not reduced, one can write $I = J_1 J_2$ for the defining ideal and use the exact sequence

$$0 \longrightarrow J_1(\mathcal{O}_M / J_2) \cong J_1 / I \longrightarrow \mathcal{O}_M / I \longrightarrow \mathcal{O}_M / J_1 \longrightarrow 0$$

and its satellite obtained by tensoring with $F = \mathcal{O}(L)$. By induction hypothesis on $\sum n_j$, we may apply the generalized Riemann–Roch Theorem to the curves $C_1 = N(J_1)$ and $C_2 = N(J_2)$ and to the bundle $L_{|C_1}$ resp. to $L_{|C_2} \otimes J_1$. By invoking once more the additivity of χ, we easily finish the *proof* of

Theorem 9.58 *Let C be a compact curve embedded in the smooth surface M. Then, for any holomorphic line bundle L on C, the generalized Riemann–Roch identity holds true:*

$$\chi(C, \mathcal{O}_C(L)) = d(L) + \chi(C)\ .$$

9.B.4 A vanishing Theorem for embedded singular curves

Under our standard assumptions in the present Appendix, the dualizing sheaf ω_C is a holomorphic line bundle whose degree can be computed by the Riemann–Roch Theorem in the last Section:

$$d(\omega_C) = \chi(\omega_C) - \chi(\mathcal{O}_C)\ .$$

But by duality, $H^0(C, \mathcal{O}_C) \cong H^1(C, \omega_C)$, $H^1(C, \mathcal{O}_C) \cong H^0(C, \omega_C)$ such that we can conclude:

Lemma 9.59 $$d(\omega_C) = 2\chi(\omega_C) = -2\chi(\mathcal{O}_C)\ .$$

Remark. The number $1 + \chi(\omega_C) = 1 - \chi(\mathcal{O}_C)$ is usually called the *arithmetic* or *arithmetical* (or *virtual*) genus of C, in symbols: $p_a(C)$ or $p(C)$. If C is smooth, then

$$p_a(C) = \dim H^1(C, \mathcal{O}_C) = \dim H^0(C, \Omega^1_C) = g(C) .$$

In the general case, the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_* \mathcal{O}_{\widetilde{C}} \longrightarrow \nu_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C \longrightarrow 0$$

yields with the dimension δ of the finite dimensional vector space $\nu_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C$:

$$p_a(C) = 1 - \chi(\nu_* \mathcal{O}_{\widetilde{C}}) + \delta = 1 - \chi(\mathcal{O}_{\widetilde{C}}) + \delta = g(\widetilde{C}) + \delta .$$

Since $g(\widetilde{C}) \ge 0$ and $\delta \ge 0$, we immediately deduce:

Lemma 9.60 *If $p_a(C) \le 0$ for a compact, embedded reduced irreducible curve C, then C is smooth and $C \cong \mathbb{P}_1$.*

For later application we finally state the following vanishing result.

Theorem 9.61 *Let C be an embedded reduced irreducible curve and L be a holomorphic line bundle on C with*

$$d(L) > 2p_a(C) - 2 .$$

Then,

$$H^1(C, \mathcal{O}(L)) = 0 .$$

Proof. Since ω_C is locally free, $L \cong L \otimes \omega_C^* \otimes \omega_C$ and $H^1(C, \mathcal{O}(L)) \cong H^0(C, \mathcal{O}(L^*) \otimes \omega_C)$ by duality. The degree is an additive function on tensor products. Thus,

$$d(L^* \otimes \omega_C) = -d(L) + d(\omega_C) < -2(p_a(C) - 1) + 2(p_a(C) - 1) = 0 .$$

Therefore, it remains to show that for any line bundle L with $d(L) < 0$ we have $H^0(C, \mathcal{O}(L)) = 0$. This is obviously correct if C is smooth since then $d(L) < 0$ implies that L has no nontrivial *holomorphic* sections. In the general case we regard a normalization $\nu_* : \widetilde{C} \to C$ and put $\widetilde{L} = \nu^* L$. There is a canonical injective homomorphism

$$L \longrightarrow \nu_* \widetilde{L} = \nu_*(\nu^* L)$$

which gives rise to an injection

$$H^0(C, \mathcal{O}(L)) \subset H^0(C, \nu_* \widetilde{L}) = H^0(\widetilde{C}, \widetilde{L}) = 0 ,$$

the last identity being a consequence of $d(\widetilde{L}) = d(L) < 0$. $\square$

Notes and References

The proof of the necessity of the main criterion follows closely the paper

[09 - 01] D. Mumford: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Publ. Math. IHES 9, 229–246 (1961),

which will play its main role in Chapter 15. For the treatment of positive/negative definite quadratic forms, see Bourbaki [08 – 06].

Most of the material concerning the opposite direction is taken from the most influential text

[09 - 02] H. Grauert: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. 146, 331–368 (1962).

The sufficient conditions for *generalized cones* are contained in his Satz 7, Corollar. The elementary reasoning in Sections 21, 22 and 23 which avoids Grauert's sophisticated cohomological arguments are due to the author.

The elementary treatment of modifications of two–dimensional manifolds is a variant of Chapter V in Laufer's book [01 - 14]. (See also loc.cit., Chapter VI d), and in particular his Proposition 6. 21, with respect to rational exceptional curves.) The very short route via the theory of rational singularities has been followed in [01 - 18].

The material on taut (and pseudotaut) singularities has its origin in

[09 - 03] G. Tjurina: On the tautness of rationally contractible curves on a surface. Math. USSR Izvestija 2, 907–934 (1968)

and has been completed by

[09 - 04] H. Laufer: Taut two dimensional singularities. Math. Ann. 205, 131–164 (1973).

Our standard source for Riemann surfaces is Forster [07 – 23]; see also

[09 - 05] R.C. Gunning: Lectures on Riemann surfaces. Princeton Mathematical Notes. Princeton, New Jersey: Princeton University Press 1966.

The material of Appendix B concerning *duality* is mostly taken from [01 - 18]. For more details, e.g. concerning the question when an *abstract* irreducible curve C has a locally free dualizing sheaf ω_C, see e.g. Chapter IV in

[09 - 06] J.P. Serre: Courbes algébriques et corps de classes. Paris: Hermann 1959.

The general case can be reduced to this one by induction on the number of components of C (for the details, see again [01 - 18]).