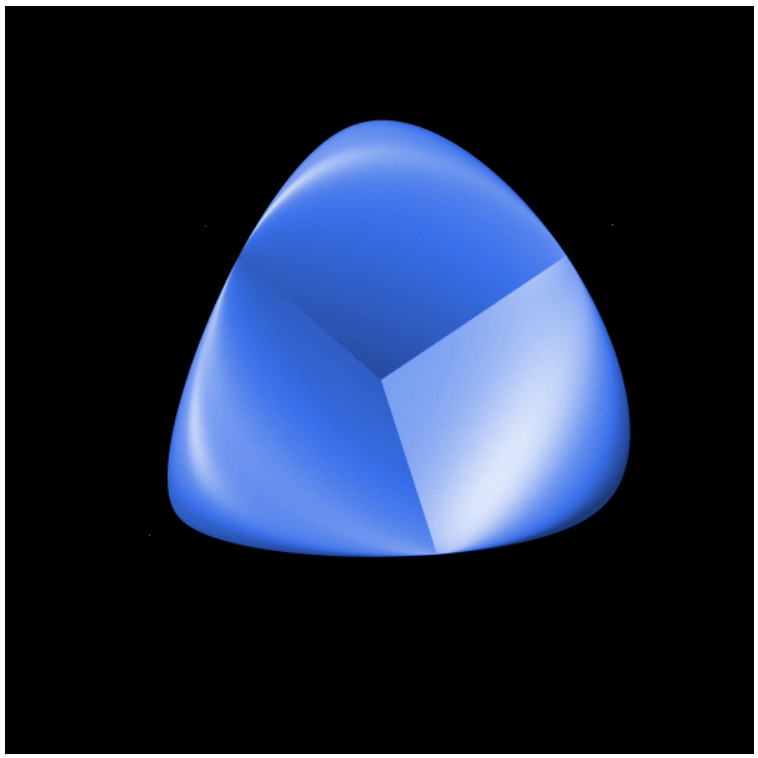
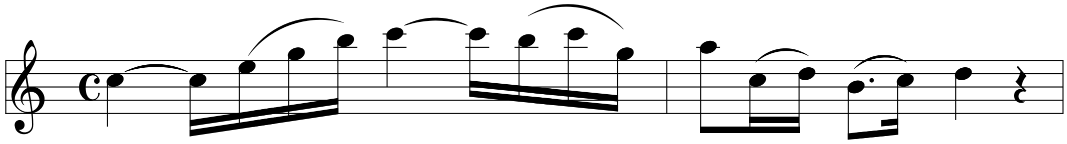
Chapter 8

Ich konnte [...] insofern auf eigene Erfahrungen zurückgreifen, als auch ich den größten Teil meines Lebens in immer kleiner werdenden Zimmern verbringe, die zu verlassen mir immer schwerer fällt. Ich hoffe aber, eines Tages ein Zimmer zu finden, das so klein ist und mich eng umschließt, daß es sich beim Verlassen von selbst mitnimmt.

(Patrick Süskind,
Remark to *Der Kontrabaß*)





Chapter 8

Quotient singularities: General theory

The purpose of Chapter 8 is to justify all the results on analytic quotients that we used already in the last Chapter and to make additional material available that is needed to complete the classification of all quotient surface singularities. We close the Chapter with a topological characterization in the surface case. Since we work in this Chapter and later on in the context of general *complex analytic spaces* and not just on germs, the reader is advised to consult the Supplement (Chapter 19) if necessary.

8.1 The formation of quotients as a universal problem

Let $(X, \mathcal{O}_X)$ be a *reduced complex analytic space* and let G be a subgroup of $\mathrm{Aut}\, X$, the group of biholomorphic transformations of X, acting from the right on X as described in Chapter 7. We want to construct a reduced complex analytic space $(Y, \mathcal{O}_Y)$ together with an open, surjective holomorphic map $\rho : X \to Y$ which is *invariant* under G, i. e. $\rho \circ \gamma = \rho$ for all $\gamma \in G$, such that the following universal properties hold:

1. For all G–stable open sets $U \subset X$ and all G–invariant holomorphic maps $\varphi : U \to Z$ there exists a uniquely determined holomorphic map $\psi : \rho(U) \to Z$ such that the diagram

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & Z \\ & \rho(U) \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. Recall that a set U is called *G–stable* or *G–invariant*, if $x \in U$ implies $\gamma(x) \in U$ for all $\gamma \in G$.

2. The mapping $\psi \mapsto \varphi := \psi \circ \rho$ gives a *surjection* between the set of *all* holomorphic maps from $\rho(U)$ to Z and the set of *all G–invariant* holomorphic maps from U to Z (and consequently, according to 1., a *bijection* of these sets).

As a universal object, Y is uniquely determined up to biholomorphic isomorphisms, if it exists. It is then called the *(analytic) quotient* of X by G. If Y is a quotient of X by G with projection ρ and if $U \subset X$ is G–stable, then, by definition, $\rho(U)$ is the quotient of U by G.

Since $\rho : X \to Y$ is assumed to be G–invariant, it maps G–orbits to points. Therefore, ρ factorizes as a *continuous* map over the *topological quotient space* $\pi : X \to X/G$ where X/G is the set of all G–orbits and $V \subset X/G$ is *open* by definition, if and only if $\pi^{-1}(V) = U$ is open in X such that π is an *open, closed* and *surjective* continuous map (see Chapter 7):

$$\begin{tikzcd} X \arrow[dr, "\pi"] \arrow[dd, "\rho"'] & \\ & X/G\,. \arrow[dl, "p"] \\ Y & \end{tikzcd}$$

Necessarily, p is continuous and open. Hence, p is a homeomorphism, if it is bijective, that is if the set of fibers $\rho^{-1}(y)$, $y \in Y$, coincides with the set of G–orbits in X. If this is the case, we write X/G also for the *analytic* quotient.

8.2 Groups acting properly discontinuously

Since we are only interested in the case, where p is bijective, we have to put strong conditions on the action of G in order to make sure that the topological quotient X/G is good enough, i. e., for instance, a Hausdorff space.

We assume in the following that G acts *properly discontinuously* on X, i.e., by definition, that for all *compact* sets $K \subset X$ the set

$$\{\,\gamma \in G : \gamma(K) \cap K \neq \emptyset\,\}$$

is *finite*. In particular, this implies that the *isotropy groups*

$$G_x := \{\,\gamma \in G : \gamma(x) = x\,\}$$

are finite for all $x \in X$. Finite groups G have always this property for trivial reasons.

Remarks. 1. Replacing the compact set K by the (compact) union of two compact sets K and L and noticing that $\gamma(K) \cap L$ is a subset of $\gamma(K \cup L) \cap (K \cup L)$, one is lead immediately to the following equivalent condition:

G acts properly discontinuously on X, if for all compact sets $K, L \subset X$ the set

$$\{\,\gamma \in G : \gamma(K) \cap L \neq \emptyset\,\}$$

is finite.

2. We leave it as an exercise to the reader to prove that under this assumption the topological quotient X/G is indeed a Hausdorff space.

A simple exercise shows that under this assumption there exist for all $x \in X$ (arbitrarily small) neighborhoods U_x of x such that

$$\begin{aligned} \gamma(U_x) &= U_x \ , \ \gamma \in G_x \ , \\ \gamma(U_x) \cap U_x &= \ \emptyset \ \ , \ \gamma \in G \setminus G_x \ . \end{aligned}$$

It is then sufficient to construct the analytic quotient of U_x by G_x for all $x \in X$ (which is obviously identical with the quotient of

$$\bigcup_{\gamma \in G} \gamma(U_x)$$

by G). In fact, these spaces may be glued together (in a uniquely determined manner) to a space which possesses the desired property, if (and only if) the underlying topological space is Hausdorff. But, as we shall see, the local analytic quotients are topologically isomorphic to U_x/G_x such that this patching method leads to the topological quotient X/G which in fact is a Hausdorff space.

8.3 The local construction of analytic quotients

According to the previous Section, we may assume that X is small with respect to $x \in X$ and that $G = G_x$ is finite. We then proceed along the lines already proposed in Chapter 7.5. Define topologically $Y = X/G$ and denote by $\rho : X \to Y$ the open projection. For V open in Y, the set $\rho^{-1}(V)$ is open and G–invariant in X such that we can form the invariant algebra

$$\mathcal{O}_Y(V) := (\mathcal{O}_X(\rho^{-1}(V)))^G ,$$

where G acts in the obvious manner on the algebra $\mathcal{O}_X(\rho^{-1}(V))$ of holomorphic functions. Hence, we have furnished the topological space Y with a *ringed* structure:

$$(Y, \mathcal{O}_Y) .$$

In order to see that $(Y, \mathcal{O}_Y)$ is indeed a (reduced) complex analytic space, we need in the next Sections the identity

$$\mathcal{O}_{Y,\rho(x)} = \mathcal{O}_{X,x}^G .$$

In fact, since ρ is finite and $\rho^{-1}(\rho(x)) = \{ x \}$, we can use Lemma 7.5 to get

$$\begin{aligned}\mathcal{O}_{Y,\rho(x)} &= \varinjlim_{V \ni \rho(x)} H^0(V, \mathcal{O}_Y) = \varinjlim_{V \ni \rho(x)} H^0(\rho^{-1}(V), \mathcal{O}_X)^G \\ &= [\varinjlim_{V \ni \rho(x)} H^0(\rho^{-1}(V), \mathcal{O}_X)]^G = \mathcal{O}_{X,x}^G .\end{aligned}$$

A substantial part of the remaining Sections of this Chapter will be devoted to derive the following more detailed statement.

Theorem 8.1 *If G acts properly discontinuously on the reduced complex analytic space X, then there exists the analytic quotient $Y = X/G$. The holomorphic quotient map $\rho : X \to Y$ is locally finite and surjective and (near any point x) isomorphic to the quotient map $X \to X/G_x$, where G_x denotes the finite stabilizer subgroup of G at x. In particular, the analytic algebra $\mathcal{O}_{Y,y}$ can be identified with the invariant algebra $\mathcal{O}_{X,x}^{G_x}$ for an arbitrary point $x \in \rho^{-1}(y)$, and $\widehat{\rho}_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is just the finite inclusion $\mathcal{O}_{X,x}^{G_x} \hookrightarrow \mathcal{O}_{X,x}$.*

Remark. That the (necessarily reduced) complex space Y solves the *universal problem* formulated at the beginning of the Chapter can be seen as follows: Let $\varphi : X \longrightarrow Z$ be a G–invariant holomorphic mapping, and let $x \in X,\, z = \varphi(x) \in Z$. Then, the corresponding local algebra homomorphism

$$\widehat{\varphi}_x : \mathcal{O}_{Z,z} \longrightarrow \mathcal{O}_{X,x}$$

is G_x–invariant which implies that the image of $\widehat{\varphi}_x$ lies in the invariant algebra $\mathcal{O}_{X,x}^{G_x} \cong \mathcal{O}_{Y,y}$. Therefore, the holomorphic map φ factorizes (locally) *uniquely* over the canonical projection $X \longrightarrow Y = X/G$. These local factorizations patch globally together to a holomorphic map from Y to Z.

Definition and *Remark*. We say that the group G acts *freely* on X, if G acts properly discontinuously and *fixpoint free*, i. e. if $\gamma(x) = x$ for some $x \in X$ and some $\gamma \in G$ implies $\gamma = \mathrm{id}$. The last assumption being the same as $G_x =$ trivial for all $x \in X$, we immediately deduce from the Theorem before:

Theorem 8.2 *If G acts freely on the reduced complex analytic space X, then the holomorphic quotient map $\rho : X \to Y = X/G$ is locally an isomorphism. In particular, X/G is a manifold if X is so.*

Example. Let Ω be the *lattice* in $\mathbb{C}^2$ generated by two $\mathbb{R}$–linearly independent elements ω_1, ω_2: $\Omega = \{ \omega = n_1\omega_1 + n_2\omega_2 ,\, n_1, n_2 \in \mathbb{Z} \}$. Ω acts *additively* on $\mathbb{C}^2$ by translations. Obviously, this is a free action such that the *complex torus* $\mathbb{C}^2/\Omega$ has a natural structure of a complex analytic manifold.

As we have already remarked in Chapter 7, an invariant algebra A^G of a *regular* algebra A by a (finite) automorphism group G may not be regular. (We give a necessary and sufficient condition on finite groups G later in this Chapter). However, if A is reduced or an integral domain, A^G is obviously reduced or an integral domain for arbitrary automorphism groups G. Since by Theorem 1 the inclusion $A^G \hookrightarrow A$ is a finite homomorphism for *finite* groups G, both algebras have the same dimension (see Theorem 3.33 and Theorem 6.14)[1]. Hence, under our standard assumptions, the quotient X/G has in $y = \rho(x)$ the same dimension as X in x. We finally note that also "normality" will be inherited from X.

