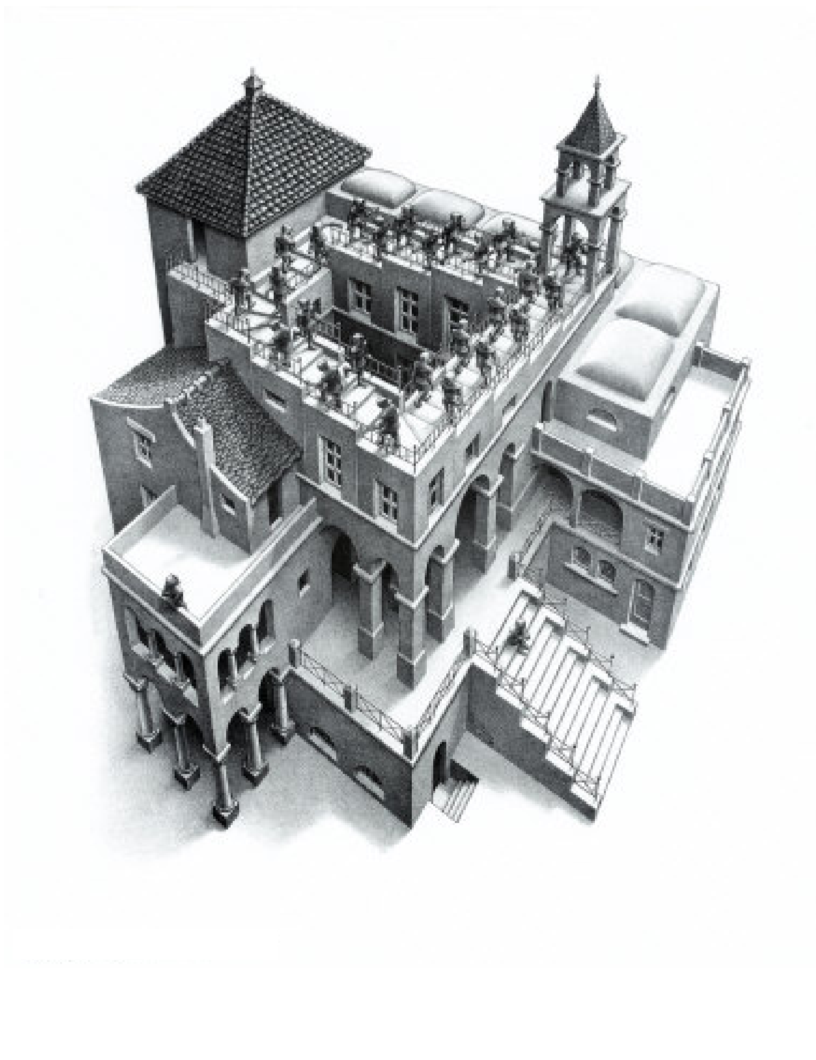
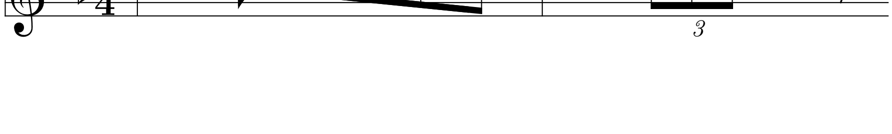
Chapter 6

... mais la vision la plus belle qui nous reste d'une oeuvre est souvent celle qui s'éleva au–dessus des sons faux tirés par des doigts malhabiles, d'un piano désaccordé.

(Marcel Proust, *A la recherche du temps perdu. Du coté de chez Swann*)





Chapter 6

Dimension and multiplicity

We are going to discuss the notion of *dimension* for complex–analytic local algebras a little further. The main purpose of this Chapter is the insight that dimension and *multiplicity* are encoded in a single algebraic object, the *Hilbert polynomial*.

6.1 Algebras of dimension zero

Let us first characterize the algebras A of dimension zero, i.e. $\dim_{\mathbb{C}} A < \infty$. Recall from *Local Algebra* that a ring is called *artinian*, if any *descending* chain of ideals becomes stationary. A prime ideal $\mathfrak{p} \subset A$ is called *isolated*, if it does not contain any proper prime ideal[1]. In particular, $\mathfrak{p} = (0)$ is the only isolated prime ideal of A if A has no zero divisors. For noetherian local rings, one has

$$(*) \qquad \bigcap_{\mathfrak{p}\ \text{isolated}} \mathfrak{p} = \bigcap_{\mathfrak{p}\ \text{prime}} \mathfrak{p} = \mathfrak{n} = \mathfrak{n}_A$$

with the *nilradical* $\mathfrak{n} = \{ f \in A : \exists t \in \mathbb{N} \text{ such that } f^t = 0 \}$. We agree to call $\operatorname{red} A := A/\mathfrak{n}_A$ the *reduction* of A. If $A \cong B/\mathfrak{b}$, then

$$\operatorname{red} A \cong B/\operatorname{rad}\mathfrak{b}\,.$$

Clearly, $\operatorname{rad}(0) = \mathfrak{n}$. An ideal $\mathfrak{q}$ is called *primary* if $\mathfrak{p} := \operatorname{rad}\mathfrak{q}$ is a prime ideal (we then also say that $\mathfrak{q}$ is $\mathfrak{p}$–*primary*). This is equivalent to the condition: if $f\,g \in \mathfrak{q}$, $g \notin \mathfrak{q}$ then $f^t \in \mathfrak{q}$ for some t.

Lemma 6.1 *Let $A \cong B/\mathfrak{b}$ be a complex analytic local algebra. Then, the following are equivalent* :

i) $\dim A = 0$;

ii) A *is artinian* ;

iii) $\mathfrak{m} = \mathfrak{m}_A$ *is an isolated (and hence the unique) prime ideal* ;

iv) $\mathfrak{m}_A = \mathfrak{n}_A$;

v) $\operatorname{red} A \cong \mathbb{C}$;

vi) $\mathfrak{b}$ *is an* $\mathfrak{m}_B$–*primary ideal* ;

vii) *there exists* $r \in \mathbb{N}$ *with* $\mathfrak{m}_B^r \subset \mathfrak{b}$;

viii) $\mathfrak{b}$ *is of finite codimension in* B.

[1] In many, if not most texts on *Commutative Algebra*, such ideals are called *minimal prime ideals*. In accordance to our main source [01 - 02], we reserve the last notion for prime ideals $\mathfrak{p} \neq 0$ in an *integral domain* R that contain the trivial ideal as the unique prime ideal.

Proof. i) $\Longrightarrow$ ii) Since $\dim_{\mathbb{C}} A < \infty$ any descending chain of ideals (which is a descending chain of $\mathbb{C}$–vector subspaces) must stop.

ii) $\Longrightarrow$ iii) Let $\mathfrak{p}$ be an isolated prime ideal of A and $B := A/\mathfrak{p}$. Then, clearly, B is artinian, as well. So, if $g \in B$, we have $gB \supset g^2B \supset \cdots$ such that $g^t = g^{t+1}h$ for some t. Since B is an integral domain, $1 = gh$ if $g \neq 0$. Therefore, B is a field, i.e. $\mathfrak{p}$ is maximal.

iii) $\Longrightarrow$ iv) follows from $(*)$.

iv) $\Longrightarrow$ v) is trivial by definition.

v) $\Longrightarrow$ vi) $\mathbb{C} \cong \operatorname{red} A \cong B/\operatorname{rad}\mathfrak{b}$ implies $\operatorname{rad}\mathfrak{b} = \mathfrak{m}_B$.

vi) $\Longrightarrow$ vii) Since $\mathfrak{m}_B$ is finitely generated the claim follows easily.

vii) $\Longrightarrow$ viii) $\mathfrak{m}_B^r \subset \mathfrak{b}$ induces an epimorphism $B/\mathfrak{m}_B^r \to B/\mathfrak{b}$.

viii) $\Longrightarrow$ i) is trivial. □

Remark. Parts of this Lemma have been included earlier in Theorem 2.13.

6.2 Parameter systems

In order to get a better insight into the true meaning of dimension we first give a new proof to the following result (see the Remark after Theorem 3.36).

Lemma 6.2 *If* $f \in \mathfrak{m}_A$, *then* $\dim A - 1 \leq \dim A/fA \leq \dim A$.

Proof. Let $d := \dim A$, $c := \dim A/fA$. From the composition of finite homomorphisms $R_d \to A \to A/fA$ we get $c \leq d$. If $\varphi : R_c \to A/fA$ is a finite homomorphism we construct a commutative diagram

$$\begin{tikzcd} R_{c+1} \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ R_c \arrow[r, "\varphi"'] & A/fA \end{tikzcd}$$

where the extra variable in R_{c+1} is sent to f via ψ. Since ψ is quasi–finite, ψ is also finite whence $c + 1 \geq d$. □

Let us recall the following definition from Chapter 3:

Definition. A set $(f_1, \ldots, f_d)$, $f_j \in \mathfrak{m}_A$, is called a *parameter system* for A if $A/(f_1, \ldots, f_d)A$ is artinian and no system of shorter length has this property.

Theorem 6.3 *If* $d = \dim A \geq 1$, *then parameter systems for* A *exist. They all have the same length* d. *In particular,*

$$\dim A = \min\{\, k : \exists f_1, \ldots, f_k \in \mathfrak{m}_A \text{ such that } A/(f_1, \ldots, f_k)A \text{ is artinian} \,\}.$$

Remark. The right hand side of the equation in the Theorem is usually called the Chevalley–dimension.

Proof. $A/(f_1, \ldots, f_k)A$ is artinian if and only if the substitution homomorphism

$$\begin{cases} R_k \longrightarrow A \\ x_j \longmapsto f_j \end{cases}$$

is quasi–finite, hence finite. Therefore, the dimension of A equals the CHEVALLEY–dimension, and parameter systems (of length $d = \dim A$) exist. $\square$

By the Theorem above, we have for any parameter system $(f_1, \ldots, f_d)$ in A with $d = \dim A \geq 1$:

$$\dim A/(f_1, \ldots, f_k)A = d - k\,, \quad 0 \leq k \leq d\,.$$

In particular, there exist elements $f \in \mathfrak{m}_A$ with $\dim A/fA = \dim A - 1$ if $\dim A \geq 1$. In the next Section we give a sufficient condition for elements $f \in \mathfrak{m}_A$ that drop the dimension (and later a stronger necessary *and* sufficient condition).

Example. Not every nontrivial element $f \in \mathfrak{m}_A$ has this property. Look for instance at the algebra $A = R_2/x_1x_2R_2$ which has dimension 1 as one can easily see by the Weierstraß' Division Theorem. But the quotient of A by the residue class of x_2 is isomorphic to R_1 which has dimension 1, too!

6.3 Active elements

The Lemma characterizing 0–dimensional algebras gives a hint how one can explicitly find elements which drop the dimension.

Definition. An element $f \in \mathfrak{m}_A$ is called *active* if its residue class $\mathrm{red}\, f \in \mathrm{red}\, A = A/\mathfrak{n}_A$ is a nonzerodivisor. In other terms: $f \notin \mathfrak{n}_A$, and if $fg \in \mathfrak{n}_A$ then $g \in \mathfrak{n}_A$.

Lemma 6.4 *Active elements exist in an analytic algebra A if and only if* $\dim A \geq 1$.

Proof. If A is artinian, all elements $f \in \mathfrak{m}_A$ are nilpotent by Lemma 1, hence not active. If, on the other hand, $\mathfrak{m} = \mathfrak{m}_A$ consists of inactive elements only, then

$$\mathfrak{m} \subset \bigcup_{\lambda=1}^{\ell} \mathfrak{p}_\lambda$$

for the isolated prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$. (It is a general fact that local noetherian rings have only *finitely many* isolated prime ideals). By an elementary argument which we elaborate in the proof of the next Lemma for the convenience of the reader, we conclude that $\mathfrak{m} \subset \mathfrak{p}_r$ for some r, whence $\mathfrak{m} = \mathfrak{p}_r$ and $\mathfrak{m}$ is an isolated prime ideal. $\square$

Lemma 6.5 *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ be prime ideals in A with $\mathfrak{p}_\lambda \not\subset \mathfrak{p}_\kappa$ for $\lambda \neq \kappa$, and let $\mathfrak{a}$ be an ideal in A that is contained in the set–theoretical union of the $\mathfrak{p}_\lambda$:*

$$\mathfrak{a} \subset \bigcup_{\lambda=!}^{\ell} \mathfrak{p}_\lambda\,.$$

Then, there exists an index $r \leq \ell$ such that $\mathfrak{a} \subset \mathfrak{p}_r$.