Lemma 8.3 *If G is any automorphism group of a normal analytic algebra A, then the invariant algebra A^G is normal.*

Proof. A and A^G being integral domains, their resp. quotient fields exist, and $Q(A^G) \subset Q(A)$. Let $q = f/g \in Q(A^G)$ be algebraic over A^G, say

$$q^n + f_1 q^{n-1} + \cdots + f_n = 0, \quad f_1, \ldots, f_n \in A^G.$$

Then, q is also algebraic over A such that, by hypothesis, $q \in A$. Hence, for all $\gamma \in G$,

$$g\,q = f = \gamma(f) = \gamma(g\,q) = \gamma(g)\,\gamma(q) = g\,\gamma(q).$$

Since $g \neq 0$, we have $\gamma(q) = q$ for all $\gamma \in G$ and therefore $q \in A^G$. □

8.4 Invariant analytic subalgebras by finite groups

Before we really can embark into the proof of the central Theorem 1, we have to concentrate separately on its *punctial* aspect. In other words, we are going to prove the following Theorem.

Theorem 8.4 *Let A be an analytic algebra, and let G be a* finite *automorphism group of A. Then, the invariant algebra A^G is also analytic, and the canonical inclusion $A^G \hookrightarrow A$ is a* finite *homomorphism.*

The *proof* affords a bit of trivial *topology* on local rings (see Appendix B of this Chapter).

Lemma 8.5 *Let A be an analytic algebra, and let G be an* arbitrary *automorphism group of A. Then, the invariant algebra A^G is* closed *in A with respect to the $\mathfrak{m}_A$–adic topology of A.*

Proof. Since each $\gamma \in G$ is *continuous* in the given topology, it follows that each set

$$\{ f \in A : \gamma(f) = f \} = \{ f \in A : \gamma(f) - f = 0 \}$$

is closed in A and, consequently, also the intersection

$$A^G = \{ f \in A : \gamma(f) = f \ \text{for all}\ \gamma \in G \} = \bigcap_{\gamma \in G} \{ f \in A : \gamma(f) = f \}.$$ □

If G is a *finite* automorphism group, we can attach to each element $f \in A$ the polynomial

$$P_{G,f}(Y) := \prod_{\gamma \in G} (Y - \gamma(f)).$$

Obviously, $P_{G,f}$ is a *monic* polynomial in Y of the order of the group G with coefficients in the invariant algebra A^G satisfying $P_{G,f}(f) = 0$.

This immediately implies the following

[1] One can also show that the *profondeur* of A and A^G coincide. Therefore, A^G for finite G is a *Cohen–Macaulay ring* if and only if A itself has the Cohen–Macaulay property.

Corollary 8.6 *Let A be an analytic algebra, and let G be a* finite *automorphism group of A. Then, the algebra A is algebraic over the invariant algebra A^G.*

So, in order to accomplish the proof of Theorem 4, we need the following result.

Theorem 8.7 *Let A be an analytic algebra, $B \subset A$ a subalgebra such that the maximal ideal $\mathfrak{m}_A$ has a system $f_1, \ldots, f_n$ of generators that are* algebraic *over B, and let B be* closed *in A with respect to the $\mathfrak{m}_A$–adic topology. Then, B is an* analytic *subalgebra of A, and the natural monomorphism $B \hookrightarrow A$ is a finite homomorphism.*

Proof. Let $\psi : R_n = \mathbb{C}\langle x_1, \ldots, x_n \rangle \longrightarrow A$ be the analytic epimorphism given by $\psi(x_j) = f_j$, $j = 1, \ldots, n$, and denote by $\widetilde{g}_1, \ldots, \widetilde{g}_N$ the system of all coefficients of the normed polynomials $P_j \in B[y_j]$ satisfying $P_j(g_j) = 0$. Let C denote the subalgebra of B which is generated by these elements. By subtracting constant terms from the $\widetilde{g}_j$ as elements in A, we can replace these generators by elements $g_1, \ldots, g_N \in \mathfrak{m}_A$ which, together with $g_0 = 1$, generate the algebra C, too. Next, define the analytic homomorphism $\overline{\gamma} : R_N = \mathbb{C}\langle y_1, \ldots, y_N \rangle \longrightarrow A$ by $\overline{\gamma}(y_k) = g_k$, $k = 1, \ldots, N$. By definition, $\overline{\gamma}(S_N) = C \subset B$, $S_N = \mathbb{C}[y_1, \ldots, y_N]$. Since S_N is dense in R_N, the image $\overline{\gamma}(S_N)$ is also dense in $\overline{\gamma}(R_N)$ such that the analytic algebra $\overline{C} := \overline{\gamma}(R_N)$ is contained in B since B is closed by assumption. Due to our construction, the elements $f_1, \ldots, f_n$ are algebraic over C and consequently also over $\overline{C}$. Hence, according to Theorem 3.23, the inclusion $\overline{C} \hookrightarrow A$ is a finite analytic homomorphism. $\overline{C}$ being a Noetherian ring, A is a Noetherian module, too, such that the induced inclusion $\overline{C} \hookrightarrow B$ is finite ($B \hookrightarrow A$ is finite for trivial reasons). If $h_1, \ldots, h_m$ denote generators of B over $\overline{C}$, there exists an obvious epimorphism

$$R_N[\xi_1, \ldots, \xi_m] \longrightarrow B$$

whose natural extension to R_{N+m} maps onto B since B is closed. $\square$

8.5 Existence proof for analytic quotient spaces

Since we have seen in the previous Section that the invariant algebra of an analytic algebra by a *finite* group G is again analytic, there exists a complex analytic space $(Z, \mathcal{O}_Z)$ and a point $z \in Z$ such that

$$\mathcal{O}_{Z,z} \cong \mathcal{O}_{X,x}^{G_x} \cong \mathcal{O}_{Y,\rho(x)} .$$

Our existence proof can henceforth be completed by showing that locally at $y = \rho(x)$ and z the ringed spaces Y and Z are isomorphic.

So, we may assume from the beginning that $G = G_x$ is finite. Then, we have finite injective homomorphisms

$$\mathcal{O}_{Z,z} \cong \mathcal{O}_{X,x}^{G} \xrightarrow{\iota_x} \mathcal{O}_{X,x}$$

such that $\gamma \circ \iota_x = \iota_x$ for all $\gamma \in G_x$. Hence, there exists (locally at x) a *finite holomorphic map*

$$\pi : X \longrightarrow Z$$

with $\pi \circ \gamma = \pi$ for all $\gamma \in G$. Since Y is *topologically* the quotient of X by G, there exists a uniquely determined *continuous* map $p : Y \to Z$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ Y \arrow[rr, "p"'] & & Z \end{tikzcd}$$

commutes. It remains to show that p is an *isomorphism of ringed spaces*.

Since π and ρ are finite and the latter is surjective it follows by Lemma 6.23 iii) that p is at least a *finite* continuous map. We will see in Appendix C of this Chapter, that the mapping π is also surjective, at least locally at x, such that p is *surjective* near $y = \pi(x)$, too. Next, we prove that p is *injective*. For this to be correct, we need at least that invariant holomorphic functions separate points locally.

Lemma 8.8 *Let X be an analytic subspace of an open subset U in $\mathbb{C}^m$ and let $G \subset \mathrm{Aut}\, X$ be a finite subgroup. Then, to each pair $x^{(1)}, x^{(2)} \in X$ with $\rho(x^{(1)}) \neq \rho(x^{(2)})$, there exist holomorphic functions $F_1, \dots, F_n$ on U such that their restrictions $f_1, \dots, f_n$ to X are invariant under G: $f_1, \dots, f_n \in H^0(X, \mathcal{O}_X)^G$, and*

$$(f_1(x^{(1)}), \dots, f_n(x^{(1)})) \neq (f_1(x^{(2)}), \dots, f_n(x^{(2)})) .$$

Proof. Denote by $x_1, \dots, x_m$, the coordinate functions of $\mathbb{C}^m$ and by $\overline{x}_1, \dots, \overline{x}_m$ their restrictions to X. For $\nu = (\nu_1, \dots, \nu_m)$, we define

$$\overline{x}^\nu := \overline{x}_1^{\nu_1} \dots \overline{x}_m^{\nu_m} \in H^0(X, \mathcal{O}_X)$$

and form the average of $\overline{x}^\nu$ with respect to G:

$$f_\nu := \sum_{\gamma \in G} \overline{x}^\nu \circ \gamma , \quad 0 \leq |\nu| = \sum_{\mu=1}^{m} \nu_\mu < 2r ,$$

where $r = \mathrm{ord}\, G$. Obviously,

$$f_\nu \in H^0(X, \mathcal{O}_X)^G \ \text{ for all } \ \nu .$$

Suppose now that the points $x^{(1)}, x^{(2)} \in X$ have distinct images in X/G, i.e. $\rho(x^{(1)}) \neq \rho(x^{(2)})$. Then there exists a linear form

$$\ell(x_1, \dots, x_m) = \sum_{\mu=1}^{m} a_\mu x_\mu$$

such that

$$\ell(x^{(1)}) \neq \ell(\gamma(x^{(1)})) \ \text{for all } \gamma \in G \setminus G_{x^{(1)}}$$

and

$$\ell(x^{(1)}) \neq \ell(\gamma(x^{(2)})) \ \text{for all } \gamma \in G .$$

The polynomial

$$Q(x_1, \dots, x_m) := \prod_{\gamma \in G \setminus G_{x^{(1)}}} (\ell(x_1, \dots, x_m) - \ell(\gamma(x^{(1)}))) \cdot \prod_{\gamma \in G} (\ell(x_1, \dots, x_m) - \ell(\gamma(x^{(2)})))$$

satisfies

$$\begin{aligned} Q(x^{(1)}) &\neq 0 , \\ Q(\gamma(x^{(1)})) &= 0 , \quad \gamma \in G \setminus G_{x^{(1)}} , \\ Q(\gamma(x^{(2)})) &= 0 , \quad \gamma \in G . \end{aligned}$$

Denote again by $\overline{Q}$ the restrictions of Q to X, and define

$$f := \sum_{\gamma \in G} \overline{Q} \circ \gamma \in H^0(X, \mathcal{O}_X)^G .$$

Then

$$f(x^{(1)}) = \mathrm{ord}\, G_{x^{(1)}} \cdot Q(x^{(1)}) \neq 0 , \quad f(x^{(2)}) = 0 .$$

Since $\deg Q < 2r$, Q is of the form

$$Q(x_1, \dots, x_m) = \sum_{|\nu| < 2r} c_\nu x^\nu , \quad c_\nu \in \mathbb{C} ,$$

such that

$$f \;=\; \sum_{\gamma \in G} \overline{Q} \circ \gamma \;=\; \sum_{|\nu|<2r} c_\nu \sum_{\gamma \in G} (\overline{x}^\nu \circ \gamma) \;=\; \sum_{|\nu|<2r} c_\nu f_\nu \;.$$

It follows from $f\,(x^{(1)}) \neq f\,(x^{(2)})$ that there exists an index ν such that $f_\nu(x^{(1)}) \neq f_\nu(x^{(2)})$. □

Lemma 8.9 *The mapping p is injective.*

Proof. Let us assume to the contrary that there exists a pair $x^{(1)}$, $x^{(2)} \in X$ with $\rho\,(x^{(1)}) \neq \rho\,(x^{(2)})$ and $\pi\,(x^{(1)}) = \pi\,(x^{(2)})$. Now, due to Lemma 8, there are sections

$$f_1, \ldots, f_n \in H^0(X, \mathcal{O}_X)^G$$

with

$$(f_1(x^{(1)}), \ldots, f_n(x^{(1)})) \neq (f_1(x^{(2)}), \ldots, f_n(x^{(2)}))\;.$$

Since the germs $f_{\nu,x}$ are contained in $\mathcal{O}^G_{X,x} = \mathcal{O}_{Z,z}$, there exist (after shrinking of Z) sections $s_1, \ldots, s_n \in H^0(Z, \mathcal{O}_Z)$ with $s_\nu \circ \pi = f_\nu$ for all ν. But then, for the specific pair $(x^{(1)}, x^{(2)})$ as above, $f_\nu(x^{(1)}) = s_\nu(\pi\,(x^{(1)})) = s_\nu(\pi\,(x^{(2)})) = f_\nu(x^{(2)})$ for all ν, which contradicts our assumption. □

Putting the puzzle pieces together we see that p is even *bijective*. However, as a finite map, it is also closed such that the inverse p^{-1} is continuous. In other words: p is a *homeomorphism*.