In particular, for each index s with $1 \leq s < \ell$ there exists an element

$$f \in \bigcap_{\lambda=!}^{s} \mathfrak{p}_\lambda \text{ with } f \notin \bigcup_{\lambda=s+1}^{\ell} \mathfrak{p}_\lambda\,.$$

Proof. Suppose that $\mathfrak{a} \not\subset \mathfrak{p}_\lambda$ for all λ. Then also

$$\mathfrak{a} \cap \bigcap_{\kappa \neq \lambda} \mathfrak{p}_\kappa \not\subset \mathfrak{p}_\lambda$$

since otherwise we get a contradiction to our assumption $\mathfrak{p}_\lambda \not\subset \mathfrak{p}_\kappa$ for $\lambda \neq \kappa$. Let $f_\lambda \in \mathfrak{a} \cap \bigcap_{\kappa \neq \lambda} \mathfrak{p}_\kappa$, $f_\lambda \notin \mathfrak{p}_\lambda$, and $f := \sum_\lambda f_\lambda$. It follows that $f \in \mathfrak{a}$, but $f \notin \mathfrak{a}_\lambda$ because of $\sum_{\kappa \neq \lambda} f_\kappa \in \mathfrak{p}_\lambda$ and

$f_\lambda \in \mathfrak{a}_\lambda$. Contradiction!

Assume to the contrary that, for some s with $1 \le s < \ell$, we have

$$\bigcap_{\lambda=!}^{s} \mathfrak{p}_\lambda \subset \bigcup_{\lambda=s+1}^{\ell} \mathfrak{p}_\lambda .$$

By the first part, there exists r with $r > s$ such that

$$\prod_{\lambda=1}^{s} \mathfrak{p}_\lambda \subset \bigcap_{\lambda=!}^{s} \mathfrak{p}_\lambda \subset \mathfrak{p}_r .$$

Since $\mathfrak{p}_r$ is a prime ideal, it is easily shown that it must contain one of the ideals $\mathfrak{p}_\lambda$, $\lambda = 1, \ldots, s$. Contradiction! □

Lemma 6.6 *An element* $f \in \mathfrak{m}_A$ *is* active *in* A *if and only if it does not belong to any isolated prime ideal* $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ *of* A.

Proof. If $f \notin \bigcup \mathfrak{p}_\lambda$, then from $f\,g \in \mathfrak{n}_A = \bigcap \mathfrak{p}_\lambda$ it follows immediately that $g \in \mathfrak{n}_A$. Hence, f is active.

On the other hand assume that $f \in \bigcup \mathfrak{p}_\lambda$ and - without loss of generality - that $f \in \mathfrak{p}_1$. If $\ell = 1$, then f is nilpotent and consequently not active. If $\ell > 1$, there exists an element $g \in \cap_{\lambda \ge 2} \mathfrak{p}_\lambda$ which does not belong to $\mathfrak{p}_1$. Therefore, $g \notin \mathfrak{n}_A$, but $f\,g \in \mathfrak{n}_A$, such that f is not active. □

Active elements really diminish the dimension (by one); this is the reason for their name (given to them by Grauert and Remmert).

Theorem 6.7 (Active Lemma) *If* $f \in \mathfrak{m}_A$ *is active then* $\dim A/fA = \dim A - 1$.

Before we embark into the *proof* of this Theorem, we draw the long overdue conclusion.

Corollary 6.8 $\dim R_n = n$.

Proof (of Corollary 8) by induction. The case $n = 0$ being trivial, assume that $\dim R_{n-1} = n - 1$. Clearly, $x_n \in R_n$ is active, since R_n is an integral domain. Thus,

$$\dim R_n = \dim R_n/x_n R_n + 1 = \dim R_{n-1} + 1 = (n-1) + 1 = n . \qquad \square$$

This might be the right place to insert here the *proof* of a *regularity criterion* we mentioned already in Chapter 3. In fact, we know sharper statements (see loc. cit.).

Theorem 6.9 *If* $\dim A = \operatorname{embdim} A = e$, *then any epimorphism* $R_e \to A$ *is an isomorphism.*

Proof. If $\varphi : R_e \to A$ is not injective we choose an element $f \in \ker \varphi$ with $f \ne 0$ and arrive at a finite morphism $R_e/fR_e \to A$. By the division theorem we find a finite morphism $R_{e-1} \to R_e/fR_e$ and hence, $\dim A \le e - 1$. □

For completing the *proof* of Theorem 7 we use another Lemma.

Lemma 6.10 *If* $\varphi : R_n \to A$ *is finite and* $f \in \mathfrak{m}_A$ *is active, then* $\varphi^{-1}(f\,A) \ne 0$.

Postponing the *proof* of this Lemma we can easily see that the "Active Lemma" is true: Due to Lemma 2, what we only have to show is that $\dim A/fA \le d - 1$, $d = \dim A$, for any active element $f \in \mathfrak{m}_A$. Take a finite homomorphism $\varphi : R_d \to A$. By the preceding Lemma there exists a nontrivial element $g \in \varphi^{-1}(f\,A) \subset R_d$. Assume that g is x_1–generic. Then, the induced homomorphism from the composite $\psi : R_d \to A \to A/fA =: \overline{A}$

$$R_d/(g, x_2, \ldots, x_d) \longrightarrow \overline{A}/(\psi(g), \psi(x_2), \ldots, \psi(x_d))\,\overline{A}$$

is finite. However, $R_d/(g, x_2, \ldots, x_d)$ is artinian such that the ring on the right hand side is artinian as well. But $\psi(g) = 0$ because of $\varphi(g) \in fA$, whence $\dim \overline{A} \le d - 1$. $\square$

Proof (of Lemma 10). $\varphi: R_n \to A$ being finite, A is integer over R_n. In particular,

$$f^t + \varphi(a_1) f^{t-1} + \cdots + \varphi(a_t) = 0, \quad a_j \in \mathfrak{m}_n.$$

Thus, there exists a minimal number s such that

$$\omega := f^s + a_1 f^{s-1} + \cdots + a_s \in \mathfrak{n} = \mathfrak{n}_A.$$

Since $\mathrm{red}\, f$ is a nonzerodivisor in $A/\mathfrak{n}_A$, we conclude that $a_s \neq 0$ (otherwise s would not be minimal). Finally, there is an $m \in \mathbb{N}$ such that $\omega^m = 0$ and consequently, $g := a_s^m \in \varphi^{-1}(fA)$. $\square$

6.4 Noether normalization

Remark. Lemma 10 is a quite weak version of a much more general result.

If $f_1, \ldots, f_d \in \mathfrak{m}_A$ are elements such that $A/(f_1, \ldots, f_d)A$ is artinian then for all $c = (c_1, \ldots, c_d) \in \mathbb{C}^d \setminus H$, H a finite union of linear subspaces of lower dimension, the element

$$f = c_1 f_1 + \cdots + c_d f_d$$

is active.

(See loc. cit., Satz II.4.10). - This implies immediately the following:

For any finite homomorphism $\varphi: A \to B$, B not artinian, there exist active elements $f \in \mathfrak{m}_A$ such that $\varphi(f)$ is active in B.

We need in the following only a special version whose proof, however, contains already the main ingredients of the general arguments.

Theorem 6.11 *Let $\varphi: R_d \hookrightarrow A$ be a finite injective homomorphism, $d \ge 1$. Then, after a* linear *change of coordinates in R_d, the element $\varphi(x_d)$ is active in A.*

Proof. By our assumption, necessarily $\dim A \ge 1$ such that the conclusion makes sense. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ denote the isolated prime ideals of A, and let π_λ be the projections $A \to A/\mathfrak{p}_\lambda$. Clearly, the tuple $(\varphi(x_1), \ldots, \varphi(x_d))$ is a weak parameter system in A, and so are, for each λ, the tuples $((\pi_\lambda \circ \varphi)(x_1), \ldots, (\pi_\lambda \circ \varphi)(x_d))$ in $A/\mathfrak{p}_\lambda$. If, for some λ, we have $(\pi_\lambda \circ \varphi)(x_1) = \cdots = (\pi_\lambda \circ \varphi)(x_d) = 0$, then, consequently, $\dim A/\mathfrak{p}_\lambda = 0$. Since the last ring is an integral domain, it is also reduced, such that it is a field. Therefore, $\mathfrak{m}_A = \mathfrak{p}_\lambda$ is an isolated prime ideal, such that $\dim A = 0$ due to Lemma 1. Contradiction! In other words: If ψ_λ denotes the $\mathbb{C}$–linear mapping which associates to any d–tuple $(c_1, \ldots, c_d) \in \mathbb{C}^d$ the residue class of $\varphi(\sum c_j x_j)$ in the ring $A/\mathfrak{p}_\lambda$, then $\ker \psi_\lambda \neq \mathbb{C}^d$. Hence, $\bigcap_\lambda (C^d \setminus \ker \psi_\lambda)$ is open and dense in $\mathbb{C}^d$, and each d–tuple $(c_1, \ldots, c_d) \neq 0$ in this set gives rise to a nontrivial linear form $\sum c_j x_j$ whose image $\varphi(\sum c_j x_j)$ is active in A. Replacing this form by x_j if $c_j \neq 0$ and interchanging x_j with x_d, if necessary, yields the result. $\square$

We now come to our central result (see also Theorem 3.34).

Theorem 6.12 (Noether normalization) *We have* $\dim A = d$ *if, and only if, there exists a finite and injective homomorphism*

$$\varphi: R_d \hookrightarrow A.$$

Proof. Only one direction needs verification. We proceed by induction on d. The case $d = 0$ being trivial, we may assume that $d \ge 1$ and that the result is true for $d - 1$. Then we may further choose coordinates $x_1, \ldots, x_d$ in R_d such that $f := \varphi(x_d)$ is active in A (Theorem 11). According to the next much more general Lemma, the induced finite homomorphism

$$\overline{\varphi}: R_{d-1} = R_d / x_d R_d \longrightarrow A / fA =: \overline{A}$$

is also injective. Hence,

$$\dim A - 1 = \dim \overline{A} = d - 1 . \qquad \square$$

Lemma 6.13 *Let* $\varphi : A \to B$ *be a finite homomorphism with* $\ker \varphi \subset \mathfrak{n}_A$ *, let* $\mathfrak{a} \subset \mathfrak{m}_A$ *be an ideal and*

$$\overline{\varphi} : \overline{A} = A / \mathfrak{a} \longrightarrow B / \mathfrak{a}B = \overline{B}$$

the associated homomorphism. Then, $\ker \overline{\varphi} \subset \mathfrak{n}_{\overline{A}}$.

Proof. Let $\overline{f}$ be an element of $\ker \overline{\varphi}$ and f a preimage of $\overline{f}$ in A. Then, $fB = \varphi(f)B \subset \mathfrak{a}B$, if B is regarded as an A–module via φ. So, setting $N = B$, $R = S = A$, $s = f$, the Dedekind Lemma 3.13 yields elements $g_1, \ldots, g_t \in \mathfrak{a}$ such that

$$f^t + g_1 f^{t-1} + \cdots + g_t \in \mathrm{An}_A B = \ker \varphi \subset \mathfrak{n}_A .$$

Calculating modulo the ideal $\mathfrak{a}$ gives $\overline{f}^{\,t} = 0$, i.e. $\overline{f} \in \mathfrak{n}_{\overline{A}}$. $\square$

6.5 Dimension and finite homomorphisms

We state in this Section various applications for finite homomorphisms. In particular, we shall see that the dimension is a function of the underlying *reduced* structure.

For an epimorphism $\varphi : A \to B$ the very definition of dimension implies $\dim B \leq \dim A$, and the dimension really drops down if $\ker \varphi$ contains active elements. We now have the following

Theorem 6.14 *For finite* $\varphi : A \to B$ *one has*

$$\dim B = \dim A / \ker \varphi \leq \dim A ,$$

hence $\dim B = \dim A$ *if* φ *is injective. On the other hand, if* $\varphi : A \to B$ *is finite,* $\dim B = \dim A$ *and* A *is an integral domain, then,* φ *is injective. Moreover,* $\dim A = \dim A / \mathfrak{a}$ *for any ideal* $\mathfrak{a} \subset \mathfrak{n} = \mathfrak{n}_A$ *; in particular,*

$$\dim A = \dim \mathrm{red}\, A .$$

Proof. Since the induced homomorphism $A / \ker \varphi \to B$ is finite, we may assume from the beginning that φ is injective. If $\dim A = 0$, then B is also a finite dimensional $\mathbb{C}$–vector space, i.e. $\dim B = 0 = \dim A$.

Suppose now that $d \geq 1$. If A is of dimension d, there exists a finite injective homomorphism $R_d \hookrightarrow A$. Composed with φ, this leads to a homomorphism $R_d \hookrightarrow B$ of the same kind, such that $\dim B = d$, too.

If $\varphi : A \to B$ is finite, A an integral domain and $\ker \varphi \neq 0$, then $\ker \varphi$ contains active elements such that $\dim A > \dim A / \ker \varphi = \dim B$.

Take a finite injective homomorphism $\varphi : R_d \to A$ and denote the composition with the projection $A \to A / \mathfrak{a}$ by $\overline{\varphi}$. If $f \in \ker \overline{\varphi}$, then $\varphi(f) \in \mathfrak{a} \subset \mathfrak{n}_A$. Hence, for some t, $\varphi(f^t) = (\varphi(f))^t = 0$ and therefore $f^t = 0$. Since R_d has no zerodivisors, $f = 0$ and $\overline{\varphi}$ is injective, too. $\square$

6.6 Lasker - Noether decomposition

The full geometric power of Rückert's Nullstellensatz, or rather its (obvious) generalization to arbitrary complex analytic algebras becomes much clearer if the Lasker–Noether *Decomposition Theorem* is brought into the play. Recall that an ideal $\mathfrak{q}$ is called *primary* if its radical $\mathrm{rad}\,\mathfrak{q}$ is a *prime* ideal $\mathfrak{p}$ (then $\mathfrak{q}$ is also called $\mathfrak{p}$–primary).

***Theorem 6.15 (Lasker - Noether)** *Any ideal $\mathfrak{a}$ in a local noetherian ring A is the intersection of finitely many primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$:*

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t .$$

Via the general rules for zero sets this so called *primary decomposition* of $\mathfrak{a}$ implies a decomposition

$$(*) \qquad \begin{aligned} N(\mathfrak{a}) &= N(\mathfrak{q}_1) \cup \ldots \cup N(\mathfrak{q}_t) = N(\mathrm{rad}\,\mathfrak{q}_1) \cup \ldots \cup N(\mathrm{rad}\,\mathfrak{q}_t) \\ &= N(\mathfrak{p}_1) \cup \ldots \cup N(\mathfrak{p}_t), \quad \mathfrak{p}_j = \mathrm{rad}\,\mathfrak{q}_j , \end{aligned}$$

and consequently,

$$\mathfrak{i}(N(\mathfrak{a})) = \mathfrak{i}(N(\mathfrak{p}_1)) \cap \ldots \cap \mathfrak{i}(N(\mathfrak{p}_t)) .$$

Since, obviously, $\mathrm{rad}\,\mathfrak{a} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_t$ it suffices to show that

$$\mathfrak{i}(N(\mathfrak{p})) = \mathfrak{p}$$

for any *prime* ideal $\mathfrak{p} \subset A$ in order to prove Rückert's Nullstellensatz for the algebra A.

To understand the geometric meaning of $(*)$, we give the following

Definition. An analytic germ X is called *irreducible* if X cannot be written in the form $X = X_1 \cup X_2$, $X_1 \not\subset X_2$, $X_2 \not\subset X_1$; otherwise X is *reducible*.

We first characterize such irreducible germs algebraically.

Lemma 6.16 *Let $\mathfrak{q} \subset A$ be an ideal in a complex analytic algebra. Then, the following are equivalent* :

i) $\mathfrak{q}$ *is primary* ;

ii) *if* $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} := \mathrm{rad}\,\mathfrak{q}$ *for ideals* $\mathfrak{a}, \mathfrak{b} \subset A$, *then* $\mathfrak{a} \subset \mathfrak{p}$ *or* $\mathfrak{b} \subset \mathfrak{p}$;

iii) *if* $N(\mathfrak{q}) \subset N(\mathfrak{a}) \cup N(\mathfrak{b})$, *then* $N(\mathfrak{q}) \subset N(\mathfrak{a})$ *or* $N(\mathfrak{q}) \subset N(\mathfrak{b})$;

vi) *if* $N(\mathfrak{q}) = N(\mathfrak{a}) \cup N(\mathfrak{b})$, *then* $N(\mathfrak{q}) = N(\mathfrak{a})$ *or* $N(\mathfrak{q}) = N(\mathfrak{b})$.

Proof. i) $\Longrightarrow$ ii). Suppose $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$. Take an arbitrary element $a \in \mathfrak{a}$ and an element $b \in \mathfrak{b} \setminus \mathfrak{p}$. Then $ab \in \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$ and thus $a \in \mathfrak{p}$.

ii) $\Longrightarrow$ iii). By assumption, $\mathfrak{i}(N(\mathfrak{a})) \cap \mathfrak{i}(N(\mathfrak{b})) \subset \mathfrak{i}(N(\mathfrak{q})) = \mathfrak{p}$ and hence, without loss of generality, $\mathrm{rad}\,\mathfrak{a} = \mathfrak{i}(N(\mathfrak{a})) \subset \mathfrak{i}(N(\mathfrak{q})) = \mathfrak{p}$. Therefore, $N(\mathfrak{q}) = N(\mathfrak{p}) \subset N(\mathrm{rad}\,\mathfrak{a}) = N(\mathfrak{a})$.

iii) $\Longleftrightarrow$ iv). If $N(\mathfrak{q}) \subset N(\mathfrak{a})$, then $N(\mathfrak{a}) \subset N(\mathfrak{a}) \cup N(\mathfrak{b}) = N(\mathfrak{q}) \subset N(\mathfrak{a})$ implies $N(\mathfrak{q}) = N(\mathfrak{a})$. If, on the other hand, $N(\mathfrak{q}) \subset N(\mathfrak{a}) \cup N(\mathfrak{b})$, then $N(\mathfrak{q}) = N(\mathfrak{q}) \cap (N(\mathfrak{a}) \cup N(\mathfrak{b})) = (N(\mathfrak{q}) \cap N(\mathfrak{a}) \cup (N(\mathfrak{q}) \cap N(\mathfrak{b})) = N(\mathfrak{a} + \mathfrak{q}) \cup N(\mathfrak{b} + \mathfrak{q})$, whence $N(\mathfrak{q}) = N(\mathfrak{a} + \mathfrak{q})$, say, such that $N(\mathfrak{q}) = N(\mathfrak{q}) \cap N(\mathfrak{a}) \subset N(\mathfrak{a})$.

iii) $\Longrightarrow$ i). It is sufficient to show that the radical $\mathfrak{p}$ of $\mathfrak{q}$ is a prime ideal. Take $f, g \in A$ with $f\,g \in \mathfrak{p}$ and set $\mathfrak{a} = fA$, $\mathfrak{b} = gA$. Then, $N(\mathfrak{p}) \subset N(fgA) \subset N(\mathfrak{a}) \cup N(\mathfrak{b})$ and, assuming $N(\mathfrak{p}) \subset N(\mathfrak{a})$, we get $f \in \mathfrak{i}(N(fA)) \subset \mathfrak{i}(N(\mathfrak{p})) = \mathfrak{p}$. □

Thus, the Lasker–Noether decomposition $(*)$ leads to a decomposition of $N(\mathfrak{a})$ into a union of finitely many irreducible germs $N(\mathfrak{p}_j)$. A prime ideal $\mathfrak{p}_j$ is called *embedded* (for $\mathfrak{a}$) if $N(\mathfrak{p}_j) \subset N(\mathfrak{p}_k)$ for some $k \neq j$, i.e. if it exists a $k \neq j$ with $\mathfrak{p}_k \subset \mathfrak{p}_j$. Clearly,

$$N(\mathfrak{a}) = N(\mathfrak{p}_1) \cup \ldots \cup N(\mathfrak{p}_s), \quad \mathfrak{p}_\sigma \text{ not embedded for } \mathfrak{a},$$

and one can show that this decomposition of the germ $N(\mathfrak{a})$ into irreducible components is *unique* (up to order). Moreover, the primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are uniquely determined by the ideal $\mathfrak{a}$, but, in general, *not* the primary ideals associated to the *embedded* prime ideals.

Suppose finally that $A = R_n/\mathfrak{a}$. Then, primary decompositions of the trivial ideal $(0) \subset A$ are in $1:1$–correspondence to primary decompositions of $\mathfrak{a}$ in R_n. Under this correspondence, the *isolated* prime ideals of A correspond to the *nonembedded* prime ideals of $\mathfrak{a}$. This remark implies

$$\dim N(\mathfrak{a}) = \max_{\substack{\mathfrak{p} \supset \mathfrak{a} \\ \text{isolated}}} \dim N(\mathfrak{p}) .$$

6.7 Krull dimension of analytic algebras

Geometrically speaking give parameter systems $f_1, \ldots, f_d$ in a d–dimensional algebra A a sequence of subspaces

$$X = X_d \supset X_{d-1} = N(f_1) \supset X_{d-2} = N(f_1, f_2) \supset \cdots \supset X_0 = N(f_1, \ldots, f_d) = \{0\}$$

with $\dim X_j = j$. So, X_{j-1} is always a *hypersurface* in X_j.

As an *Example* we regard $A = R_3/f R_3$ with $f = x_1^2 + x_2^2 - x_3^2$ and take f_1, f_2 to be the residue classes of x_1 and x_2 in A. Then, the geometric picture is the following

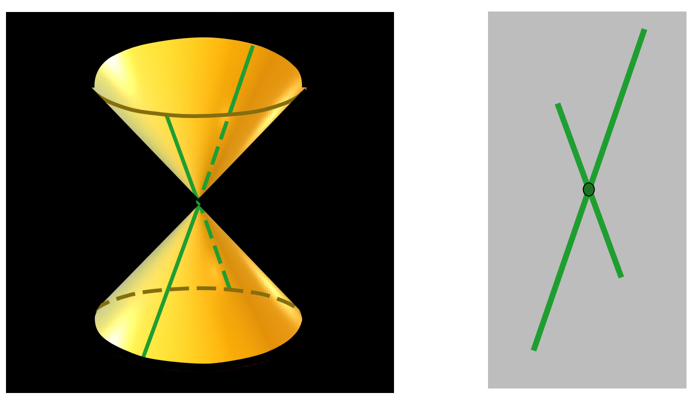


Figure 6.1

As we see, the algebras $A_{d-j} := A/(f_1, \ldots, f_j)$ may neither be integral domains nor reduced.

The geometric idea behind the Krull *dimension* is to find a maximal sequence of *irreducible* subspaces

$$X \supset X_d \supset \cdots \supset X_0 = \{0\} .$$

Here, we cannot start always with $X = X_d$ since X may not be irreducible. If we e.g. have $X \subset \mathbb{K}^3$ given by $x_3 x_1 = x_3 x_2 = 0$, and take f_1 and f_2 as above we get the sequence of pictures

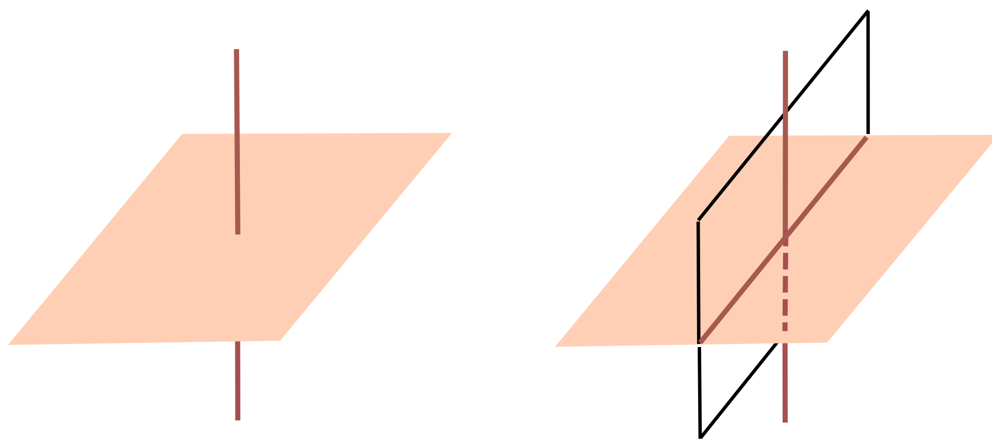


Figure 6.2

Hence, the dimension is 2, i.e. equal to the dimension of the highest dimensional *component*. It is easy to see in both examples how to find sequences of irreducible subspaces of length 2 (if one neglects the trivial last one, a point). This is no coincidence.

Definition. Let A be a noetherian local algebra. Then, a *chain* of prime ideals in A of length k is a sequence

$$\mathfrak{p}_d \subset \cdots \subset \mathfrak{p}_1 \subset \mathfrak{p}_0 = \mathfrak{m}$$

of *prime* ideals $\mathfrak{p}_j$ with $\mathfrak{p}_{j-1} \neq \mathfrak{p}_j$ for all $j = 1, \ldots, d$. The *Krull dimension* of A is the maximum of all possible chains of prime ideals.