It remains to prove that the canonical map

$$\widehat{p} : \, p_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Z$$

is an isomorphism of sheaves of local rings. To see this, we first remark that π is G–invariant, such that there is a natural action of G on the coherent $\mathcal{O}_Z$–module sheaf $\pi_*\mathcal{O}_X$. The invariant $\mathcal{O}_Z$–module $(\pi_*\mathcal{O}_X)^G$ is locally finitely generated: Choose, locally, sections $s_1, \ldots, s_t$ in $\pi_*\mathcal{O}_X$ which generate each fiber $(\pi_*\mathcal{O}_X)_{z'}$ over $\mathcal{O}_{Z,z'}$. Then, defining the *mean value* of s_τ as

$$\mu\,(s_\tau) \;:=\; \frac{1}{\mathrm{ord}\ G} \sum_{\gamma \in G} s_\tau \circ \gamma \;,$$

we produce sections of the sheaf $(\pi_*\mathcal{O}_X)^G$. If now $f \in (\pi_*\mathcal{O}_X)^G_{z'}$, then there exist elements $a_1, \ldots, a_t \in \mathcal{O}_{Z,z'}$, such that

$$f \;=\; \sum_{\tau=1}^{t} a_\tau s_{\tau,z'} \;.$$

Since f is G–invariant, we have for all $\gamma \in G$:

$$f \;=\; f \circ \gamma \;=\; \sum_{\tau=1}^{t} a_\tau \cdot (s_{\tau,z'} \circ \gamma) \;=\; \sum_{\tau=1}^{t} a_\tau \cdot (s_\tau \circ \gamma)_{z'} \;.$$

Summation over all $\gamma \in G$ yields the identity

$$f \;=\; \sum_{\tau=1}^{t} a_\tau \cdot (\mu\,(s_\tau))_{z'} \;.$$

Hence, $(\pi_*\mathcal{O}_X)^G$ is locally finitely generated, and therefore it is, as an $\mathcal{O}_Z$–subsheaf of the coherent sheaf $\pi_*\mathcal{O}_X$, a coherent $\mathcal{O}_Z$–module. But

$$(\pi_*\mathcal{O}_X)^{G_z} \;=\; \mathcal{O}^G_{X,x} \;=\; \mathcal{O}_{Z,z}\;,$$

such that (again after shrinking of Z):

$$(\pi_*\mathcal{O}_X)^G \;\cong\; \mathcal{O}_Z\;.$$

From the G–invariance of ρ and the definition of the structure sheaf $\mathcal{O}_Y$, we finally get the isomorphisms

$$p_*\mathcal{O}_Y \;\cong\; p_*(\rho_*\mathcal{O}_X)^G \;\cong\; (p_*\rho_*\mathcal{O}_X)^G \;\cong\; (\pi_*\mathcal{O}_X)^G \;\cong\; \mathcal{O}_Z\;.$$ □

8.6 Quotient singularities

In general, X/G is not smooth at $\rho(x)$, if x is a regular point of X. Such singularities (including the regular one) are called *quotient singularities*. By the foregoing, they are completely determined by the normal invariant algebra

$$A = R_n^G ,$$

where $R_n = \mathbb{C}\langle x_1, \ldots, x_n \rangle$ denotes the convergent power series ring, and G is a finite subgroup of $\mathrm{Aut}\, R_n$ acting by substitution:

$$R_n \times \mathrm{Aut}\, R_n \ni (f, \gamma) \longmapsto f \circ \gamma^{-1} \in R_n .$$

We also write f^γ or $\gamma(f)$ instead of $f \circ \gamma^{-1}$.

Our first goal is to show that the action of G can be *linearized* by introducing new holomorphic coordinates on $\mathbb{C}^n$ near the origin. Each element $\gamma \in G$ is automatically local such that it induces a map

$$\gamma' : \mathfrak{m}_n/\mathfrak{m}_n^2 \longrightarrow \mathfrak{m}_n/\mathfrak{m}_n^2 ,$$

where $\mathfrak{m}_n$ = maximal ideal of R_n. We can think of γ' as being an automorphism of R_n which is linear in the sense that it is a substitution by the degree one part of γ^{-1}. We represent γ' by an $n \times n$–matrix, called also γ', satisfying

$$f^{\gamma'}(x) = \gamma'(f)(x_1, \ldots, x_n) = f(\gamma'^{-1}(x_1, \ldots, x_n)) = f((x_1, \ldots, x_n)\gamma') .$$

Then we have

$$\begin{aligned} (\gamma_1 * \gamma_2)' &= (\gamma_2 \circ \gamma_1)' = \gamma_2' \circ \gamma_1' = \gamma_1' \cdot \gamma_2' && \text{(matrix multiplication)} ,\\ \mathrm{id}' &= E_n && (\text{unit } n \times n-\text{matrix}) , \end{aligned}$$

so that $'$ defines a group representation

$$G \longrightarrow \mathrm{GL}(n, \mathbb{C}) .$$

Since $h \equiv h' \bmod \mathfrak{m}_n^2$ for all group elements $h \in G$, it is clear that for the n–tuple $y = (y_1, \ldots, y_n)$ with

$$y_j = \frac{1}{b} \sum_{h \in G} (h' \circ h^{-1})(x_j) = \frac{1}{b} \sum_{h \in G} h^{-1}(h'(x_j)) , \quad b = \mathrm{ord}\, G ,$$

the Jacobi determinant

$$\det \left. \frac{\partial y}{\partial x} \right|_{x=0} = 1 .$$

Therefore, $(y_1, \ldots, y_n)$ is a new holomorphic coordinate system near 0, and for each $\gamma \in G$ we have with

$$\gamma'(x_j) = \sum_{k=1}^n a_{kj} x_k$$

the identity

$$\begin{aligned} \gamma(y_j) &= \frac{1}{b} \sum_{h \in G} \gamma((h' \circ h^{-1})(x_j)) = \frac{1}{b} \sum_{h \in G} (h^{-1} \circ \gamma)((\gamma^{-1} \circ h)'\gamma'(x_j)) \\ &= \sum_{k=1}^n a_{kj} \Big(\frac{1}{b} \sum_{h \in G} (\gamma^{-1} \circ h)^{-1} (\gamma^{-1} \circ h)'(x_k) \Big) = \sum_{k=1}^n a_{kj} y_k , \end{aligned}$$

i.e.: G acts *linearly* in the coordinate system $(y_1, \ldots, y_n)$. - We thus have proven:

Theorem 8.10 *Each quotient singularity is isomorphic to a quotient $\mathbb{C}^n/G$, where G is a finite subgroup of* $\mathrm{GL}(n, \mathbb{C})$.

This result can be applied to show that quotient singularities are *algebraic* in all dimensions though they need not be isolated for $n \geq 3$. In fact, assuming without loss of generality that G acts linearly, it acts canonically on the polynomial ring $S_n = \mathbb{C}[x_1, \ldots, x_n]$, and by *Hilbert's Basis Theorem*, the invariant ring S_n^G is a finitely generated algebra: there exist finitely many polynomials $P_1, \ldots, P_e \in S_n^G$ such that the image of the substitution homomorphism

$$\begin{cases} S_e = \mathbb{C}[y_1 \ldots, y_e] \longrightarrow S_n \\ \qquad\qquad y_\varepsilon \qquad\qquad \longmapsto P_\varepsilon \end{cases}$$

equals S_n^G. The proof of Theorem 7 shows that the polynomials $P_1, \ldots, P_e$ generate the analytic algebra R_n^G *analytically*, i.e. that the substitution homomorphism

$$\begin{cases} R_e = \mathbb{C}\langle y_1, \ldots, y_e \rangle \longrightarrow R_n \\ \qquad\qquad y_\varepsilon \qquad\qquad \longmapsto P_\varepsilon \end{cases}$$

is an epimorphism onto R_n^G. Moreover, if $\mathfrak{a}$ denotes the kernel of $S_e \to S_n$, then

$$\ker(R_e \to R_n) = \mathfrak{a}R_e ,$$

such that $R_n^G = R_e / \mathfrak{a}R_e$, and $\mathbb{C}^n / G$ may be realized by the vanishing of any finite set of polynomials generating $\mathfrak{a}$, i.e. it may be realized as an *affine algebraic variety*. Finally, if $P_1, \ldots, P_e$ is a *minimal* set of generating polynomials, then e is the embedding dimension of the algebra R_n^G.

8.7 Hilbert's Syzygy Theorem

In Chapter 7 we found some examples of cyclic quotients which were nonsingular. In these cases, the groups were generated by reflections. In the next Section, we want to make a further step towards the classification of quotient singularities by proving that this is a general property of reflection groups (which, moreover, characterizes them as we will see in Section 9).

The result just mentioned can easily be deduced from an algebraic statement about the invariant algebra S^G which is due to Chevalley (see Theorem 17). For the sake of completeness, we give here another proof that uses the basic characterization of *regular* local noetherian rings by Serre as those rings which satisfy *Hilbert's Syzygy Theorem*. Let us briefly gather the material we need in the following.

For a finitely generated module M over a local noetherian ring R, a j–th *syzygy module* for M is by definition equal to $\operatorname{im} \varphi_j$ for an exact sequence of type

$$(*) \qquad F_j \xrightarrow{\varphi_j} F_{j-1} \xrightarrow{\varphi_{j-1}} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

with finitely generated *free* R–modules F_k. In particular, M is the only 0–th syzygy module of itself. A syzygy module is called *minimal*, if the defining sequence $(*)$ is minimal in the sense that none of the free modules F_k can be replaced by one with fewer generators, i.e. if the rank of F_k coincides with the minimal number of generators of $\operatorname{im} \varphi_k$ for all k, which is equivalent to the condition $\ker \varphi_k \subset \mathfrak{m}F_k$ for all k, $\mathfrak{m}$ the maximal ideal of R. It is not difficult to see that minimal syzygy modules exist for all j and are uniquely determined (up to isomorphisms). Moreover, each j–th syzygy module decomposes into a direct sum of the minimal one - which we call $\operatorname{syz}^j M$ - and a free module. In particular, if $\operatorname{syz}^j M$ is free itself, we can modify $(*)$ to a finite exact sequence

$$(**) \qquad 0 \longrightarrow \operatorname{im} \varphi_j \longrightarrow F_{j-1} \xrightarrow{\varphi_{j-1}} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

which we call for obvious reasons a *finite free resolution* of M (of length j). Under the same assumption, there also exist *minimal* free resolutions for M having all the same length

$$\operatorname{syl}_R M := \min\{ k : \operatorname{syz}^{k+1} M = 0 \} .$$

We call $\operatorname{syl}_R M$ the *syzygy length* or the *homological dimension* of M.

The *Syzygy Theorem* can now be stated as follows (Hilbert's classical theorem is the analogous result for the nonlocal ring S_n) - a proof can be found in Chapter 11.5:

Theorem 8.11 *If R is a regular local noetherian ring of dimension n, then $\mathrm{syl}_R M \le n$ for any finitely generated R–module M.*

For us, it is at the moment more important to have the converse to Theorem 11 which is due to Serre.

***Theorem 8.12** *If, for a local noetherian ring R, the inequality $\mathrm{syl}_R M \le \dim R$ holds for all finitely generated R–modules M, then R is regular.*

The way, we want to use this characterization, is the following: We start with a subring S of a regular ring R and would like to relate free resolutions of modules over S to such resolutions over R, which in general is impossible since tensoring is only right–exact. Therefore, we assume that R is a flat S–module. (Recall that an S–module N is flat if the functor $\cdot \otimes_S N$ is exact). Then each exact sequence

$$F_j \xrightarrow{\varphi_j} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

with finitely generated free S–modules F_k gives rise to an exact sequence

$$F_j \otimes_S R \xrightarrow{\varphi_j \otimes \mathrm{id}} \cdots \xrightarrow{\varphi_1 \otimes \mathrm{id}} F_0 \otimes_S R \xrightarrow{\varphi_0 \otimes \mathrm{id}} M \otimes_S R \longrightarrow 0$$

with finitely generated free S–modules $G_k = F_k \otimes_S R$. If, moreover, S is local and R is finitely generated over S, then R is even *faithfully flat* over S; that is, if

$$M_1 \longrightarrow M_2 \longrightarrow M_3$$

is a sequence of S–modules such that

$$M_1 \otimes_S R \longrightarrow M_2 \otimes_S R \longrightarrow M_3 \otimes_S R$$

is exact, then the original sequence was already exact. From this property it is easily deduced that, for a finitely generated S–module M, the j–th syzygy module is free if $\mathrm{syz}^j_R(M \otimes_S R)$ is a free R–module. Hence,

$$\mathrm{syl}_S M \le \mathrm{syl}_R(M \otimes_S R) \le \dim R = \dim S,$$

and S must be regular by Theorem 12.