We only state without proof the important result that the Chevalley dimension agrees with the Krull dimension for complex analytic local algebras (for complete details see loc. cit.).

***Theorem 6.17** *For complex analytic algebras $A = \mathbb{C}\langle x_1, \ldots, x_n \rangle / \mathfrak{a}$ of dimension d there exist chains of prime ideals of length d, and no chain has larger length.*

It is clear that any maximal chain must stop with an *isolated* prime ideal $\mathfrak{p}_d$. Moreover, $\dim A/\mathfrak{p}_j = j$. For no prime ideal $\mathfrak{p}$ we can have $\dim A/\mathfrak{p} > d$ (since otherwise we can construct a chain of prime ideals in A of length $> d$ ending with $\mathfrak{p}$). This consideration implies the geometrically reasonable

Corollary 6.18 $\dim A = \max\{\dim A/\mathfrak{p}, \ \mathfrak{p} \text{ isolated prime ideal}\}$.

This result gives rise to the following

Definition. A local algebra A is called *pure–dimensional* if

$$\dim A/\mathfrak{p} = \dim A$$

for *all* isolated prime ideals $\mathfrak{p}$ of A.

If A is in particular an integral domain, then, $\mathfrak{p} = 0$ is the unique isolated prime ideal and A is, consequently, pure–dimensional.

6.8 Noether normalization, once more

It is not difficult to deduce from Remark [**??**] the following result by induction.

Theorem 6.19 *Let $g_1, \ldots, g_k \in \mathfrak{m}_A$ be elements such that $A/(g_1, \ldots, g_k)\,A$ is artinian. Then, for all $c \in \mathbb{C}^{kd} \setminus H$, H a finite union of hyperplanes, the d elements*

$$f_1 = \sum_{j=1}^{r} c_{1j}\, g_j, \ldots, f_d = \sum_{j=1}^{r} c_{dj}\, g_j$$

form a system of parameters.

By this result we can prove that our definition of the dimension in Chapter 1 coincides with that given in this Chapter.

Lemma 6.20 *Let $\mathfrak{a}$ be an ideal in R_n and denote by d the smallest natural number such that there exist d linear forms*

$$\ell_j = \sum_{k=1}^{n} c_{jk}\, x_k\,, \quad j = 1, \ldots, d$$

with the property that the ideal $\mathfrak{a} + (\ell_1, \ldots, \ell_d)\, R_n$ contains a power of the maximal ideal $\mathfrak{m}_n$. Then,

$$d = \dim R_n/\,\mathfrak{a}\,.$$

Proof. Suppose that $\mathfrak{a} + (\ell_1, \ldots, \ell_k)\, R_n$ contains a power of $\mathfrak{m}_n$ for some linear forms $\ell_1, \ldots, \ell_k$. Then, we have for the residue classes $\overline{\ell}_1, \ldots, \overline{\ell}_k$ in $A = R_n/\,\mathfrak{a}$ that

$$A/\,(\overline{\ell}_1, \ldots, \overline{\ell}_k)\, A \cong R_n/\,\mathfrak{a} + (\ell_1, \ldots, \ell_k)\, R_n$$

is finite dimensional over $\mathbb{C}$. Consequently, $k \geq \dim A = \dim R_n/\,\mathfrak{a}$.

To prove the opposite direction notice that for the residue classes $\overline{x}_1, \ldots, \overline{x}_n$ in A the quotient ring $A/\,(\overline{x}_1, \ldots, \overline{x}_n)\, A$ is artinian. By the preceding Theorem, A contains a parameter system $g_1, \ldots, g_d$ consisting of linear combinations of the $\overline{x}_j$ with constant coefficients. These can be lifted to linear forms $\ell_1, \ldots, \ell_d$ in $x_1, \ldots, x_n$, and since

$$R_n/\,\mathfrak{a} + (\ell_1, \ldots, \ell_d)\, R_n \cong A/\,(g_1, \ldots, g_d)\, A$$

is artinian, the ideal $\mathfrak{a} + (\ell_1, \ldots, \ell_d)\, R_n$ contains a power of $\mathfrak{m}_n$. $\square$

Remark. Since the zero set of d (independent) linear forms is a plane of dimension $n - d$, i.e. of codimension d, this result is equivalent to the statement that $N(\mathfrak{a})$ is d-dimensional at the origin if and only if there exist planes E through 0 of dimension $n - d$ and no of higher dimension such that 0 is an isolated point of the intersection $N(\mathfrak{a}) \cap E$.

The Lemma stated above allows another interpretation. It is clear that the linear forms $\ell_1, \ldots, \ell_d$ must be linearly independent. Therefore, we can choose them as the first d of a set of new coordinates $x_1, \ldots, x_n$ of R_n. Now, in the following diagram

$$\begin{tikzcd} & R_n \arrow[d] \\ R_d \arrow[ur] \arrow[r] & A \end{tikzcd}$$

with canonical inclusion $R_d \hookrightarrow R_n$ the composition $R_d \to A$ is quasi–finite and thus finite (and necessarily injective). So, we see that not only such *Noether normalizations* $R_d \hookrightarrow A$ exist abstractly but that they are realized by "almost all" projections to d–dimensional planes where $d = \dim A$.

Theorem 6.21 *If $R_n \to A$ is an epimorphism onto an analytic algebra A of dimension d then, after generic change of variables in R_n, the natural inclusion $R_d \hookrightarrow R_n$ induces a Noether normalization $R_d \hookrightarrow A$.*

In order to illustrate this result we look again at the example of the ideal $\mathfrak{a} = (x_1 x_3, x_2 x_3)$ in R_3. The mapping $R_2 \to R_3/\mathfrak{a} = A$ given by

$$x_1 \longmapsto \overline{x}_1\,, \quad x_2 \longmapsto \overline{x}_2$$

is not finite since

$$A/(\overline{x}_1, \overline{x}_2)\,A \cong R_3/(x_1 x_3, x_2 x_3, x_1, x_2)\,R_3 \cong R_3/(x_1, x_2)\,R_3$$

is not artinian. But after an arbitrary small change of variables

$$(x_1, x_2, x_3) \longmapsto (x_1 + \varepsilon_1 x_3, x_2 + \varepsilon_2 x_3, x_3)\,, \quad \varepsilon_1 \varepsilon_2 \neq 0\,,$$

everything is alright since

$$R_3/((x_1 + \varepsilon_1 x_3)\,x_3, (x_2 + \varepsilon_2 x_3)\,x_3, x_1, x_2)\,R_3 \cong R_3/(x_1, x_2, x_3^2)\,R_3$$

is an artinian ring.

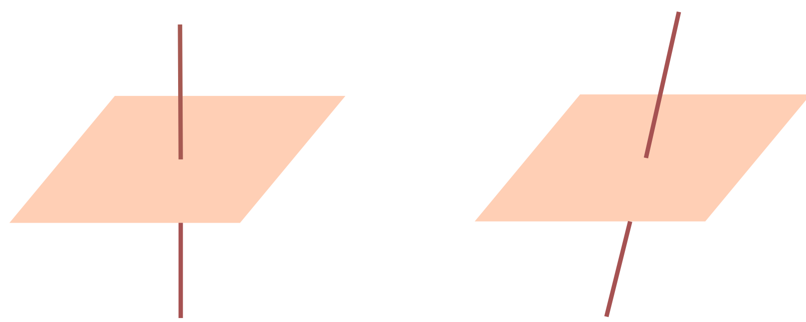


Figure 6.3

6.9 Finite algebra homomorphisms and finite holomorphic map germs

We call a holomorphic map germ between analytic sets *finite* if it has a finite representative (recall the definition for finite topological maps from Chapter xxx). Our main result is the following

Theorem 6.22 *Let* $f : N(\mathfrak{a}) \to N(\mathfrak{b})$ *be a holomorphic map germ associated to a diagram*

$$\begin{array}{ccc} R_m & \xrightarrow{\psi} & R_n \\ \downarrow & & \downarrow \\ R_m/\mathfrak{b} = B & \xrightarrow[\varphi]{} & A = R_n/\mathfrak{a} \end{array}$$

as in Chapter xx.xx. Then, the following are equivalent :

i) f *is finite* ;

ii) $f^{-1}(0)$ *is isolated in* $N(\mathfrak{a})$;

iii) $\varphi : B \to A$ *is a finite homomorphism* ;

iv) $\varphi : B \to A$ *is a quasi–finite homomorphism.*

The *proof* of this Theorem will occupy the rest of this Section. We start with showing ii) $\Longleftrightarrow$ iv). The claim that $f^{-1}(0) = N(\psi(\mathfrak{m}_m))$ is *isolated* in $N(\mathfrak{a})$ is obviously the same as to say that $N(\mathfrak{a}) \cap N(\psi(\mathfrak{m}_m)) = N(\mathfrak{a} + \psi(\mathfrak{m}_m))$ is a point. By Rückert's Nullstellensatz this is equivalent to the fact that the ideal $\mathfrak{a} + \psi(\mathfrak{m}_m)$ is of finite codimension in R_n. But this, in turn, is equivalent to $A/\mathfrak{m}_B A$ being a finite dimensional $\mathbb{C}$–vector space, i.e. to the quasi–finiteness of φ. It is also immediately clear by the definitions that i) $\Longrightarrow$ ii) and from the Quasi–finiteness Theorem that iii) $\Longleftrightarrow$ iv).

Thus, it remains to show that, e.g., iii) $\Longrightarrow$ i). This conclusion will be established in several steps. To begin with, we make a few useful comments on finite maps.

Lemma 6.23 *Let $f : X \to Y$, $g : Y \to Z$ be continuous mappings of locally compact Hausdorff spaces. Then, the following are true* :

i) *If f and g are finite, then $g \circ f$ is finite* ;

ii) *If $g \circ f$ and g are finite, then f is finite* ;

iii) *If $g \circ f$ is finite, and if f is finite and surjective, then g is finite.*

Proof. i) follows directly by the definition. For ii) we use the alternative characterization of finite maps. The only fact to be proven is that f is a *proper* map. Now, if $K \subset Y$ is compact the preimage $f^{-1}(K)$ is closed and a subset of the *compact* set $(g \circ f)^{-1}(L)$ since $L := g(K)$ is compact in Z. Consequently, $f^{-1}(K)$ is compact. For iii): By the assumptions, it is clear that all fibers of g are finite; moreover each closed set $B \subset Y$ is the image under f of the closed set $A := f^{-1}(B)$. Hence, $g(B) = (g \circ f)(A)$ is closed in Z. $\square$

Remark. Again by the second characterization of finiteness it is plain that the restriction of a finite map $f : X \to Y$ to $f^{-1}(V)$, V open in Y, is finite. This observation has the consequence that the implications of the Lemma above remain valid for map *germs*.

For the rest of the Section we say that $(*)$ is satisfied for a *finite* homomorphism $\varphi : B \to A$ if the implication iii) $\Longrightarrow$ i) is valid. If φ *factorizes* over an epimorphism $\varepsilon : B \to C$ we can reduce the proof for $(*)$ to the proof of $(*)$ for the remaining map $C \to A$ by separating our basic diagram into two parts:

$$\begin{tikzcd} R_m \arrow[r, "\mathrm{id}"] \arrow[d] & R_m \arrow[r] \arrow[d] & R_n \arrow[d] \\ B \arrow[r, "\varepsilon"'] & C \arrow[r] & A \end{tikzcd}$$

This follows from conclusion ii) in the Lemma on finite continuous maps. Indeed, if $B = R_m/\mathfrak{b}$ then there exists an ideal $\mathfrak{c} \supset \mathfrak{b}$ such that $R_m/\mathfrak{c} \cong C$ and the identity $\mathrm{id} : R_m \to R_m$ induces the commutative diagram

$$\begin{tikzcd} R_m/\mathfrak{b} = B \arrow[rr, "\varphi"] \arrow[dr] & & A = R_n/\mathfrak{a} \\ & R_m/\mathfrak{c} \cong C \arrow[ur] & \end{tikzcd}$$

which leads to a factorization $N(\mathfrak{a}) \to N(\mathfrak{c}) \subset N(\mathfrak{b})$. Since $N(\mathfrak{c})$ is a closed subset of $N(\mathfrak{b})$, the inclusion is clearly finite. $\square$

As a Corollary, we get

Corollary 6.24 $(*)$ *holds for any epimorphism* $\varphi : B \to A$.