In the next Section, we apply this fact to $S = R_n^G$, $R = R_n$, G a finite group generated by reflections.

8.8 Quotients by reflection groups

An element $\gamma \in \mathrm{Aut}\, M$, M a connected complex manifold, is called a *reflection* (or, perhaps more precisely, a *pseudoreflection*) if it is of finite order and if the (analytic) fixpoint set

$$\mathrm{Fix}\,(\gamma) = \{\, x \in M : \gamma\,(x) = x \,\}$$

is of pure codimension 1 in M. A finite group $G \subset \mathrm{Aut}\, M$ is called a *reflection group* if it is generated by the reflections contained in G. Of course, an element γ of finite order in $\mathrm{GL}\,(n, \mathbb{C})$ is a reflection if and only if it leaves a hyperplane in $\mathbb{C}^n$ pointwise fixed; this is equivalent to g having the eigenvalues 1 (of multiplicity $n-1$) and ζ_k, $k \ge 2$, a k–th root of unity.

We will prove that quotients M/G, G a reflection group, are manifolds. This, of course, is equivalent to the following claim.

Theorem 8.13 *The invariant algebra R_n^G is isomorphic to the convergent power series ring R_n if $G \subset \mathrm{GL}\,(n, \mathbb{C})$ is a (finite) reflection group.*

In the previous Section, we have already reduced the problem to the following:

Lemma 8.14 *For a (finite) reflection group* $G \subset \mathrm{GL}(n, \mathbb{C})$, R_n *is a flat* R_n^G*–module.*

Proof. In order to get rid of convergence questions, we remark that G acts not only on the polynomial ring S_n , but also on the $\mathfrak{m}$–adic completion $\widehat{R}_n$ of R_n , i.e. on the ring

$$\widehat{R}_n = \mathbb{C}\{\{ x_1, \ldots, x_n \}\}$$

of formal power series. Since G acts linearly, the invariant ring $\widehat{R}_n^G$ is the completion of R_n^G with respect to the maximal ideal $\mathfrak{m}(R_n^G) = \mathfrak{m} \cap R_n^G$ and also of the localization of S_n^G with respect to the maximal ideal $\mathfrak{m}(R_n^G) \cap S_n = \mathfrak{m} \cap S_n^G$. Since flatness is preserved under localization and completion, we may also prove

Lemma 8.15 *For a finite reflection group* $G \subset \mathrm{GL}(n, \mathbb{C})$ *, the polynomial ring* S_n *is a flat* S_n^G*–module.*

Using Bourbaki's *criterion for flatness*, we are finally reduced to show under the above condition:

Lemma 8.16 *If* $\sum_{\lambda=1}^{\ell} p_\lambda q_\lambda = 0$ *for elements* $p_\lambda \in S_n^G$, $q_\lambda \in S_n$ *, then there exist elements* $p_{\mu\lambda} \in S_n^G$, $\widetilde{q}_\mu \in S_n$, $\mu = 1, \ldots, m$, $\lambda = 1, \ldots, \ell$ *, such that*

$$\sum_{\lambda=1}^{\ell} p_\lambda p_{\mu\lambda} = 0 \,, \quad \mu = 1, \ldots, m \,, \quad q_\lambda = \sum_{\mu=1}^{m} \widetilde{q}_\mu p_{\mu\lambda} \,, \quad \lambda = 1, \ldots, \ell \,.$$

Remark. Bourbaki's *criterion for flatness* is a necessary and sufficient condition for certain rings A to be *flat* over a subring $B \subset A$ in terms of *linear relations* of elements in A as a module over B: If $\sum_{\lambda=1}^{\ell} b_\lambda a_\lambda = 0$ for elements $a_\lambda \in A$, $b_\lambda \in B$, then after a certain transformation

$$a_\lambda = \sum_{\mu=1}^{m} \widetilde{a}_\mu b_{\mu\lambda} \,, \quad \widetilde{a}_\mu \in A \,,\ b_{\mu\lambda} \in B \,,$$

the original relation

$$\sum_{\mu=1}^{m} \Big(\sum_{\lambda=1}^{\ell} b_\lambda \, b_{\mu\lambda} \Big) \widetilde{a}_\mu = \sum_{\lambda=1}^{\ell} b_\lambda \Big(\sum_{\mu=1}^{m} b_{\mu\lambda} \widetilde{a}_\mu \Big) = \sum_{\lambda=1}^{\ell} b_\lambda a_\lambda = 0$$

is already determined by the relations

$$\sum_{\lambda=1}^{\ell} b_\lambda \, b_{\mu\lambda} = 0$$

in B . - Or in other words: *Each solution in* A *of a homogeneous linear equation with coefficients in* B *is an* A*–linear combination of solutions in* B .

The *proof* of Lemma 16 is carried out by induction on

$$s := \max_{\lambda=1,\ldots,\ell} (\deg q_\lambda) \,.$$

If $s = 0$, then all $q_\lambda \in \mathbb{C} \subset S_n^G$ such that we may put $m := 1$, $p_{1\lambda} := q_\lambda$, $\widetilde{q}_1 := 1$. Suppose now

$$\sum_{\lambda=1}^{\ell} p_\lambda q_\lambda = 0 \,, \tag{1}$$

$p_\lambda \in S_n^G$, $q_\lambda \in S_n$, $\max\limits_{\lambda=1,\ldots,\ell} (\deg q_\lambda) = t \geq 1$, and assume that the claim of the Lemma is proven for all $s < t$. We may further assume that ($m = m_s$ depending on s):

$$1 = m_0 \leq m_1 \leq \cdots \leq m_{t-1} .$$

For any reflection $\gamma \in G$, its fixpoint set $\mathrm{Fix}(\gamma)$ can be described by a linear form $L_\gamma \neq 0$: $\mathrm{Fix}(\gamma) = \{ x \in \mathbb{C}^n : L_\gamma(x) = 0 \}$. Since, for a polynomial p, the function $\gamma(p) - p$ vanishes on $\mathrm{Fix}(\gamma)$, the polynomial $\gamma(p) - p$ must be divisible by L_γ: for all $p \in S_n$ there exists a polynomial $r_\gamma(p)$ with $\deg r_\gamma(p) < \deg p$, such that

$$\gamma(p) - p = r_\gamma(p) \cdot L_\gamma .$$

Applying γ to (1) and subtraction of (1) yields

$$L_\gamma \cdot \sum_{\lambda=1}^{\ell} p_\lambda r_\gamma(q_\lambda) = \sum_{\lambda=1}^{\ell} p_\lambda(\gamma(q_\lambda) - q_\lambda) = 0$$

and hence

$$\sum_{\lambda=1}^{\ell} p_\lambda r_\gamma(q_\lambda) = 0 , \quad \max_{\lambda=1,\ldots,\ell} (\deg r_\gamma(q_\lambda)) \leq t - 1 .$$

By induction hypothesis, there exist elements

$$\left.\begin{array}{r} p_{\mu\lambda}^{(\gamma)} \in S_n^G \\ \widehat{q}_\mu^{(\gamma)} \in S_n \end{array}\right\} \quad \mu = 1, \ldots, m_{t-1} , \ \lambda = 1, \ldots, \ell ,$$

such that

$$\sum_{\lambda=1}^{\ell} p_{\mu\lambda}^{(\gamma)} p_\lambda = 0 , \qquad \mu = 1, \ldots, m_{t-1} , \tag{2}$$

$$r_\gamma(q_\lambda) = \sum_{\mu=1}^{m_{t-1}} \widehat{q}_\mu^{(\gamma)} p_{\mu\lambda}^{(\gamma)} , \quad \lambda = 1, \ldots, \ell . \tag{3}$$

Equation (3) implies with $\widetilde{q}_\mu^{(\gamma)} := L_\gamma \cdot \widehat{q}_\mu^{(\gamma)}$:

$$\gamma(q_\lambda) = q_\lambda + \sum_{\mu=1}^{m_{t-1}} \widetilde{q}_\mu^{(\gamma)} p_{\mu\lambda}^{(\gamma)} , \quad \lambda = 1, \ldots, \ell .$$

G is generated by finitely many reflections, say $\gamma_1, \ldots, \gamma_r$. Then, putting

$$p_{\mu\lambda}^{(\rho)} := p_{\mu\lambda}^{(\gamma_\rho)} , \quad \rho = 1, \ldots, r ,$$

it is clear that for all $\gamma \in G$ there exist elements $\widetilde{q}_{\mu\lambda}^{(\gamma)} \in S_n$ such that

$$g(q_\lambda) = q_\lambda + \sum_{\rho=1}^{r} \sum_{\mu=1}^{m_{t-1}} \widetilde{q}_{\mu\rho}^{(\gamma)} p_{\mu\lambda}^{(\rho)} , \quad \lambda = 1, \ldots, \ell . \tag{4}$$

Denote again the mean value of a polynomial p by $\mu(p)$. If we define

$$\widetilde{q}_{\mu\rho} := -\frac{1}{\mathrm{ord}\, G} \sum_{\gamma \in G} \widetilde{q}_{\mu\rho}^{(\gamma)} ,$$

equation (4) implies

(5) $$\mu(q_\lambda) = q_\lambda - \sum_{\rho,\mu} \widetilde{q}_{\mu\rho} p^{(\rho)}_{\mu\lambda} .$$

We now put $m_t = r m_{t-1} + 1$; using any bijection $\{1, \ldots, r\} \times \{1, \ldots, m_{t-1}\} \xrightarrow{\sim} \{1, \ldots, r \cdot m_{t-1}\}$, we define

$$\begin{aligned} p_{\sigma\lambda} &= p^{(\rho)}_{\mu\lambda} , \qquad \sigma = 1, \ldots, m_t - 1 , \ \lambda = 1, \ldots, \ell , \\ p_{m_t\lambda} &= \mu(q_\lambda) , \quad \lambda = 1, \ldots, \ell , \\ \widetilde{q}_\sigma &= \widetilde{q}_{\mu\rho} , \qquad \sigma = 1, \ldots, m_t - 1 , \\ \widetilde{q}_{m_t} &= 1 , \end{aligned}$$

such that $p_{\sigma\lambda} \in S_n^G$, $\sigma = 1, \ldots, m_t$, $\lambda = 1, \ldots, \ell$, and

$$\sum_{\lambda=1}^{\ell} p_{\sigma\lambda} p_\lambda = \sum_{\lambda=1}^{\ell} p^{(\rho)}_{\mu\lambda} p_\lambda = 0 , \qquad \sigma = 1, \ldots, m_t - 1 ,$$

$$\sum_{\lambda=1}^{\ell} p_{m_t\lambda} p_\lambda = \sum_{\lambda=1}^{\ell} \mu(q_\lambda) p_\lambda = \mu\Big(\sum_{\lambda=1}^{\ell} p_\lambda q_\lambda\Big) = \mu(0) = 0 , \qquad \sigma = m_t .$$

Finally,

$$q_\lambda = \sum_{\rho,\mu} \widetilde{q}_{\mu\rho} p^{(\rho)}_{\mu\lambda} + 1 \cdot \mu(q_\lambda) = \sum_{\sigma=1}^{m_t} \widetilde{q}_\sigma p_{\sigma\lambda} , \quad \lambda = 1, \ldots, \ell . \qquad \square$$

Applying the previous results to the invariant algebra S_n^G, one can easily derive the before mentioned theorem of Chevalley:

Theorem 8.17 *If $G \subset \mathrm{GL}(n, \mathbb{C})$ is a finite reflection group, then S_n^G is generated by n algebraically independent homogeneous polynomials.*

The degrees d_j of the generating polynomials are uniquely determined by the reflection group G up to order. They are sometimes called the *degrees* of G and linked to other invariants of G by

***Theorem 8.18** *Let $d_1, \ldots, d_n$ denote the degrees of a finite reflection group $G \subset \mathrm{GL}(n, \mathbb{C})$. Then*

(i) $\prod_{j=1}^{n} d_j$ *is the order of G,*

(ii) $\sum_{j=1}^{n} (d_j - 1)$ *equals the number of reflections contained in G.*

8.9 Classification of quotient singularities by conjugacy classes of small groups

Theorem 13 enables us to restrict our attention to the considerably narrower class of *small* subgroups $G \subset \mathrm{GL}(n, \mathbb{C})$ which, by definition, do not contain any pseudoreflections. The classification of n–dimensional quotient singularities is now provided by the following two theorems.