Proof. Exploiting the same diagram as above, we can take $C = A$ and $\varepsilon = \varphi$. By Lemma 02.xx, we know that the map $N(\mathfrak{a}) \to N(\mathfrak{c})$ is a homeomorphism, hence finite. $\square$

After dividing out the kernel of φ we are reduced via part i) of the preceding Lemma to the case where φ is *injective* (but, as we will see, for this we need again the result for general *surjective* homomorphisms). Using Noether normalization $R_d \hookrightarrow B$ and invoking part ii) of the preceding Lemma again, we are immediately reduced to the case $B = R_d \hookrightarrow A$. Thus, we see that our Theorem follows from its

Corollary 6.25 *For a Noether normalization* $R_d \longrightarrow A$ *the corresponding map germ* $N(\mathfrak{a}) \to (\mathbb{C}^d)_0$ *is finite.*

Remark. It is not too difficult to show that a representative of $N(\mathfrak{a}) \to (\mathbb{C}^d)_0$ is in fact outside a lower dimensional analytic subset of $(\mathbb{C}^d)_0$ an unbranched holomorphic covering (see the next Section). If A is normal, this covering is even connected.

Proof of Corollary. Finiteness of the homomorphism $\varphi : R_d \hookrightarrow A = R_n / \mathfrak{a}$ implies that each residue class $f_j = \overline{y}_j \in A$, $j = 1, \ldots, n$, is algebraic over R_d where $y_1, \ldots, y_n$ denote a holomorphic coordinate system for R_n. Therefore, we find Weierstraß polynomials $\omega_j = \omega_j(x, Y_j) \in R_d[Y_j]$ such that $\omega_j(x, f_j) = 0$ or, equivalently, $\omega_j(x, y_j) \in \mathfrak{a} \cdot R_{d+n}$. The canonical factorization $R_d \hookrightarrow R_{d+n} \to R_n$ of a lifting $R_d \to R_n$ of φ then yields a factorization of the diagram we started with:

$$\begin{tikzcd}
R_d \arrow[r] \arrow[d, "\mathrm{id}"'] & R_{d+n} \arrow[r] \arrow[d] & R_n \arrow[d] \\
R_d \arrow[r] & R_{d+n}/(\omega_1, \ldots, \omega_n) \arrow[r] & A
\end{tikzcd}$$

in which the arrow on the lower right hand side is surjective by construction. So, we are finally reduced to the homomorphism

$$R_d \longrightarrow R_{d+n}/(\omega_1, \ldots, \omega_n)\, R_{d+n}$$

which is clearly (quasi–) finite. Geometrically, this homomorphism corresponds to the restriction of the projection $\mathbb{C}^d \times \mathbb{C}^n \to \mathbb{C}^d$ to

$$\Omega := \{ (x, y_1, \ldots, y_n) : \omega_j(x, y_j) = 0 ,\ j = 1, \ldots, n \} \subset (\mathbb{C}^d \times \mathbb{C}^n)_0$$

which is indeed (in a neighborhood of $0 \in \mathbb{C}^d$) a finite map (a straightforward generalization of the case of Weierstraß hypersurfaces). $\square$

6.10 Topological properties of Noether normalization

For a Noether normalization $R_d \longrightarrow A$ the corresponding map germ $N(\mathfrak{a}) \to (\mathbb{C}^d)_0$ is finite. We are going to get more information concerning the topological properties of this finite map.

We first apply former considerations in Chapter xx.xx to maps $\varphi : R_m \to A$, in particular to the Noether normalization, by exploiting the following diagram

$$\begin{tikzcd} & R_n \arrow[d] \\ R_m \arrow[ur] \arrow[r] \arrow[d] & A = R_n/\,\mathfrak{a} \\ R_m/\ \ker\varphi \arrow[ur] & \end{tikzcd}$$

Then, $f(N(\mathfrak{a})) \subset N(\ker\varphi) \subset N(0) = (\mathbb{C}^m)_0$, and $N(\ker\varphi)$ is a *proper* analytic set germ of $(\mathbb{C}^m)_0$ if φ is not injective. Of course, this occurs for the composition

$$R_d \hookrightarrow A \longrightarrow A/\,\mathfrak{p}\ ,$$

where $R_d \hookrightarrow A$ is finite (such that $d = \dim A$) and $\mathfrak{p}$ is an isolated prime ideal of A such that $\dim A/\,\mathfrak{p} < d$.

Corollary 6.26 *Under a Noether normalization homomorphism $R_d \hookrightarrow A = R_n/\,\mathfrak{a}$, lower dimensional components of $N(\mathfrak{a})$ are mapped into lower dimensional analytic germs of $(\mathbb{C}^d)_0$.*

Remarks. 1. In fact, the lower dimensional components are mapped *onto* such analytic germs.

2. By the preceding Corollary, understanding a Noether normalization *topologically* "up to analytic germs of lower dimension" in $(\mathbb{C}^d)_0$ depends only on the components of top dimension d. Moreover, it is geometrically obvious that intersections $N(\mathfrak{p}_1) \cap N(\mathfrak{p}_2)$ of two different top–dimensional components are also of dimension $< d$. Thus, one must study only *one* component of dimension d outside suitable hypersurfaces in $\mathbb{C}^d$, and this can be seen to be an *unbranched covering* (c.f. below).

We are going to make these Remarks on the "topological" properties of a Noether normalization $R_d \hookrightarrow A$ more precise. If $\mathfrak{p} \subset A$ denotes an isolated prime ideal with $\dim A/\,\mathfrak{p} = d$, the given Noether normalization induces also a Noether normalization of $A/\,\mathfrak{p}$ by composition: $R_d \hookrightarrow A \to A/\,\mathfrak{p}$. If $\mathfrak{a} \subset A$ is any other ideal such that $\dim A/\,\mathfrak{p} + \mathfrak{a} = d$ the canonical epimorphism

$$A/\,\mathfrak{p} \longrightarrow A/\,\mathfrak{p} + \mathfrak{a}$$

is injective due to Lemma xxx and henceforth an isomorphism, i.e. $\mathfrak{p} = \mathfrak{p} + \mathfrak{a} \supset \mathfrak{a}$. In particular, if $\mathfrak{p}'$ is a second prime ideal with the property stated above, but different from $\mathfrak{p}$, the composition

$$R_d \longrightarrow A/\,\mathfrak{p} + \mathfrak{p}'$$

is *not* injective such that $N(\mathfrak{p}) \cap N(\mathfrak{p}') = N(\mathfrak{p} + \mathfrak{p}')$ is again mapped into a lower dimensional analytic germ of $(\mathbb{C}^d)_0$.

Thus, to understand the finite map germ $N(\mathfrak{a}) \to (\mathbb{C}^d)_0$ "up to codimension 1", we can restrict ourselves to the case where A is an *integral domain* in which we can exploit general results of the theory of finite field extensions by going over to the respective quotient fields.

6.11 Existence of universal denominators

A little more generally we regard in this Section a finite injective homomorphism $\varphi : B \hookrightarrow A$ of integral domains A and B and denote by K and L the quotient fields of A and B, resp.. Clearly, φ induces an embedding

$$L \hookrightarrow K\ .$$

If g is any nontrivial element in A, it satisfies an equation $g^\ell + b_1 g^{\ell-1} + \cdots + b_\ell = 0$, $b_\lambda \in B$. If ℓ is chosen to be minimal, we have $b_\ell \neq 0$ since B has no zerodivisors. Therefore, to each $g \in A$ we find a *nontrivial* $h \in A$ with $g\,h \in B$; in particular, each element in K can be written in the form f/g with $f \in A$, $g \in B$. Hence, $K = A \cdot L$, and elements $f_1, \ldots, f_r \in m_A$ generating A as an *algebra* over B also generate K over L. In other words: the field K is a *finite algebraic extension* of L.

Since we are working in characteristic zero we can now apply the *Theorem of Primitive Elements*: there exists $\widetilde{\vartheta} \in K$ such that K is the simple extension of L associating $\widetilde{\vartheta}$; in symbols

$$K = L\,[\,\widetilde{\vartheta}\,]\,.$$

Moreover, $\widetilde{\vartheta}$ can be chosen to be a "generic" linear combination of any set of generators.

Remark. If we regard a Noether normalization coming from a generic projection

$$\begin{tikzcd} & R_n \arrow[d] \\ R_d \arrow[ur] \arrow[r] & A \end{tikzcd}$$

then the elements $\overline{x}_{d+1}, \ldots, \overline{x}_n \in A$ form a set of algebra generators. Hence, a primitive element $\widetilde{\vartheta}$ is given by the residue class of a suitably chosen linear combination $c_{d+1}x_{d+1} + \cdots + c_n x_n$. After a linear change of coordinates we can even assume that this is *equal* to x_{d+1}.

Corollary 6.27 *If $A = R_n/\mathfrak{p}$, $\mathfrak{p}$ a prime ideal different from zero, then, after generic linear coordinate change of R_n, the canonical embedding $R_d \hookrightarrow R_n$ induces a Noether normalization of A such that $Q\,(A) = Q\,(R_d)\,[\,\overline{x}_{d+1}\,]$.*

As a next step we prove the existence of *universal denominators*.

Lemma 6.28 *If $B \hookrightarrow A$ is a finite homomorphism of integral domains, then there exists a nontrivial element $\Delta \in m_B$ and an element $\vartheta \in A$ such that*

$$\Delta\,A \subset B\,[\,\vartheta\,] \subset A\,.$$

Proof. Let $\widetilde{Q} = Y^m + q_1 Y^{m-1} + \cdots + q_m \in L\,[\,Y\,]$ be the *minimal* polynomial of a primitive element $\widetilde{\vartheta}$, i.e. $\widetilde{Q}\,(\widetilde{\vartheta}) = 0$, m minimal, and suppose $\widetilde{\vartheta} \in A$. Then,

$$\widetilde{P}\,(Y) := b\,\widetilde{Q}\,(Y) \in B\,[\,Y\,]\,,$$

if b denotes the product of the denominators of $q_1, \ldots, q_m \in L = Q\,(B)$. Clearly, $\widetilde{P}\,(\widetilde{\vartheta}) = 0$, and $\widetilde{P}$ is indecomposable (otherwise, $\widetilde{Q}$ would not be minimal). Define now $\vartheta = b\widetilde{\vartheta}$ and

$$P\,(Y) := b^{m-1}\,\widetilde{P}\,(Y/\,b)\,.$$

Then, P is a monic polynomial in $B\,[\,Y\,]$ which is minimal for ϑ as an element of $L\,[\,Y\,]$. In particular, P is *indecomposable*. Let now $f_1, \ldots, f_r$ generate the B*–module* A. Since $A \subset K = L\,[\,\vartheta\,]$ we find polynomials $P_j \in B\,[\,Y\,]$ and nontrivial elements $\delta_j \in B$ such that

$$f_j = P_j(\vartheta)/\,\delta_j\,, \quad j = 1, \ldots, r\,.$$

Therefore, $\Delta := \delta_1 \cdot \ldots \cdot \delta_r \neq 0$ and $\Delta\,f_j \in B\,[\,\vartheta\,]$ which implies $\Delta\,A \subset B\,[\,\vartheta\,]$. $\square$

Applying the preceding Lemma to a Noether normalization $R_d \to A$, A an integral domain, and the associated holomorphic map germ $N\,(\mathfrak{a}) \to (\mathbb{C}^d)_0$, the following is more or less an immediate consequence (for more details, cf. Chapter xxx).

Corollary 6.29 *There are arbitrarily small connected neighborhoods* $V = V(0) \subset \mathbb{C}^d$ *and representatives* X *of* $N(\mathfrak{a})$ *over* V *such that the following are satisfied* :

i) $\Delta \in \mathcal{O}(V)$;

ii) *over* $V_0 := V \setminus \{ x \in V : \Delta(x) = 0 \}$ *the map*

$$\pi : X_0 = \pi^{-1}(V_0) \longrightarrow V_0$$

is a (holomorphic) covering. In particular, π *is open at the origin, i.e. there are arbitrary small neighborhoods* U *of* $0 \in N(\mathfrak{a})$ *such that* $\pi(U)$ *is a neighborhood of* $0 \in \mathbb{C}^d$.

We leave it as an Exercise to the reader to conclude the following.

Theorem 6.30 *Let* A *be an analytic subset of an open set* $G \subset \mathbb{C}^n$ *. Then, the* dimension function $A \ni x \mapsto \dim_x A$ *is* upper semicontinuous, *i.e.*

$$\dim_x A \le \dim_{x^{(0)}} A$$

for all x *in a neighborhood* $U \cap A$ *of* $x^{(0)}$.

For each top–dimensional component A_j *of* A *at* $x^{(0)}$ *, the dimension* $\dim_x A_j$ *is* constant *in a neighborhood of* $x^{(0)}$.