Theorem 8.19 *Every quotient singularity is isomorphic to a quotient by a small group $G \subset \mathrm{GL}(n, \mathbb{C})$.*

It is trivial that the quotients $\mathbb{C}^n / G_1$ and $\mathbb{C}^n / G_2$ are biholomorphically equivalent, if G_1 is conjugate to G_2 in $\mathrm{GL}(n, \mathbb{C})$. Theorem 20 gives the converse for small groups.

Theorem 8.20 *If G_1 and G_2 are small subgroups of $\mathrm{GL}(n, \mathbb{C})$ such that $\mathbb{C}^n/G_1 \cong \mathbb{C}^n/G_2$, then G_1 and G_2 are conjugate in $\mathrm{GL}(n, \mathbb{C})$.*

Proof of Theorem 19. Let X be equal to $\mathbb{C}^n/G$, $G \subset \mathrm{GL}(n, \mathbb{C})$, and let $H \subset G$ be the subgroup of G generated by the reflections contained in G. For a reflection h and a general element γ in G one checks immediately that

$$\mathrm{Fix}(\gamma h \gamma^{-1}) = \gamma(\mathrm{Fix}(h)) .$$

This implies that H is in fact a normal subgroup of G, and therefore, that the quotient group $\overline{G} = G/H$ operates on $\mathbb{C}^n/H \cong \mathbb{C}^n$. But $\mathbb{C}^n/\overline{G} \cong (\mathbb{C}^n/H)/(G/H) \cong \mathbb{C}^n/G$; hence, it suffices to show that $\overline{G}$ - which operates linearly in a suitable coordinate system - is a small group.

Thus, take an element $\overline{\gamma} = \gamma H \in \overline{G}$, $\gamma \notin H$. Then, for all $h \in H$, we have $\gamma h \notin H$ such that the codimension of the fixpoint set $\mathrm{Fix}(\gamma h)$ is ≥ 2. If we denote by τ the finite, open holomorphic map $\mathbb{C}^n \to \mathbb{C}^n/H$, the fixpoint set of $\overline{\gamma}$ equals

$$\bigcup_{h \in H} \tau(\mathrm{Fix}(\gamma h)) .$$

Since H is finite and τ preserves dimension, the codimension of the fixpoint set of $\overline{\gamma}$ is at least 2. □

Proof of Theorem 20. Denote the quotient maps $\mathbb{C}^n \to \mathbb{C}^n/G_i$ by ρ_i, $i = 1, 2$. We want to construct a holomorphic map $\psi : \mathbb{C}^n \to \mathbb{C}^n$ making the diagram

$$\begin{tikzcd} C^n \arrow[r, dashed, "\psi"] \arrow[d, "\rho_1"'] & \mathbb{C}^n \arrow[d, "\rho_2"] \\ \mathbb{C}^n/G_1 \arrow[r, "\sim", "\varphi"'] & \mathbb{C}^n/G_2 \end{tikzcd}$$

commutative. In fact, we will replace all spaces by their germs at the distinguished points such that we have a local diagram

$$\begin{tikzcd} \rho_1^{-1}(U_1) = V_1 \arrow[d, "\rho_1"'] & V_2 = \rho_2^{-1}(U_2) \arrow[d, "\rho_2"] \\ U_1 \arrow[r, "\sim", "\varphi"'] & U_2 \end{tikzcd}$$

where, without loss of generality, V_1 may be assumed connected and simply connected.

We denote by H_i the union of all fixpoint sets $\mathrm{Fix}(\gamma)$, $\gamma \in G_i \setminus \{\mathrm{id}\}$, $i = 1, 2$ which, by assumption, is an analytic subset of V_i of codimension ≥ 2. Now, φ is biholomorphic and ρ_1 and ρ_2 are open and finite; thus, the set

$$\widetilde{H}_1 = H_1 \cup \rho_1^{-1}(\varphi^{-1}(\rho_2(H_2)))$$

is of codimension at least 2 such that

$$V_1^- := V_1 \setminus \widetilde{H}_1$$

is still connected and simply connected. Since V_1^- is locally biholomorphic to $U_1 \setminus (\rho_1(H_1) \cup \varphi^{-1}(\rho_2(H_2)))$ and $V_2^- = V_2 \setminus H_2$ is locally biholomorphic to $U_2 \setminus \rho_2(H_2)$, we can locally construct maps

$$\psi : V_1^- \longrightarrow V_2^- \subset \mathbb{C}^n$$

with the desired property (by $\psi = \rho_2^{-1} \circ \varphi \circ \rho_1$). But, by the topological properties of V_1^-, these local maps - if suitably chosen - patch together to a globally defined locally biholomorphic map $\psi : V_1^- \to V_2^- \subset \mathbb{C}^n$, which by Riemann's second Extension Theorem may be extended to a holomorphic map

$$\psi : V_1 \longrightarrow V_2$$

satisfying $\rho_2 \circ \psi = \varphi \circ \rho_1$. Since $\psi | V_1 \setminus \widetilde{H}_1$ is locally biholomorphic, the Jacobi determinant $J_\psi(x)$ does not vanish for $x \in V_1 \setminus \widetilde{H}_1$. But the zero set of the holomorphic function J_ψ being purely 1–codimensional or empty, we may conclude that $J_\psi(x) \neq 0$ for all $x \in V$, especially for $x = 0$. Hence, ψ is an isomorphism (after shrinking U_1 and U_2, if necessary).

Now, for $\gamma \in G_1$, we have

$$\rho_2 \circ (\psi \circ \gamma \circ \psi^{-1}) = \varphi \circ \rho_1 \circ \gamma \circ \psi^{-1} = \varphi \circ \rho_1 \circ \psi^{-1} = \rho_2$$

such that $\psi \circ \gamma \circ \psi^{-1} \in G_2$. This implies that, by sending γ to $\psi \circ \gamma \circ \psi^{-1}$, we obtain an isomorphism $G_1 \to G_2$. If we finally take into consideration that γ and $\psi \circ \gamma \circ \psi^{-1}$ are linear automorphisms, we get the same isomorphism by the assignment

$$\gamma \longmapsto \psi' \circ \gamma \circ \psi'^{-1} ,$$

where ψ' denotes the linear part of ψ at 0. $\square$

As a Corollary, Theorem 20 yields the converse to Theorem 13:

Theorem 8.21 *If $R_n^G \cong R_n$ for a finite group G, then G is a reflection group.*

Proof. Theorem 20 amounts to saying that, for a small group $\overline{G}$, the isomorphism $R_n^{\overline{G}} \cong R_n$ implies $\overline{G} = \{ \mathrm{id} \}$. But, taking the reflection group $H \subset G$ as in the proof of Theorem 19, we have for the small group $\overline{G} = G/H$:

$$R_n \cong R_n^G \cong (R_n^H)^{\overline{G}} = R_n^{\overline{G}} . \qquad \square$$

8.10 The local fundamental group of a normal singularity

The rest of the present Chapter will be devoted to Prill's characterization of quotient *surface* singularities by the *finiteness* of their (local) *fundamental group*. The two main ingredients of this result are Mumford's smoothness criterion to be proven in Chapter 15 and an existence theorem for complex analytic structures on certain branched coverings due to Grauert and Remmert whose proof shall be outlined in the next Section.

Following Prill we call a neighborhood U of a point x in a topological space X *good with respect to the subspace* $Y \subset X$ if there exists a neighborhood basis $\{ U_\alpha \}$ of x such that $U_\alpha \setminus Y$ is a *deformation retract* of $U \setminus Y$ for all α. ($W \subset Z$ is a deformation retract of Z if there exists a continuous map $\Phi : Z \times [0, 1] \to Z$ satisfying $\Phi(z, 0) = z$ for all $z \in Z$, $\Phi(w, t) = w$ for all $w \in W$, $t \in [0, 1]$ and $\{ \Phi(z, 1) : z \in Z \} = W$).

It is easily shown that for two good neighborhoods U, V with associated families $\{ U_\alpha \}$, $\{ V_\beta \}$ as in the definition, all U_α and V_β are good neighborhoods of x with respect to Y and $U \setminus Y$ and $V \setminus Y$ have the same *homotopy type*. In particular, the homotopy type of $U \setminus Y$ is an invariant of the triple (X, Y, x).

Thus, if any good neighborhood U with *connected* complement $U \setminus Y$ exists, the system of all good neighborhoods with this property is cofinal, and we can define

$$\pi_1(X, Y, x) := \varprojlim_{\substack{U \text{good} \\ U \ni x}} \pi_1(U \setminus Y) ,$$

the *local fundamental group* of X at x with respect to Y. Of course, $\pi_1(X, Y, x) \cong \pi_1(U \setminus Y)$ for any good neighborhood U such that $U \setminus Y$ is connected.

If V is a *simplicial complex* with underlying space $X = |V|$, and if W is a subcomplex of V with $|W| = Y$ and x a vertex of W, then the *open star* of x is a good neighborhood with respect to Y.

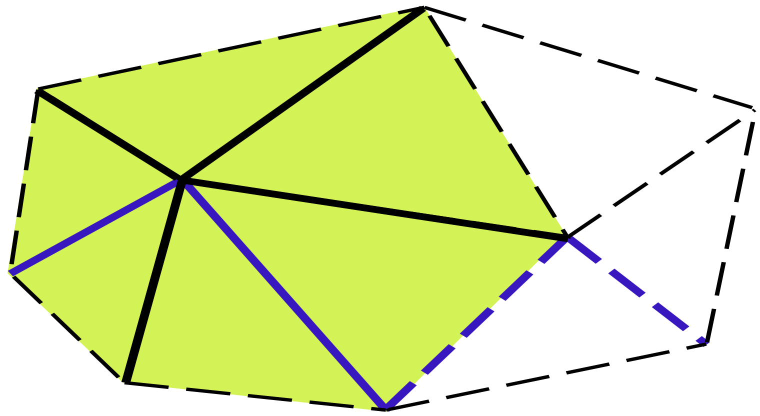


Figure 8.1

By a theorem due to Lojasiewicz and Giesecke, every real analytic space (with countable topology) may be triangulated, having any prescribed locally finite collection of real analytic subvarieties as support of a subcomplex. Moreover, by *Zariski's Connectedness Theorem*, if X is a normal complex space, $U \subset X$ is connected and $Y \subset X$ is of codimension at least 2, the complement $U \setminus Y$ is still connected. Consequently, we may formulate:

Theorem 8.22 *For any normal complex analytic space X and any complex analytic subspace Y of codimension ≥ 2, the local fundamental group $\pi_1(X, Y, x)$ exists.*

In the special case, where X is normal complex analytic and Y is the singular variety $\operatorname{sing} X$, we define

$$\pi_1(X, x) := \pi_1(X, \operatorname{sing} X, x)$$

to be the *local fundamental group* of X at x.

If x is an isolated normal singularity of X, in particular, when X is two–dimensional, the proof of the existence of $\pi_1(X, x)$ is much simpler, since in this case X is by *Milnor's Theorem* (see Chapter 15) topologically a cone with vertex x.

Of course, $\pi_1(X, x)$ measures to some extent, whether x is a smooth point of X, since obviously x smooth implies that π_1 is trivial. The converse is true for normal surfaces by *Mumford's criterion* which will also be proved in Chapter 15.

A necessary condition for an isolated normal singularity to be a quotient is given by the finiteness of the local fundamental group $\pi_1(X, x)$ because of the following

Lemma 8.23 *Let $f : (Y, y) \to (X, x)$ be a finite branched covering of isolated normal singularities. Then*

$$\operatorname{ord} \pi_1(X, x) \leq \operatorname{ord}_y f \cdot \operatorname{ord} \pi_1(Y, y)\,.$$

Here, $\operatorname{ord}_y f$ denotes the number of sheets of the covering f near y.