6.12 Proof of Rückert's Nullstellensatz

Recall that Rückert's Nullstellensatz is equivalent to the statement

$$(+) \qquad \mathfrak{i}(N(\mathfrak{p})) = \mathfrak{p}$$

for all prime ideals $\mathfrak{p} \subset R_n$ if we take the Lasker–Noether decomposition for granted. The purpose of this Section is a proof of (+) and a short sketch for a somewhat weaker formulation and derivation of the Lasker–Noether decomposition.

In the following, we always refer to a Noether normalization $R_d \hookrightarrow A = R_n/\mathfrak{p}$, $\mathfrak{p}$ a prime ideal, induced by a substitution homomorphism $R_d \hookrightarrow R_n$ with (necessarily) $\mathfrak{p} \cap R_d = (0)$. - We start with an easy

Lemma 6.31 *For all* $f \in R_n$ *there exist* $g \in (R_n \setminus \mathfrak{p})$ *and* $h \in R_d$ *such that*

$$g\,f + h \in \mathfrak{p} .$$

Proof. Since A is a finite R_d–homomorphism, we have a relation

$$f^m + \sum_{j=0}^{m-1} a_j f^j \in \mathfrak{p} , \quad a_j \in R_d .$$

If m is minimal, $a_0 \neq 0$ since $\mathfrak{p}$ is prime. So, it suffices to set

$$g = f^{m-1} + \sum_{j=1}^{m-1} a_j f^{j-1} \text{ and } h = a_0 . \qquad \square$$

Corollary 6.32 *For all prime ideals* $\mathfrak{p} \subset R_d$ *we have* $\mathfrak{i}(N(\mathfrak{p})) = \mathfrak{p}$.

Proof. We know that $\mathfrak{p} \subset \mathfrak{i}(N(\mathfrak{p}))$. So, let $f \in \mathfrak{i}(N(\mathfrak{p}))$. Using a decomposition $g\,f + h \in \mathfrak{p}$ as above, we conclude that $h \in \mathfrak{i}(N(\mathfrak{p})) \cap R_d$. Due to the Corollary at the end of the preceding Section, $h \equiv 0$ outside the *discriminant set* $\{ \Delta(x) = 0 \} \subset V \subset \mathbb{C}^d$. Therefore, $h = 0$, and $g\,f \in \mathfrak{p}$, $g \notin \mathfrak{p}$ implies $f \in \mathfrak{p}$. Hence, $\mathfrak{i}(N(\mathfrak{p})) \subset \mathfrak{p}$. $\square$

If we do not want to use the general Lasker–Noether decomposition Theorem, we can proceed along the following lines. First prove that $N(\mathfrak{a}) = N(\operatorname{rad}\mathfrak{a})$ is indecomposable if and only if $\operatorname{rad}\mathfrak{a}$ is prime, i.e. $\mathfrak{a}$ is primary, without using Rückert's Nullstellensatz. Secondly, by the Noether property of R_n we immediately derive that *descending* chains of zero sets

$$N(\mathfrak{a}_0) \supset N(\mathfrak{a}_1) \supset \cdots$$

must become *stationary*. From this it is easy to conclude that

$$(**) \qquad N(\mathfrak{a}) = A_1 \cup \cdots \cup A_r$$

for any ideal $\mathfrak{a} \subset R_n$ where the analytic set germs $A_j = N(\mathfrak{a}_j)$ are indecomposable such that $\mathfrak{q}$ is primary. If we put $\mathfrak{p}_j = \operatorname{rad}\mathfrak{q}_j$ we get

$$\mathfrak{i}(N(\mathfrak{a})) = \mathfrak{i}(A_1) \cap \cdots \cap \mathfrak{i}(A_r) = \bigcap_{j=1}^{r} \mathfrak{i}(N(\mathfrak{q}_j)) = \bigcap_{j=1}^{r} \mathfrak{i}(N(\mathfrak{p}_j)) = \bigcap_{j=1}^{r} \mathfrak{p}_j .$$

Corollary 6.33 *For any ideal $\mathfrak{a} \subset \mathfrak{m}_n \subset R_n$, we have a decomposition*

$$\mathfrak{i}(N(\mathfrak{a})) = \bigcap_{j=1}^{r} \mathfrak{p}_j$$

with prime ideals $\mathfrak{p}_j$.

Remark. Since $\mathfrak{a} \subset \mathfrak{i}(N(\mathfrak{a}))$, it is clear that the set $\{\mathfrak{p}_j : 1 \le j \le r\}$ consists of all prime ideals $\mathfrak{p} \subset R_n$ satisfying $\mathfrak{a} \subset \mathfrak{p}$, i.e. of all *isolated* prime ideals of $\mathfrak{a}$.

6.13 Multiplicity and degree

The multiplicity of a function germ f can be interpreted as an invariant of the projective tangent cone $T_{X,0}$ for $X = \{f(x) = 0\}$. We will use this fact to define the multiplicity of a general singularity in a similar vein.

We have seen above that for a function germ f of multiplicity m there is an identity

$$T_{X,0} = \{x = [x_1 : \cdots : x_n] \in \mathbb{P}_{n-1} : f_m(x) = 0\},$$

where $f_m = \operatorname{in}(f)$ is a homogeneous polynomial of degree m in n variables. The number m associated to the projective algebraic hypersurface $T_{X,0} \subset \mathbb{P}_{n-1}$ has a simple algebro–geometric interpretation: For most lines $\ell \subset \mathbb{P}_{n-1}$, the restriction $f_{m|\ell}$ is a nonzero polynomial of degree m in one variable such that its number of roots counted with multiplicities equals m.

This is a special example of what is called the *degree* $d = \deg Y$ of a variety $Y \subset \mathbb{P}_N$. In the general situation, one regards, if V is of dimension r, say, all r–codimensional linear spaces in $\mathbb{P}_N$ which form the *Graßmann manifold*

$$\operatorname{Grass}(N - r, \mathbb{P}_N).$$

One can show that the *generic* r–codimensional linear space L intersects the underlying set of Y in finitely many points $y^{(1)}, \ldots, y^{(s)}$. (Here, generic means that all such linear spaces are parametrized by an open dense subset of the manifold $\operatorname{Grass}(N - r, \mathbb{P}_N)$). If $Y = N(F_1, \ldots, F_t)$ with homogeneous polynomials F_j and if $x_1, \ldots, x_N$ are local coordinates near a point $y^{(k)}$ such that L can be described by $\{x_1 = \cdots = x_r = 0\}$, then the ring

$$\mathbb{C}\langle x_{r+1}, \ldots, x_N\rangle / (F_1(0, \ldots, 0, x_{r+1}, \ldots, x_N), \ldots, F_t(0, \ldots, 0, x_{r+1}, \ldots, x_N))$$

is artenian by assumption. We call its $\mathbb{C}$–vector space dimension the *multiplicity* of the intersection $Y \cap L$ at $y^{(k)}$, in symbols:

$$\operatorname{mult}_{y^{(k)}}(Y \cap L).$$

Finally, the total number

$$\sum_{k=1}^{s} \mathrm{mult}_{y^{(k)}}(Y \cap L)$$

can be shown to be constant for generic $L \in \mathrm{Grass}\,(N - r, \mathbb{P}_N)$. Its value is by definition the degree $\deg Y$ of the variety Y. By an easy exercise the reader can convince himself that this definition generalizes the old one for hypersurfaces. One should also notice that the number $\deg Y$ does only depend on the top–dimensional irreducible components of Y.

By definition, the *multiplicity* $\mathrm{mult}_{x^{(0)}}$ of a complex space X at a point $x^{(0)}$ is given by the degree of the projective tangential cone:

$$\mathrm{mult}_0 X = \deg T_{X,0} .$$

Sometimes, a germ $(X, x^{(0)})$ is called *simple* (with respect to multiplicity) if $\mathrm{mult}_{x_0} X = 1$. Of course, regular points are simple in this sense. The converse is also true for normal surface singularities (see Section 7). Similarly, the terms *double point, triple point* etc. refer to singular points of multiplicity $2, 3$, etc.

There exists an extensive literature concerning the *algebraic* theory of degree and multiplicity which we briefly recall in the following.

If A denotes the local ring $\mathcal{O}_{\mathbb{C}^n,0}/I_0 =: R_n/\mathfrak{a}$ of $X = N(I)$ at 0, then the homogeneous coordinate ring $\overline{A}$ of $T_{X,0}$ is isomorphic to the algebra $S_n/\,\mathrm{in}(\mathfrak{a})$, where $S_n = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathrm{in}(\mathfrak{a})$ denotes the ideal generated in S_n by all initial forms $\mathrm{in}(f)$, $f \in \mathfrak{a}$. Certainly, if $\mathfrak{m} = \mathfrak{m}_A$, then

$$\overline{A} = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1} .$$

$\overline{A}$ is a graded ring, elements in $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ being of grade i, and $\overline{A}$ will be generated as an $A_0 = A/\mathfrak{m} = \mathbb{C}$–algebra by elements of grade 1. In particular, $\overline{A}$ is a $\mathbb{C}[x_1, \ldots, x_n]$–module.

Even more generally, one has the following existence theorem for the *Hilbert polynomial*:

***Theorem 6.34** *Let M be a finitely generated graded module over the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, i.e.*

$$M = \bigoplus_{k \geq k_0} M_k$$

with submodules $M_k \subset M$ satisfying $f \cdot M_k \subset M_{k+\ell}$ for all $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree ℓ. Then there exists a polynomial $H_M(t)$ of degree $\leq n - 1$ with rational coefficients, such that

$$H_M(k) = \dim_{\mathbb{C}} M_k$$

for all sufficiently large k.

If $Y \subset \mathbb{P}_{n-1}$ is a projective algebraic variety of dimension r and degree d, then the polynomial $H_{\overline{A}}$ of its homogeneous coordinate ring $\overline{A}$ (regarded as $\mathbb{C}[x_0, \ldots, x_{n-1}]$–module) has the form

$$H_{\overline{A}}(t) = d \cdot \frac{t^r}{r!} + \text{ lower order terms} .$$

6.14 The Artin-Rees Lemma and Krull's Intersection Theorem

In the following let A be a noetherian ring with a distinguished ideal $\mathfrak{q}$ and M a finitely generated A–module with a fixed *filtration*

$$(M_j) \qquad M = M_0 \supset M_1 \supset \cdots .$$

We call this a $\mathfrak{q}$–*filtration* if $\mathfrak{q}\, M_j \subset M_{j+1}$ for all $j \in \mathbb{N}$. It is called $\mathfrak{q}$–*stable* if $\mathfrak{q}\, M_j = M_{j+1}$ for sufficiently large j. Since we fix the ideal $\mathfrak{q}$ and the filtration we write

$$\mathrm{Gr}\, A = \mathrm{Gr}\,(\mathfrak{q}, A) = \bigoplus_{j=0}^{\infty} \mathfrak{q}^j/\mathfrak{q}^{j+1}$$

and

$$\mathrm{Gr}\ M \;=\; \bigoplus_{j=0}^{\infty} M_j / M_{j+1}\ .$$

When (M_j) is a $\mathfrak{q}$–filtration, $\mathrm{Gr}\ M$ is a $\mathrm{Gr}\ A$–module in a natural way.

Lemma 6.35 *If the filtration (M_j) is $\mathfrak{q}$–stable, then* $\mathrm{Gr}\ M$ *is a finitely generated module over* $\mathrm{Gr}\ A$.

Proof. Suppose that $\mathfrak{q}\, M_j \;=\; M_{j+1}$ for all $j \;\geq\; n$. Since

$$(\mathfrak{q}/\,\mathfrak{q}^2)\,(M_j/\,M_{j+1}) \;=\; M_{j+1}/\,M_{j+2} \ \text{ for } j \;\geq\; n\ ,$$

the union of any finite sets of generators of the modules $M_j/\,M_{j+1}$, $0 \;\leq\; j \;\leq\; n$, will generate $\mathrm{Gr}\ M$ over $\mathrm{Gr}\ A$. □

We now introduce the *blowup algebra* of $\mathfrak{q}$ in A:

$$B_{\mathfrak{q}}A \;:=\; A \oplus \mathfrak{q} \oplus \mathfrak{q}^2 \oplus \cdots$$

which can be regarded as an A–subalgebra of the polynomial ring $A\,[\,t\,]$. Notice that

$$B_{\mathfrak{q}}A/\,\mathfrak{q}\,B_{\mathfrak{q}}A \;\cong\; \mathrm{Gr}\ A$$

and

$$B\,M \;:=\; M_0 \oplus M_1 \oplus \cdots$$

becomes a graded module over $B_{\mathfrak{q}}A$.

Lemma 6.36 *The $\mathfrak{q}$–filtration (M_j) is stable if and only if the $B_{\mathfrak{q}}A$–module $B\,M$ is finitely generated.*

Proof. Assuming stability of the $\mathfrak{q}$–filtration yields $\mathfrak{q}^j M_n \;=\; M_{n+j}$ for $j \;\geq\; 0$. Then, $B\,M$ is generated by the union of sets of generators for $M_0, \ldots, M_n$.