Proof. Since f is finite, we may assume that $f^{-1}(x) = \{ y \}$ and that we have good neighborhoods U_1 and U_2 of x and V_1 of y, resp. such that

$$f(V_1) \subset U_1\,,\ U_2 \subset U_1\,,\ f^{-1}(U_2) \subset V_1\,.$$

We put $V_2 := f^{-1}(U_2)$, $U_i^- = U_i \setminus \{ x \}$, $V_i^- = V_i \setminus \{ y \}$ and denote by $\widetilde{U}_1 \to U_1^-$ and $\widetilde{V}_1 \to V_1^-$ the corresponding universal coverings (which again are locally biholomorphic). The fiber products

$$\widetilde{V}_2 := V_2^- \times_{V_1^-} \widetilde{V}_1 = \widetilde{V}_{1|V_2^-}$$

$$\widetilde{U}_2 := U_2^- \times_{U_1^-} \widetilde{U}_1 = \widetilde{U}_{1|U_2^-}$$

and

$$\widehat{V}_i = \widetilde{V}_i \times_{U_i^-} \widetilde{U}_i$$

are complex analytic manifolds. Since $\widetilde{U}_1 \to U_1^-$ is an unbranched covering, the canonical projection

$$p_1 : \widehat{V}_1 \longrightarrow \widetilde{V}_1$$

is also locally biholomorphic, and since $\widetilde{V}_1$ is simply connected, there exists a holomorphic section s_1 to p_1, i.e. a holomorphic map $s_1 : \widetilde{V}_1 \to \widehat{V}_1$ with $p_1 \circ s_1 = \mathrm{id}$. Such a section s_2 exists also for $p_2 : \widehat{V}_2 \to \widetilde{V}_2$. Composing s_2 with the projection $\widehat{V}_2 \to \widetilde{U}_2$ yields a commutative diagram of holomorphic maps

$$\begin{tikzcd}
\widetilde{V}_2 \arrow[r] \arrow[d] & \widetilde{U}_2 \arrow[d] \\
V_2^- \arrow[r] & U_2^-
\end{tikzcd}$$

Since U_2^- is connected, the same property must hold for the restriction $\widetilde{U}_2$ of $\widetilde{U}_1$ to U_2^-. The holomorphic map $\widetilde{V}_2 \to \widetilde{U}_2$ being finite, its image has the same dimension as $\widetilde{U}_2$ and coincides therefore with a connected component, i.e. with $\widetilde{U}_2$ itself. Hence, this map is surjective.

Since $\operatorname{ord} \pi_1(X, x)$, $\operatorname{ord} \pi_1(Y, y)$ and $\operatorname{ord}_y f$ are the branch numbers of the coverings $\widetilde{U}_2 \to U_2^-$, $\widetilde{V}_2 \to V_2^-$ and $V_2^- \to U_2^-$, resp., we are done. $\square$

In order to establish the opposite direction, we shall make use of a deep result of Grauert and Remmert that characterizes complex analytic spaces as certain topological covering spaces of complex analytic manifolds. In the next Section, we give a simple proof in the two–dimensional case.

8.11 The Grauert - Remmert Theorem for normal surfaces

We introduced complex analytic spaces via the Oka–Cartan–Serre approach as ringed spaces that are locally isomorphic to analytic subsets in $\mathbb{C}^n$ (with an appropriate structure sheaf). By the Noether Normalization Theorem, these spaces can (locally) be realized as branched coverings of manifolds. Historically, however, singular spaces came at the beginning into the play as such coverings and the abstract theory was thus based on the corresponding local models. The theorem of Grauert and Remmert says that both theories agree (in the normal case).

Let us call the triple (Y, ρ, M) an *analytic covering* of the complex analytic manifold M, if Y is a topological space (which is always - by assumption - locally compact, Hausdorff, with countable topology), if $\rho : Y \to M$ is a finite surjective map, and if there exists a nowhere dense analytic subset $A \subset M$ such that $\rho^{-1}(A)$ does not disconnect Y locally and ρ maps $Y \setminus \rho^{-1}(A)$ locally topologically onto an open set in M. An analytic covering is called connected, if Y is connected.

It is easy to see that for an analytic covering (Y, ρ, M) the restriction $\rho_{|Y\setminus\rho^{-1}(A)}$ is an open map from $Y \setminus \rho^{-1}(A)$ onto $M \setminus A$, and that $Y \setminus \rho^{-1}(A) \to M \setminus A$ is an unbranched topological covering. Moreover, each point in Y has a countable neighborhood basis $\{V_\alpha\}$ such that the triples $(V_\alpha, \rho_{|V_\alpha}, \rho(V_\alpha))$ are analytic coverings.

Next, one has to introduce the concept of holomorphic functions in the obvious fashion: A complex–valued continuous function f in an open set $V \subset Y$ is called holomorphic, if f is holomorphic on $V \setminus \rho^{-1}(A)$ with respect to the natural complex analytic manifold structure on $Y \setminus \rho^{-1}(A)$ which makes ρ into a locally biholomorphic map.

We denote by $\mathcal{H}_\rho$ the sheaf of germs of holomorphic functions on Y with respect to $\rho : Y \to M$. Clearly, $\mathcal{H}_\rho$ makes Y into a locally ringed space whose stalks $\mathcal{H}_{\rho,y}$ are integral domains such that

$$(\rho_* \mathcal{H}_\rho)_x = \bigoplus_{y \in \rho^{-1}(x)} \mathcal{H}_{\rho,y} \,.$$

Moreover, using elementary symmetric functions and Riemann's Extension Theorem (for bounded functions on M), it is easy to prove that $\mathcal{H}_{\rho,y}$ is the algebraic closure of $\mathcal{O}_{M,x}$, $x = \rho(y)$, in $\mathcal{C}_{Y,y}$ under the natural inclusion homomorphism $\mathcal{C}_{M,x} \hookrightarrow \mathcal{C}_{Y,y}$. More precisely, each element $f \in \mathcal{H}_{\rho,y}$ satisfies an integral equation

$$f^b + a_1 f^{b-1} + \cdots + a_b = 0 \,, \quad a_\beta \in \mathcal{O}_{M,x} \,,$$

where the number b is independent of f (viz. the local number of sheets of the covering $Y \setminus \rho^{-1}(A) \to M \setminus A$ near y). - We can now formulate the theorem of Grauert and Remmert:

***Theorem 8.24** *Under the conditions as above, the locally ringed space* $(Y, \mathcal{H}_\rho)$ *is a normal complex analytic space, and* $\rho : Y \to M$ *is a (finite) holomorphic map.*

We prove only *normality* in that general situation (assuming that $(Y, \mathcal{H}_\rho)$ is a complex analytic space and ρ is holomorphic). Necessarily, Y is a reduced space, holomorphic functions being continuous by construction. Since Y is locally irreducible, the normalization map $\nu : \widehat{Y} \to Y$ is a homeomorphism of the underlying topological spaces. In particular, elements h_y in the normalization $\mathcal{O}_{\widehat{Y},y} = \widehat{\mathcal{H}}_{\rho,y}$ may be regarded as continuous functions on Y. Since $\mathcal{H}_{\rho,y} = \mathcal{O}_{Y,y}$ is finite over $\mathcal{O}_{M,x}$, it follows that each such element h_y satisfies an integral equation over $\mathcal{O}_{M,x}$. By the preceding remarks, $h_y \in \mathcal{O}_{Y,y}$. $\square$

Obviously, Theorem 24 implies:

Theorem 8.25 *Under the conditions as above,* $\rho_* \mathcal{H}_\rho$ *is a coherent analytic sheaf on* M. *(Hence,* Y *is isomorphic to the analytic spectrum of* $\rho_* \mathcal{H}_\rho$*).*

Remark. Conversely, Theorem 24 may also be deduced from Theorem 25: Just recall that the existence proof for analytic spectra (see Chapter 19.15) is valid in the category of arbitrary locally ringed spaces lying finitely branched over $(M, \mathcal{O}_M)$. Therefore, assuming $\mathcal{A} = \rho_* \mathcal{H}_\rho$ to be a coherent $\mathcal{O}_M$–algebra and forming the analytic spectrum $(Z, \mathcal{O}_Z)$ with respect to $\mathcal{A}$, it follows that $(Y, \mathcal{H}_\rho) \cong (Z, \mathcal{O}_Z)$ over M as ringed spaces. Consequently, Y is a complex analytic space, and ρ is holomorphic.

Henceforth, it is sufficient for our purposes to show Theorem 24 in the *surface* case. This is done by modifying the analytic set $A \subset M$ by means of Theorem 5.3: Take a monoidal transformation $\sigma : N \to M$ such that $B = \sigma^{-1}(A)$ has only normal crossings in N. Then let Z be the closure of the fiber product

$$(Y \setminus \rho^{-1}(A)) \times_{(M \setminus A)} (N \setminus B) \text{ in } Y \times_M N \,,$$

and denote by π the canonical projection $Z \to N$. In the resulting commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ \tau\ } & Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \rho} \\ N & \xrightarrow{\ \sigma\ } & M \end{array}$$

the map τ is proper and surjective, and (Z, π, N) is an analytic covering. - We claim:

It is sufficient to prove Theorem 24 *and* Theorem 25 *for analytic coverings with a normal crossing divisor* A.

Indeed: If $\mathcal{H}_\pi$ denotes the structure sheaf of Z, then we have, by construction of Z, the identity $\tau_* \mathcal{H}_\pi \cong \mathcal{H}_\rho$. Thus,

$$\rho_* \mathcal{H}_\rho \cong \rho_* (\tau_* \mathcal{H}_\pi) \cong \sigma_* (\pi_* \mathcal{H}_\pi) \,.$$

To finish the *proof* of Theorem 24 in the *normal crossing case*, we notice that we have a local problem with respect to M. Moreover, we encountered above the fact that the ringed space $(Y, \mathcal{H}_\rho)$ is *weakly normal* in the sense that each (germ of a) continuous function on Y which is holomorphic on $Y \setminus \rho^{-1}(A)$ is automatically a section of $\mathcal{H}_\rho$. In the classification of normal Jung singularities as the quotients X_{nq} (Chapter 7.7), it is precisely that property of normal singularities which makes the proof go through. So, we conclude that, in the given situation, the ringed space $(Y, \mathcal{H}_\rho)$ over $(M, \mathcal{O}_M)$ is locally isomorphic to a normal Jung singularity with a canonical projection ρ. □

For the rest of the Section, assume that X is a connected normal surface with precisely one singular point x, and let $p' : Y' \to X' = X \setminus \{ x \}$ be a finite connected topological covering (with its canonical complex analytic manifold structure) such that ${p'}^{-1}(U)$ is connected for all sufficiently small connected neighborhoods U of x in X. By an easy exercise, one checks immediately that $Y := Y' \cup \{ y \}$, suitably topologized, forms under $p : Y \to X$, $p_{|X'} = p'$, $p(y) = x$, a finite covering of X (branched only at x). We call Y the *topological completion* with respect to X.

Theorem 8.26 *Under the assumptions as above, the topological completion Y with respect to X carries a natural normal complex analytic structure extending the manifold structure on Y' such that $p : Y \to X$ is a finite holomorphic map.*

Proof. Composing p locally with a Noether normalization $\nu : X \to M$, we obtain an analytic covering $\rho : Y \to M$ with $\{ y \} = \rho^{-1}(\rho(y))$ which is branched exactly over the branch locus A of ν. $\rho^{-1}(A) \setminus \{ y \}$ being one–dimensional in Y', we have for all open sets V in Y:

$$(*) \qquad H^0(V \setminus \{ y \}, \mathcal{O}_{Y'}) \cap H^0(V, \mathcal{C}_Y) = H^0(V, \mathcal{H}_\rho) .$$

Therefore, $\mathcal{O}_Y = \mathcal{H}_\rho$ is the desired structure sheaf. Clearly, $\mathcal{O}_Y$ is uniquely determined by $(*)$. □

8.12 A topological characterization of quotient surface singularities

We are now able to prove

Theorem 8.27 *Let X be a normal surface singularity. Then the following conditions are equivalent* :

(i) X *is a quotient singularity* ;

(ii) $\pi_1(X, x)$ *is finite* ;

(iii) X *has a* smooth *finite branched covering.*

Proof. By definition, (i) $\Longrightarrow$ (iii) is clear, and (iii) $\Longrightarrow$ (ii) is trivial because of Lemma 23. It remains to show that X is a quotient singularity under the assumption (ii).

Suppose without loss of generality that X is a good neighborhood of x. Then $X' = X \setminus \{ x \}$ is connected. Take the universal covering space $p' : Y' \to X'$ and form the topological completion Y (which is obviously possible). Since Y carries by Theorem 26 a natural normal complex analytic structure and

$$\pi_1(Y, y) = \pi_1(Y \setminus \{ y \}) = \pi_1(Y') = 1 ,$$

we infer from Mumford's criterion (Chapter 15.6) that y is a smooth point of Y.

The finite group $G := \pi_1(X, x)$ acts holomorphically on Y by deck–transformations, and this operation extends to Y (leaving y fixed). The spaces Y/G and X are homeomorphic and biholomorphic outside $x = p(y)$. Since Y/G and X are normal spaces, they are in fact biholomorphically equivalent, and $X \cong Y/G$ is a quotient of a smooth space. □

The following is evident from Theorem 27:

Corollary 8.28 *Let $\rho : X \to Y$ be a finite branched covering of normal surface singularities. If X is a quotient singularity, then so is Y.*

8.A Appendix A: The classification of quotient surface singularities

In anticipation of later results, we classify in this Appendix the quotient surface singularities by their resolutions and by the defining small groups. Still undefined notions like "minimal resolution", "dual resolution graph", "plumbing", "tautness" etc. will be explained in Chapter 9. As we know all details in the cyclic case, we can restrict our investigation to noncyclic groups.