Conversely, let $B\,M$ be a finitely generated $B_{\mathfrak{q}}A$–module. Then, any finite set of generators is contained in the direct sum of the first n terms for some n. Replacing them by their homogeneous components, we infer that $B\,M$ is generated by elements in $M_0, \ldots, M_n$. Therefore,

$$M_n \oplus M_{n+1} \oplus \cdots$$

is generated as a $B_{\mathfrak{q}}A$–module by M_n. This implies $M_{n+j} \;=\; \mathfrak{q}^j M_n$ for all $j \;\geq\; 0$, so (M_j) is $\mathfrak{q}$–stable. □

We now formulate and prove the Artin–Rees *Lemma*.

Lemma 6.37 *Let A be a noetherian ring, $\mathfrak{q} \subset A$ an ideal, M a finitely generated A–module, $M' \subset M$ a submodule. If (M_j) is a $\mathfrak{q}$–stable filtration of M, then the induced filtration $(M'_j \;=\; M_j \cap M')$ is also $\mathfrak{q}$–stable. In other words: there exists a number n such that*

$$M_{j+n} \cap M' \;=\; \mathfrak{q}^j (M_n \cap M')\ , \quad j \;\geq\; 0\ .$$

Proof. By the Hilbert Basis Theorem, $B_{\mathfrak{q}}A$ is a finitely generated A–algebra as a subalgebra of $A\,[\,t\,]$, hence noetherian (for more details, see below). Then, $B\,M'$ is, as a submodule of the *noetherian* $\mathfrak{q}A$–module $B\,M$ (use the preceding Lemma), finitely generated. Invoking the Lemma again gives the result. □

Krull's *Intersection Theorem* is an easy consequence of the Artin–Rees Lemma: Take a local noetherian ring A, a proper ideal $\mathfrak{q} \subset \mathfrak{m}_A$ and a finitely generated A–module M. Define a $\mathfrak{q}$–stable filtration (M_j) by $M_j \;:=\; \mathfrak{q}^j M$ and set $M' \;=\; \bigcap_{j=1}^{\infty} \mathfrak{q}^j M$. Due to the Artin–Rees Lemma,

$$M' \;=\; M_{n+1} \cap M' \;=\; \mathfrak{q}\,(M_n \cap M') \;=\; \mathfrak{q}\,M' \subset \mathfrak{m}_A M'\ ,$$

and Nakayama's Lemma implies $M' \;=\; 0$, i.e.

$$\bigcap_{j=1}^{\infty} \mathfrak{q}^j M = 0 . \qquad \square$$

Besides Hilbert's Basis Theorem itself which we do not want to prove we used a Corollary of it which can be stated as follows:

Corollary 6.38 *If A is a noetherian ring and B a finitely generated A–algebra, then B is noetherian.*

Proof. Since B is a finitely generated A–module there exists an A–algebra epimorphism $A[x_1, \ldots, x_t] \to B$. By Hilbert's Theorem,

$$A[x_1, \ldots, x_t] = A[x_1, \ldots, x_{t-1}][x_t]$$

is noetherian, and epimorphic images of noetherian rings are again noetherian. $\square$

6.15 Multiplicities of local rings with respect to $\mathfrak{m}$ - primary ideals

Now we change our point of view and some notations slightly. In the local algebra A (of dimension d), we take an $\mathfrak{m}$–primary ideal $\mathfrak{q}$, i.e. an ideal $\mathfrak{q}$ satisfying $\mathfrak{m}^\ell \subset \mathfrak{q} \subset \mathfrak{m}$ for some ℓ or, equivalently, $\operatorname{rad} \mathfrak{q} = \mathfrak{m}$. Then again, the vector spaces $\mathfrak{q}^i / \mathfrak{q}^{i+1}$ are finite dimensional, and we can form the graded ring

$$\operatorname{Gr}(\mathfrak{q}, A) = \bigoplus_{i=0}^{\infty} \mathfrak{q}^i / \mathfrak{q}^{i+1} ,$$

thus generalizing the definition of $\overline{A} = \operatorname{Gr}(\mathfrak{m}, A)$.

Using Theorem 2, it is easy to see that there exists a polynomial

$$H_{A,\mathfrak{q}}(t) = m(\mathfrak{q}) \cdot \frac{t^d}{d!} + \text{lower order terms} ,$$

such that

$$H_{A,\mathfrak{q}}(i) = \dim_{\mathbb{C}} A / \mathfrak{q}^i , \quad i >> 0 .$$

We call the number $\operatorname{mult}(\mathfrak{q}, A) := m(\mathfrak{q})$ the *multiplicity* of the local analytic algebra A with respect to the $\mathfrak{m}$–primary ideal $\mathfrak{q}$. Of course, $\operatorname{mult}(\mathfrak{m}(\mathcal{O}_{X,x}), \mathcal{O}_{X,x})$ is the multiplicity of X at x.

In the following, we want to give a geometric interpretation of the multiplicity by comparing $\operatorname{mult}(\mathfrak{m}, A)$ and $\operatorname{mult}(\mathfrak{q}, A)$ for certain ideals $\mathfrak{q}$. We proceed by induction on the dimension of A. Recall that for two ideals $\mathfrak{a}$, $\mathfrak{b} \subset A$ the symbol $(\mathfrak{a} : \mathfrak{b})$ denotes the *ideal quotient*

$$\{ x \in A : x\, \mathfrak{b} \subset \mathfrak{a} \} .$$

In particular, $(0 : \mathfrak{b}) = \operatorname{Ann} \mathfrak{b}$.

Lemma 6.39 *Let $\mathfrak{q} \subset A$ be an $\mathfrak{m}$–primary ideal, $x \in \mathfrak{q}$. Then*

$$H_{A/xA, \mathfrak{q}/x\mathfrak{q}}(i) = H_{A,\mathfrak{q}}(i) - \dim_{\mathbb{C}}(A / (\mathfrak{q}^i : x A)) .$$

Proof. By the canonical isomorphism

$$(A / x A) / (\mathfrak{q} / x\, \mathfrak{q})^i \cong A / (\mathfrak{q}^i + x A) ,$$

we get the identity

$$H_{A,\mathfrak{q}}(i) - H_{A/xA, \mathfrak{q}/x\mathfrak{q}}(i) = \dim_{\mathbb{C}}(A / \mathfrak{q}^i) - \dim_{\mathbb{C}}(A / (\mathfrak{q}^i + x A)) .$$

The exact sequence of Artin rings

$$0 \longrightarrow (\mathfrak{q}^i + x\,A)/\,\mathfrak{q}^i \longrightarrow A/\,\mathfrak{q}^i \longrightarrow A/\,(\mathfrak{q}^i + x\,A) \longrightarrow 0$$

implies that the difference in question is in fact equal to

$$\begin{aligned}\dim_{\mathbb{C}}((\mathfrak{q}^i + x\,A)/\,\mathfrak{q}^i) &= \dim_{\mathbb{C}}(x\,A/\,(\mathfrak{q}^i \cap x\,A)) \\ &= \dim_{\mathbb{C}}(x\,A/\,x\,A\,(\mathfrak{q}^i : x\,A)) \\ &= \dim_{\mathbb{C}}(A/\,(\mathfrak{q}^i : x\,A)) \quad .\end{aligned}$$

We call an element $x \in \mathfrak{q} \setminus \mathfrak{q}^2$ to be of *good reduction* (with respect to $\mathfrak{q}$), if there exists a natural number c such that

$$(\mathfrak{q}^i : x\,A) \cap \mathfrak{q}^c = \mathfrak{q}^{i-1}$$

for sufficiently large i.

Lemma 6.40 *Let A, $\mathfrak{m}$, $\mathfrak{q}$ be given as above, and let x be of good reduction with respect to $\mathfrak{q}$. Then, with $H = H_{A,\mathfrak{q}}$, $\overline{H} = H_{A/xA,\mathfrak{q}/x\mathfrak{q}}$, there exist inequalities:*

$$H\,(i) - H\,(i-1) \le \overline{H}\,(i) \le H\,(i) - H\,(i-1) + H\,(c)\,.$$

Proof. Since x has good reduction, it exists a number $c \in \mathbb{N}$ such that

$$\begin{aligned}\dim_{\mathbb{C}}((\mathfrak{q}^i : x\,A)/\,\mathfrak{q}^{i-1}) &= \dim_{\mathbb{C}}((\mathfrak{q}^i : x\,A)/\,(\mathfrak{q}^i : x\,A) \cap \mathfrak{q}^c) \\ &= \dim_{\mathbb{C}}((\mathfrak{q}^c + (\mathfrak{q}^i : x\,A))/\,\mathfrak{q}^c) \\ &\le \dim_{\mathbb{C}}(A/\,\mathfrak{q}^c) = H\,(c)\,.\end{aligned}$$

Hence, $0 \le H\,(i-1) - \dim_{\mathbb{C}}(A/\,(\mathfrak{q}^i : x\,A)) \le H\,(c)$, and the result follows from Lemma 3. □

Let us remark that an element $x \in \mathfrak{q} \setminus \mathfrak{q}^2$ has good reduction, if the leading term of x in $\mathrm{Gr}\,(\mathfrak{q}, A)$ has only zerodivisors of bounded degree. In this case, $\mathrm{mult}\,(\mathfrak{q}, A) = \mathrm{mult}\,(\mathfrak{q}/\,x\mathfrak{q}, A/\,x\,A)$. The situation is much better, if $\mathrm{Gr}\,(\mathfrak{q}, A)$ is a Cohen–Macaulay ring (see Chapter 13.2). In that case, we have more precisely the equality

$$H_{A,\mathfrak{q}}(i) = \sum_{j=0}^{i} H_{A/xA,\mathfrak{q}/x\mathfrak{q}}(j)\,.$$

If x has good reduction, then $\dim A/\,x\,A = \dim A - 1$.

We want to proceed by induction with the help of elements of good reduction. For this purpose we need an Existence Theorem.

Theorem 6.41 *Let A be a noetherian local ring of dimension $d > 0$ such that the residue field $k = A/\,\mathfrak{m}$ has infinitely many elements. Then, to each $\mathfrak{m}$–primary ideal $\mathfrak{q}$, there exist elements x of good reduction.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal ideals of $\mathrm{Gr}\,(\mathfrak{q}, A)$, where the numbering is chosen in such a way that

$$\mathfrak{a} := \bigoplus_{i=1}^{\infty} \mathfrak{q}^i/\,\mathfrak{q}^{i+1}$$

is contained in $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_r$, but not in $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Hence, $\mathfrak{p}_\sigma \cap (\mathfrak{q}/\,\mathfrak{q}^2)$ is a proper vector subspace of $\mathfrak{p}$, $\sigma = 1, \ldots, s$. Since k has infinitely many elements, we can find an element $x \in \mathfrak{q}$ such that its residue class $\overline{x}$ in $\mathfrak{q}/\,\mathfrak{q}^2$ is not contained in $\cup_{\sigma=1}^{s} \mathfrak{p}_\sigma$.

According to the remark before Theorem 5, we want to show that there exists a natural number $c \in \mathbb{N}$

such that $\alpha\overline{x} = 0$ and $\deg \alpha \geq c$ implies $\alpha = 0$. Now, by the *Lasker–Noether Decomposition Theorem*, we find ideals $\mathfrak{q}_\rho$ with $\operatorname{rad} \mathfrak{q}_\rho = \mathfrak{p}_\rho$ such that

$$\bigcap_{\rho=1}^{r} \mathfrak{q}_\rho = (0) .$$

Furthermore, to each $\rho = s + 1, \ldots, r$, there exists a number c_ρ satisfying

$$\mathfrak{a}^{c_\rho} \subset \mathfrak{q}_\rho ,$$

hence

$$\mathfrak{a}^c \subset \bigcap_{\rho=s+1}^{r} \mathfrak{q}_\rho , \quad c = \max c_\rho .$$

To finish the proof, let $\alpha \in \operatorname{Gr}(\mathfrak{q}, A)$ be a zerodivisor of $\overline{x}$. Since $\overline{x} \notin \mathfrak{p}_\sigma$, $\sigma = 1, \ldots, s$, we must have $\alpha \in \mathfrak{q}_\sigma$, $\sigma = 1, \ldots, s$. But, if the degree of α would exceed c, then we would have $\alpha \in \mathfrak{a}^c$, and hence

$$\alpha \in \bigcap_{\rho=1}^{r} \mathfrak{q}_\rho = (0) . \qquad \square$$

A central result in the theory of multiplicities is the following

Theorem 6.42 *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$ and infinite residue class field $k = A/\mathfrak{m}$, and let $\mathfrak{q}$ be an $\mathfrak{m}$–primary ideal. Then there exists a parameter system*

$$x_1, \ldots, x_d \in \mathfrak{q}$$

(*i.e. a system $x_1, \ldots, x_d$ such that $A/(x_1, \ldots, x_d)A$ is artinian*) with

$$\operatorname{mult}((x_1, \ldots, x_d), A) = \operatorname{mult}(\mathfrak{q}, A) .$$

Proof (by induction on $d = \dim A$). In the case $d = 0$, each ideal $\mathfrak{a} \subset \mathfrak{m}$ is nilpotent. Hence, $H_{A,\mathfrak{q}} = \text{const} = \dim_k A$. So, it suffices to take the empty parameter system.