By a straightforward group theoretical argument, it is even possible to go back to the binary polyhedral groups $G \subset \mathrm{SL}(2, \mathbb{C})$ which will be studied carefully in Chapter 11. Noticing that we have a surjective group homomorphism

$$\psi : \mathrm{ZGL}_2 \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{GL}(2, \mathbb{C})$$

(ZGL_2 denoting the center of $\mathrm{GL}(2, \mathbb{C})$ consisting of all multiples aE, $a \neq 0$, of the unit matrix) defined by multiplication, it is not difficult to convince oneself that the following is true:

Lemma 8.29 *Each noncyclic finite subgroup G of $\mathrm{GL}(2, \mathbb{C})$ may be obtained from a quadruple $(G_1, N_1; G_2, N_2)$, where*

(a) *$G_1 \subset \mathrm{ZGL}_2$ and $G_2 \subset \mathrm{SL}(2, \mathbb{C})$ are finite subgroups, G_2 not cyclic,*

(b) *$N_1 \subset G_1$ and $N_2 \subset G_2$ are normal subgroups such that there exists an isomorphism*

$$\varphi : G_2/N_2 \xrightarrow{\sim} G_1/N_1 ,$$

by the following fiber product construction :

$$G := \psi(G_1 \times_\varphi G_2), \; G_1 \times_\varphi G_2 = \{ (g_1, g_2) \in G_1 \times G_2 : \overline{g}_1 = \varphi(\overline{g}_2) \} ;$$

here, $\overline{g}_i$ denotes the residue class of g_i in G_i/N_i, $i = 1, 2$.

The conjugacy class of G in $\mathrm{GL}(2, \mathbb{C})$ does not depend on the specific isomorphism φ. Therefore, we use the symbol

$$(G_1, N_1; G_2, N_2)$$

also as a name for the *class* containing the groups $\psi(G_1 \times_\varphi G_2)$.

In Chapter 11, we will determine the finite noncyclic subgroups of $\mathrm{SL}(2, \mathbb{C})$ (up to conjugacy). These *binary polyhedral groups* are in one–to–one correspondence with the *Platonian triples*, i.e. the triples $(n_1, n_2, n_3) \in \mathbb{N}^3$ satisfying the inequality

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1 , \quad 2 \leq n_1 \leq n_2 \leq n_3 .$$

We denote them by $\mathbb{D}_{k+2}$, $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$, when the triples are $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$, respectively. One can in fact classify the conjugacy classes of small subgroups in $\mathrm{GL}(2, \mathbb{C})$ by Lemma 29 in combination with some rudimentary knowledge of these groups. In the approach presented here we mix group theory and the resolution theory of surface singularities developed thus far.

In Chapter 11.13 we will find the resolution of a Klein singularity $\mathbb{C}^2/G_2$, $G_2 \subset \mathrm{SL}(2, \mathbb{C})$, by blowing up the origin 0 in $Y = \mathbb{C}^2$ and forming the canonical quotient $\widetilde{Y}/G_2$ which is a modification of Y/G_2. The action of G_2 on $\sigma^{-1}(0) = \mathbb{P}_1$ has three special orbits at whose points the stabilizer groups act in local coordinates cyclically by

$$u_0 \longmapsto \zeta_{n_i} u_0 , \quad v_0 \longmapsto \zeta_{2n_i}^{-1} v_0 , \quad i = 1, 2, 3 ,$$

where u_0 is a local coordinate on $\mathbb{P}_1$ and v_0 is a fiber coordinate of the bundle $\widetilde{Y} \to \mathbb{P}_1$. Hence, $\widetilde{Y}/G_2$ contains the curve $\mathbb{P}_1/G_2 \cong \mathbb{P}_1$ on which we find exactly three cyclic singularities of $\widetilde{Y}/G_2$ (of order n_1, n_2 and n_3 only since we have to divide out the action of the reflection $u_0 \mapsto u_0$, $v_0 \mapsto -v_0$).

Now, for a general group G, we must also investigate how the group $G_1 \subset \mathrm{ZGL}_2$ acts on $\widetilde{Y}$. Clearly, G_1 is a cyclic group generated by an element ψ of the form $\zeta_\ell E$ which (in global coordinates of $\widetilde{Y}$) has to act by $u_0 \mapsto u_0$, $v_0 \mapsto \zeta_\ell v_0$ and $u_1 \mapsto u_1$, $v_1 \mapsto \zeta_\ell^{-1} v_1$. In particular, ψ leaves $\mathbb{P}_1$ pointwise fixed such that G has the same three special orbits on $\mathbb{P}_1$ as G_2 has.

Consequently, $\widetilde{Y}/G$ contains at most three singularities on $\mathbb{P}_1/G \cong \mathbb{P}_1$ which are quotients by groups whose elements are products of powers of

$$\begin{pmatrix} \zeta_{n_i} & 0 \\ 0 & \zeta_{n_i}^{-1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & \zeta_\ell^2 \end{pmatrix} .$$

Eliminating the action of the normal subgroup generated by the reflections, we obtain an action of type (n_i, q_i), $0 \le q_i < n_i$ (but $\gcd(n_i, q_i)$ not necessarily equal to 1). In any case, we can state:

Theorem 8.30 *The minimal resolution of a noncyclic quotient singularity is determined by a system of numbers*

$$(b;\, n_1,\, q_1,\, n_2,\, q_2,\, n_3,\, q_3)$$

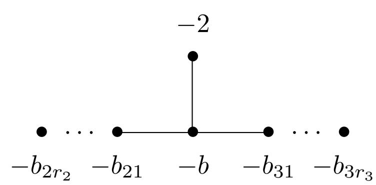
with

(i) $b \ge 2$,

(ii) $n_1 \le n_2 \le n_3$ a Platonian triple*,*

(iii) $1 \le q_i < n_i$, $\gcd(n_i, q_i) = 1$, $i = 1, 2, 3$,

by forming the plumbed manifold with smooth rational components of the exceptional set associated to the star



where

$$\frac{n_i}{q_i} = b_{i1} - 1 \,\big\lceil\, b_{i2} - \cdots - 1 \,\big\lceil\, b_{ir_i} \,, \quad i = 2, 3 .$$

(Of course, the pair $n_1 = 2$, $q_1 = 1$ belongs to the upper (short) arm).

The various parts of Theorem 30 can be elaborated by different techniques as we will see in subsequent Chapters. In particular, each surface singularity admitting a good $\mathbb{C}^*$–action is resolved by a plumbed manifold with dual graph a star, all curves being rational with the possible exception of the central curve (Chapter 10). That the central curve for quotient singularities is rational follows e.g. from the fact that they also belong to the wider class of rational singularities (Chapter 12). The same result can be deduced (together with the precise form of the stars given in Theorem 30) from the characterization of quotients by the finiteness of their fundamental groups (see Chapter [**??**]).

Furthermore, the last argument implies that the graphs in Theorem 30 indeed give rise to finite fundamental groups, and finally, the plumbed manifolds attached to these graphs are analytically determined by the given data. Putting everything together, we get

Theorem 8.31 *The (isomorphism classes of) noncyclic quotient surface singularities are in one–to–one correspondence to the systems* $(b;\, n_1,\, q_1,\, n_2,\, q_2,\, n_3,\, q_3)$ *as in* Theorem 30*. The associated dual resolution graphs are taut.*

Carrying out the program sketched above carefully, one is led to explicit abstract presentations of the fundamental groups, thereby arriving at the conclusion that different graphs yield abstractly different groups (which, a priori, is not clear at all). Thus, in order to find a complete set of conjugacy classes of finite small subgroups of $\mathrm{GL}\,(2,\,\mathbb{C})$, one just has to write down a list of such groups with the correct presentation. The result is encoded in the following table ($\mathbf{Z}_\ell$ denotes the group $\langle\,\zeta_\ell\,\rangle$):

$G = (G_1,\,N_1;\,G_2,\,N_2)$	$(b;\,n_1,\,q_1,\,n_2,\,q_2,\,n_3,\,q_3)$		
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{D}_n, \mathbb{D}_n)$	$(b; 2, 1, 2, 1, n, q)$	$m = (b-1)\,n - q =$	odd
$(\mathbb{Z}_{4m}, \mathbb{Z}_{2m}; \mathbb{D}_n, C_{2n})$			even
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{T}, \mathbb{T})$	$(b; 2, 1, 3, 2, 3, 2)$	$m = 6\,(b-2) +$	1
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{T}, \mathbb{T})$	$(b; 2, 1, 3, 1, 3, 1)$		5
$(\mathbb{Z}_{6m}, \mathbb{Z}_{2m}; \mathbb{T}, \mathbb{D}_2)$	$(b; 2, 1, 3, 1, 3, 2)$		3
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$	$(b; 2, 1, 3, 2, 4, 3)$	$m = 12\,(b-2) +$	1
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$	$(b; 2, 1, 3, 1, 4, 3)$		5
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$	$(b; 2, 1, 3, 2, 4, 1)$		7
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$	$(b; 2, 1, 3, 1, 4, 1)$		11
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 2, 5, 4)$	$m = 30\,(b-2) +$	1
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 2, 5, 3)$		7
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 1, 5, 4)$		11
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 2, 5, 2)$		13
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 1, 5, 3)$		17
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 2, 5, 1)$		19
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 1, 5, 2)$		23
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$(b; 2, 1, 3, 1, 5, 1)$		29

Without relying on the fundamental groups, one can alternatively prove Theorem 31 and the correctness of the table by computing for all given groups the stabilizer subgroups with respect to the action of G on the blow up $\widetilde{Y}$ of $Y = \mathbb{C}^2$ (see the remarks before Theorem 30) and the self–intersection number of the central curve. The first task is easy: Take for instance the last sequence of groups $(\mathbf{Z}_{2m},\,\mathbf{Z}_{2m};\,\mathbb{I},\,\mathbb{I})$, and look for a point on $\mathbb{P}_1 \subset \widetilde{Y}$, where the binary icosahedral group $\mathbb{I}$ acts locally via the matrix

$$A = \begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{pmatrix} .$$

$\mathbf{Z}_{2m}$ operates there, as we have seen above, through the reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_m \end{pmatrix} .$$

In local coordinates $u,\,v$, invariants with respect to $\mathbf{Z}_{2m}$ are

$$u = u\,,\ w = v^m\,,$$

and on these coordinates A operates as

$$u \longmapsto \zeta_5 u\,,\ w \longmapsto {\zeta_5}^{-m} w\,,$$

i.e. by the matrix

$$\begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{q} \end{pmatrix} ,$$

where $q + m \equiv 0 \bmod 5$. This consideration determines the entry n_3, q_3: we obtain

$$(n_3,\,q_3) = \left\{ \begin{array}{l} (5,\,1) \\ (5,\,2) \\ (5,\,3) \\ (5,\,4) \end{array} \right. \quad \text{for} \quad m \equiv \left\{ \begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \end{array} \right. \quad \bmod 5\,.$$

Similarly, one can compute n_1, q_1 and n_2, q_2.

The reason for the numbers m being prime to 30 in that sublist is also evident: If $\gcd(m, 30) \neq 1$, then at least one of the stabilizer subgroups contains reflections after reduction modulo $\mathbf{Z}_{2m}$, and we are in fact considering a case already listed before (or even a cyclic quotient).

The self–intersection number b of the central curve is computable with the help of Corollary 10.20. Perhaps surprisingly, one finds a formula that covers all cases:

***Theorem 8.32** *Let h denote the order of the image of the group $G = (G_1, N_1; G_2, N_2)$ in $\mathrm{PGL}(2, \mathbb{C})$. Then, for all m in the table above,*

$$b = \frac{q_1}{n_1} + \frac{q_2}{n_2} + \frac{q_3}{n_3} + \frac{2m}{h} .$$

We sketch a third method in the vein of Sections 10, 11 and 12 in Chapter 11 for proving Theorem 31. Notice that the group N_2 is just the intersection

$$G \cap \mathrm{SL}(2, \mathbb{C}) ,$$

when $G = (G_1, N_1; G_2, N_2)$ as in Lemma 29. Hence, N_2 is a normal subgroup of G, and there exists an obvious isomorphism $G \overset{\sim}{\to} G_1/N_2$. Clearly, if x_1, x_2, x_3 are homogeneous generators for $\mathbb{C}\langle u, v\rangle^{N_2}$ (see Chapter 11, where it is shown that always three generators suffice) and if G_1 is generated by $\zeta_\ell E$, then the corresponding generator of G/N_2 acts by

$$x_i \longmapsto \zeta_\ell^{\deg x_i} x_i , \quad i = 1, 2, 3 .$$

In view of the results in Chapter 10, that amounts to saying the following:

Theorem 8.33 *Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be a finite small subgroup, and let X_0 be the Klein singularity $\mathbb{C}^2/G \cap \mathrm{SL}(2, \mathbb{C})$ equipped with its (natural) good $\mathbb{C}^*$–action. Then there exists a finite (necessarily cyclic) subgroup $H \subset \mathbb{C}^*$ such that*

$$X = \mathbb{C}^2/G \cong X_0/H .$$

Now, the resolution $\widetilde{X}_0 \to X_0$ to be constructed in Chapter 10 is $\mathbb{C}^*$–equivariant. So, we can form the commutative diagram

$$\begin{tikzcd} \widetilde{X}_0 \arrow[r] \arrow[d] & \widetilde{X}_0/H \arrow[d] \\ X_0 \arrow[r] & X_0/H \cong X \end{tikzcd}$$

in which the right vertical arrow represents a modification. Since the $\mathbb{C}^*$–action on $\widetilde{X}_0$ respects the plumbing construction, it is completely determined by its action on the normal bundle of one of the components of the exceptional set. Moreover, if there exists a central curve E_0, E_0 itself is pointwise fixed by $\mathbb{C}^*$ such that there is exactly one such action on $\widetilde{X}_0$. In any case, $H \subset \mathbb{C}^*$ can have nontrivial stabilizer subgroups only at intersection points $E_j \cap E_k$. Hence, $\widetilde{X}_0/H$ has only cyclic quotient singularities which are easily computable and resolvable. However, the resulting resolution of X may not be minimal, since self–intersection numbers can decrease (in absolute value) during this process. Take, for instance, a line bundle $\mathcal{O}(-b)$ on $\mathbb{P}_1$ with homogeneous coordinates u_0, u_1, and let $H = \langle \zeta_\ell \rangle$ act by

$$u_0 \longmapsto \zeta_\ell u_0 , \ v_0 \longmapsto v_0 .$$

Then, necessarily,

$$u_1 \longmapsto \zeta_\ell^{-1} u_1 , \ v_1 \longmapsto \zeta_\ell^{b} v_1 .$$

Consequently, in order to have a trivial action on the fibers of $\mathcal{O}(-b)$, we assume $b \equiv 0 \bmod \ell$, and in this case it follows from a trivial calculation that $\mathcal{O}(-b)/H$ is isomorphic to the line bundle $\mathcal{O}(-b/\ell)$ over $\mathbb{P}_1/H \cong \mathbb{P}_1$ (with homogeneous coordinates $w_0 = u_0^\ell$, $w_1 = u_1^\ell$). The opposite effect can simply be achieved, as we saw at former instances, by the action $u_0 \mapsto u_0$, $v_0 \mapsto \zeta_\ell v_0$. - The details are left to the reader.

8.B Appendix B: The $\mathfrak{m}$–adic topology on analytic algebras and modules

If R is a ring and $\mathfrak{a}$ is an ideal in R one can define a *basis of neighbourhoods* of $0 \in R$ by the powers $\mathfrak{a}^k$ of $\mathfrak{a}$. Translating these neighbourhoods *additively* to other elements $r \in R$ one comes up with a *topology* on R that makes R into a *topological ring*. It is called the $\mathfrak{a}$–*adic* topology on R. If M is a module over R, one can similarly start with the basis $\mathfrak{a}^k M$ of neighbourhoods of $0 \in M$. This defines the $\mathfrak{a}$–*adic* topology on M which provides it with the structure of a *topological* module over R with its $\mathfrak{a}$–*adic* topology. Since all these topologies satisfy the *first axiom of countability*, one can test most of the topological properties by testing the behaviour of *convergent series*. Remark also that if $\mathfrak{b}$ is another ideal in R such that a certain power of $\mathfrak{b}$ is contained in $\mathfrak{a}$ then the $\mathfrak{b}$–*adic* topology and the $\mathfrak{a}$–*adic* topology on M coincide.

A well known result (and simple exercise) says that the $\mathfrak{a}$–*adic* topology on M is a *Hausdorff* or a *separated* topology if and only if

$$\bigcap_k \mathfrak{a}^k M = 0 .$$

This is, in particular, satisfied for any proper ideal $\mathfrak{a} \subset \mathfrak{m}$ in an *analytic* algebra A and any noetherian module M over A.

We are particularly interested in the main corpus of this Chapter in *analytic algebras* A and their $\mathfrak{m}_A$–*adic* topology. In this topology, however, analytic algebras are <u>not</u> *complete*. Of course, for a *regular* analytic algebra R_n of *convergent* power series, the *completion* of R_n with respect to the $\mathfrak{m}_n$–*adic* topology of R_n is the ring of *formal* power series. Since analytic homomorphisms $\varphi : A \longrightarrow B$ are *local* they are *continuous* in the resp. $\mathfrak{m}$–adic topologies. Moreover, if φ is *surjective*, then $\varphi(\mathfrak{m}_A) = \mathfrak{m}_B$ such that φ is an *open* homomorphism.

Remark. There are other topologies on analytic algebras over *locally compact* fields $\mathbb{K}$ and modules over such algebras which reflect much better the *analytic* aspect of *convergent power series*. Since we do not need them in this text, we omit more details and refer to [01 - 02].

8.C Appendix C: A local openness criterion for finite holomorphic maps

We still need to prove the *local openness criterion* which we applied to show that that the map p is *surjective* at $y = \rho(x)$.

Definition and *Remark*. Let $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ be complex analytic spaces and $(\pi, \widehat{\pi})$ a holomorphic map between X and Z (for details see the Supplement). We call π *open* at a point $x \in X$, if π maps a basis of neighbourhoods of x onto such a basis of the image $z = \pi(x)$, that is to say that the germ of the image $\pi(X)$ at z coincides with the whole germ Z_z.

Warning. Openness at a point is in general <u>not</u> an "open condition".

Remark. The situation, however, is easier to handle when x lies *discrete* in its fiber $\pi^{-1}(\pi(x))$, i.e. when locally near x the mapping π is *finite*. In this case, the germ $(\pi(X))_z$ is an *analytic* subgerm of Z_z defined by the coherent *Annulator* ideal of the direct image sheaf $\pi_*(\mathcal{O}_X)$. Hence, π is open at x if and only if

$$(N(\mathrm{Ann}_{\mathcal{O}_Z}\pi_*\mathcal{O}_X))_z = Z_z = (N(0))_z ,$$

which by Rückerts Nullstellensatz is equivalent to

$$(\mathrm{Ann}_{\mathcal{O}_Z}\pi_*\mathcal{O}_X)_z \subset \mathfrak{n}_{Z,z}$$

or, in other words, all elements in $(\mathrm{Ann}_{\mathcal{O}_Z}\pi_*\mathcal{O}_X)_z$ are *nilpotent*. But in our situation $(\pi_*\mathcal{O}_X)_z \cong \mathcal{O}_{X,x}$, and the annulator ideal in question is nothing else but the *kernel* of the analytic morphism $\widehat{\pi}_x : \mathcal{O}_{Z,z} \longrightarrow \mathcal{O}_{X,x}$. - So, we have proved the following result.

Theorem 8.34 *A* finite *holomorphic mapping* $(\pi, \widehat{\pi}) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ *of complex spaces is* open *at* $x \in X$, *if and only if each element in the kernel of* $\widehat{\pi}_x : \mathcal{O}_{Z,z} \longrightarrow \mathcal{O}_{X,x}$ *is nilpotent. In particular, if* Z *is a* reduced *space, this condition is equivalent to the injectivity of* $\widehat{\pi}_x$.

Notes and References

The first existence proof for analytic quotients by groups of biholomorphic maps acting properly discontinuously on complex spaces is due to

[08 - 01] H. Cartan: Quotient d'un espace analytique par un groupe d'automorphismes. In: Algebraic geometry and topology. A symposium in honour of S.Lefschetz, pp. 90–102. Princeton: Princeton University Press 1957.

A concise treatment including more general *holomorphic equivalence relations* may be found in [01–09]. From this book we borrowed also the Criterion of Openness in Appendix C. Theorem 4 and the proof of Theorem 7 are taken from [01–02], Kapitel III, paragraph 3. Theorem 11 (Hilbert's Syzygy Theorem) is Satz III.2.6 in loc. cit. for the regular analytic algebra R_n (with an interpretation of the difference $n - \mathrm{syl}_{R_n}$ as the *profondeur* of M; see also Chapter 13.2). For general local noetherian rings R, and the converse to Theorem 11, see

[08 - 02] J.–P. Serre: Algèbre locale - Multiplicités. Redigé par P. Gabriel. Lecture Notes in Mathematics 11. Berlin–Heidelberg–New York: Springer 1965.

The original Hilbert Syzygy Theorem appeared in the famous paper [08–03], after which invariant theory came to a halt for a long time, because of the widely held opinion that Hilbert had solved all relevant problems ever raised in this field:

[08 - 03] D. Hilbert: Über die Theorie der algebraischen Formen. Math. Ann. 36, 473–534 (1890).

Chevalley's Theorem (Theorem 17 and Theorem 18) was stated and proved in

[08 - 04] C. Chevalley: Invariants of finite groups generated by reflections. Amer. J. Math. 77, 778–782 (1955).

Another algebraic treatment can be found in

[08 - 05] T. A. Springer: Invariant Theory. Lecture Notes in Mathematics 585. Berlin–Heidelberg–New York: Springer 1977.

(see Theorem 4.2.5 and Corollary 4.2.12 there). This brochure will be also our general reference for invariants of finite groups acting on S_2.

The proof of Theorem 13 presented here is taken from an unpublished manuscript of the author which was written just before the appearance of

[08 - 06] N. Bourbaki: Groupes et algèbres de Lie. Élements de mathématique, Fascicules XXVI, XXXVII, XXXIV, XXXVIII. Hermann, Paris 1971, 1972, 1968, 1975,

where a similar proof appeared in paragraph 5, no 5, Théorème 4. Both work also more generally for arbitrary algebraically closed ground fields k of characteristic $p > 0$ and finite groups G satisfying $\mathrm{ord}\, G \not\equiv 0 \bmod p$.

Theorem 19 and 20 are due to Gottschling and Prill:

[08 - 07] E. Gottschling: Die Uniformisierbarkeit der Fixpunkte eigentlich diskontinuierlicher Gruppen von biholomorphen Abbildungen. Math. Ann. 169, 26–54 (1967),

[08 - 08] D. Prill: Local classification of quotients of complex manifolds by discontinuous groups. Duke Math. Journal 34, 375–386 (1967),

[08 - 09] E. Gottschling: Invarianten endlicher Gruppen und biholomorphe Abbildungen. Inventiones math. 6, 315–326 (1969).

That the same is true in case $p \nmid \mathrm{ord}\, G$ was shown in

[08 - 10] D. Denneberg, O. Riemenschneider: Verzweigung bei Galois–Erweiterungen und Quotienten regulärer analytischer Raumkeime. Inventiones math. 7, 111–119 (1969).

Most of the material on the local fundamental group is contained in [08–08]; see also

[08 - 11] D. Denneberg: Universell–endliche Erweiterungen analytischer Algebren. Math. Ann. 200, 307–326 (1973).

The two papers concerning triangulations of real–analytic spaces are:

[08 - 12] S. Lojasiewicz: Triangulations of semi–analytic sets. Ann. Scuola Norm. Sup. Pisa 18, 449–474 (1964)

and

[08 - 13] B. Giesecke: Simpliziale Zerlegungen abzählbarer analytischer Räume. Math. Z. 83, 177–213 (1964).

Grauert and Remmert established the equivalence between the older definition of (normal) complex spaces as certain branched coverings of complex manifolds - due to Stein and others - and the Cartan–Serre definition adopted in this book in the important paper

[08 - 14] H. Grauert, R. Remmert: Komplexe Räume. Math. Ann. 136, 245–318 (1958).

The characterization of quotient surface singularities as normal two–dimensional singularities finitely covered by a smooth surface is due to

[08 - 16] E. Brieskorn: Rationale Singularitäten komplexer Flächen. Invent. Math. 4, 336–358 (1968).

We will have the opportunity to cite this manuscript at various other places in our book. For instance, the classification of all quotients $\mathbb{C}^2/G$, $G \subset \mathrm{GL}(2, \mathbb{C})$ is contained in [08–16], using the group–theoretical material in

[08 - 17] P. Du Val: Homographies, Quaternions and Rotations. Oxford: Oxford University Press 1957.

Let us close these notes with the remark that it has escaped by no means our attention that a certain number is missing from the list of references.