In the case $d = 1$, $\mathfrak{m}$ is the only nonminimal prime ideal of A. There are two possibilities:

a) $\mathfrak{m}$ is not embedded. By the same arguments as in the proof of Theorem 5, we can then show that there exists an element $x \in \mathfrak{m}$ with good reduction with respect to $\mathfrak{q}$ which does not belong to any prime ideal associated to A. So, we find a number c such that

$$(\mathfrak{q}^i : x\,A) \cap \mathfrak{q}^c = \mathfrak{q}^{i-1} \text{ for all } i$$

and consequently (see [02–06], vol. III, Chap. VIII, Paragraph 5, Theorem 13, Corollary 1),

$$(\mathfrak{q}^i : x\,A) \subset \mathfrak{q}^{s(i)} , \quad \lim_{i\to\infty} s(i) = \infty .$$

This implies $(\mathfrak{q}^i : x\,A) = \mathfrak{q}^{i-1}$ for large i, and therefore, by Lemma 4,

$$H_{A/xA,\mathfrak{q}/x\mathfrak{q}}(i) = H_{A,\mathfrak{q}}(i) - H_{A,\mathfrak{q}}(i-1) .$$

Since $H_{A,\mathfrak{q}}$ is linear, the difference on the right hand side is precisely $\operatorname{mult}(\mathfrak{q}, A)$. Since $A/x\,A$ has dimension 0, the left hand side equals the length $\ell(A/x\,A) = \dim_{\mathbb{C}}(A/x\,A)$ for large i. Now $A/x\,A \cong x^{i-1}A/x^iA$ for all i, whence

$$\begin{aligned} H_{A,xA}(i) - H_{A,xA}(i-1) &= \dim(x^{i-1}A/x^iA) \\ &= H_{A,\mathfrak{q}}(i) - H_{A,\mathfrak{q}}(i-1) , \end{aligned}$$

and

$$\operatorname{mult}(x\,A,\,A) \,=\, \operatorname{mult}(\mathfrak{q},\,A)\;.$$

b) $\mathfrak{q}$ is an embedded prime ideal of A. Choose again an element x of good reduction with respect to $\mathfrak{q}$, and define $\mathfrak{a} := \operatorname{Ann}_A(x)$. $\mathfrak{a}$ is a finite $A/\,x\,A$–module, and since $A/\,x\,A$ is artinian, $\mathfrak{a}$ is of finite length. Since

$$\bigcap_{i\geq 0}(\mathfrak{q}^i\cap\mathfrak{a})\subset\bigcap_{i\geq 0}\mathfrak{q}^i \,=\, (0)$$

and $\mathfrak{a}$ is of finite length, we get

$$\mathfrak{q}^i\cap\mathfrak{a} \,=\, (0)\;,\quad i >> 0\;.$$

Now, we put

$$A^* \,:=\, A/\,\mathfrak{a}\;,\quad \mathfrak{q}^* \,:=\, (\mathfrak{q}\,+\,\mathfrak{a})/\,\mathfrak{a}\;.$$

Then, by definition and the exactness of

$$0 \longrightarrow (\mathfrak{q}^i\,+\,\mathfrak{a})/\,\mathfrak{q}^i \longrightarrow A/\,\mathfrak{q}^i \longrightarrow A/\,(\mathfrak{q}^i\,+\,\mathfrak{a}) \longrightarrow 0\;,$$

we have

$$\begin{aligned}
H_{A^*,\mathfrak{q}^*}(i) &= \ell\,(A^*/\,{\mathfrak{q}^*}^i) \,=\, \ell\,((A/\,\mathfrak{a})/\,(\mathfrak{q}^i\,+\,\mathfrak{a})/\,\mathfrak{a})\\
&= \ell\,(A/\,(\mathfrak{q}^i\,+\,\mathfrak{a}))\\
&= \ell\,(A/\,\mathfrak{q}^i) \,-\, \ell\,((\mathfrak{q}^i\,+\,\mathfrak{a})/\,\mathfrak{q}^i)\\
&= \ell\,(A/\,\mathfrak{q}^i) \,-\, \ell\,(\mathfrak{a}/\,\mathfrak{q}^i\cap\mathfrak{a})\\
&= H_{A,\mathfrak{q}}(i) \,-\, \ell\,(\mathfrak{a})\;,\quad i >> 0\;,
\end{aligned}$$

and therefore,

$$\operatorname{mult}(\mathfrak{q},\,A) \,=\, \operatorname{mult}(\mathfrak{q}^*,\,A^*)\;.$$

By the same arguments, one can show that

$$\operatorname{mult}(x\,A,\,A) \,=\, \operatorname{mult}(x^*A^*,\,A^*)\;,$$

where x^* is the image of x under the restriction map $A \to A/\,\mathfrak{a} \,=\, A^*$. Because of the case a), we know

$$\operatorname{mult}(x^*A^*,\,A^*) \,=\, \operatorname{mult}(\mathfrak{q}^*,\,A^*)\;,$$

which implies

$$\operatorname{mult}(x\,A,\,A) \,=\, \operatorname{mult}(\mathfrak{q},\,A)\;.$$

Finally, let d be greater than 1, and suppose that the statement is proven for all rings of dimension $d\,-\,1$. Take an element $x\in\mathfrak{q}\setminus\mathfrak{q}^2$ of good reduction and define

$$A^* \,:=\, A/\,x\,A\;,\quad \mathfrak{q}^* \,:=\, \mathfrak{q}/\,x\,\mathfrak{q}\;,\quad H \,:=\, H_{A,\mathfrak{q}}\;,\quad H^* \,:=\, H_{A^*,\mathfrak{q}^*}\;.$$

As an immediate consequence of Lemma 4, we get

$$\operatorname{mult}(\mathfrak{q}^*,\,A^*) \,=\, \operatorname{mult}(\mathfrak{q},\,A)\;.$$

By induction hypothesis, we find a parameter system

$$(x_1^*,\ldots,x_{d-1}^*)\subset\mathfrak{q}^*$$

of A^* with

$$\operatorname{mult}((x_1^*,\ldots,x_{d-1}^*),\,A) \,=\, \operatorname{mult}(\mathfrak{q}^*,\,A^*)\;.$$

Choose representatives $x_1, \ldots, x_{d-1}$ of $x_1^*, \ldots, x_{d-1}^*$ resp. in $\mathfrak{q}$. Then the set $x_1, \ldots, x_{d-1}, x$ forms a parameter system in $\mathfrak{q}$ and generates an $\mathfrak{m}$–primary ideal $\mathfrak{q}' \subset \mathfrak{q}$. By Lemma 3 and the inclusion $(\mathfrak{q}')^{i-1} \subset ((\mathfrak{q}')^i : x\, A)$, we deduce

$$\begin{aligned} H_{A/xA,\mathfrak{q}'/x\mathfrak{q}'}(i) &= H_{A,\mathfrak{q}'}(i) - \dim_{\mathbb{C}}(A/({\mathfrak{q}'}^i : x\, A)) \\ &\geq H_{A,\mathfrak{q}'}(i) - H_{A,\mathfrak{q}'}(i-1)\, . \end{aligned}$$

Thus,

$$\mathrm{mult}\,((x_1^*, \ldots, x_{d-1}^*),\, A^*) = \mathrm{mult}\,(\mathfrak{q}'/x\,\mathfrak{q}',\, A/x\,A) \geq \mathrm{mult}\,(\mathfrak{q}',\, A)$$

and

$$\mathrm{mult}\,(\mathfrak{q},\, A) \geq \mathrm{mult}\,(\mathfrak{q}',\, A)\, .$$

Since $\mathfrak{q}' \subset \mathfrak{q}$, the opposite estimate holds trivially. □

6.16 A geometric interpretation for Cohen - Macaulay rings

For Cohen–Macaulay rings A one can get a very precise formula for the Hilbert polynomial with respect to a *regular* sequence $x_1, \ldots, x_d$, $d = \dim A$, i.e. a sequence in which $x_1 \in \mathfrak{m}$ is a nonzerodivisor in A and $x_j \in \mathfrak{m}$ is a nonzerodivisor in $A/(x_1, \ldots, x_{j-1})\,A$, $j = 2, \ldots, d$. (For more details on Cohen–Macaulay rings, see Chapter 13). We shall prove in Theorem 14.1 that for the ideal $\mathfrak{q}$ generated by $x_1, \ldots, x_d$ the quotient $\mathfrak{q}/\mathfrak{q}^2$ is a free $A/\mathfrak{q}$–module of rank d. Then it is certainly true that $\mathfrak{q}^i/\mathfrak{q}^{i+1}$ is isomorphic to the i–th symmetric power of $\mathfrak{q}/\mathfrak{q}^2$, $i \geq 1$. Consequently,

$$\mathrm{Gr}\,(\mathfrak{q},\, A) = \bigoplus_{i\geq 0} \mathfrak{q}^i/\mathfrak{q}^{i+1} = \bigoplus_{i\geq 0} (S_{A/\mathfrak{q}})_i(\mathfrak{q}/\mathfrak{q}^2) \cong (A/\mathfrak{q})[\xi_1, \ldots, \xi_d]\, .$$

(Notice that, in general, $\mathrm{Gr}\,(\mathfrak{q},\, A)$ is an epimorphic image of the polynomial ring $(A/\mathfrak{q})[\xi_1, \ldots, \xi_d]$ for any parameter system $x_1, \ldots, x_d \in \mathfrak{m}$).

By assumption, $A/\mathfrak{q}$ is zero–dimensional such that $N(\mathfrak{q}) = N(\mathfrak{m})$ and $\mathrm{rad}\,\mathfrak{q} = \mathrm{rad}\,\mathfrak{m} = \mathfrak{m}$ by Rückert's Nullstellensatz. Therefore, $\mathfrak{q}$ is an $\mathfrak{m}$–primary ideal, and we can conclude:

Theorem 6.43 *Let A be a d–dimensional local noetherian Cohen–Macaulay ring, and let $\mathfrak{q}$ be the $\mathfrak{m}$–primary ideal generated by a regular sequence $x_1, \ldots, x_d$. Then*

$$H_{A,\mathfrak{q}}(i) = (\dim_{\mathbb{C}} A/\mathfrak{q}) \sum_{j=0}^{i} \binom{j+d-1}{d-1}\, .$$

In particular,

$$\mathrm{mult}\,(\mathfrak{q},\, A) = \dim_{\mathbb{C}} A/\mathfrak{q}\, .$$

Combining Theorem 7 and Theorem 6, we find a parameter system $x_1, \ldots, x_d$ for any Cohen–Macaulay ring A satisfying

$$\dim_{\mathbb{C}} A/\mathfrak{q} = \mathrm{mult}\,(\mathfrak{q},\, A) = \mathrm{mult}\,(\mathfrak{m},\, A)\,, \quad \mathfrak{q} = (x_1, \ldots, x_d)A\, .$$

But the length $\ell = \ell\,(A/\mathfrak{q})$ has a nice geometric interpretation for local analytic Cohen–Macaulay rings $A = \mathcal{O}_{X,x}$. The sequence $x_1, \ldots, x_d$ defines a finite injective morphism

$$K_d = \mathbb{C}\,\langle\, x_1, \ldots, x_d\,\rangle \longrightarrow A\,,$$

which makes A into a free K_d–module (see Theorem 13.20) and induces (locally) a finite branched covering $\rho : X \to \mathbb{C}^d$. Then the coherent analytic sheaf $\rho_*\mathcal{O}_X$ is free near $\rho\,(x^{(0)}) = 0$ of rank

$$\dim_{\mathbb{C}} A/(x_1, \ldots, x_d)\,A = \ell\,,$$

and therefore, ρ is outside the discriminant set an ℓ–sheeted covering. On the other hand,

$$\mathrm{mult}\,(\mathfrak{m},\, A) \leq \mathrm{mult}\,(\mathfrak{q},\, A)$$

for all $\mathfrak{m}$–primary ideals $\mathfrak{q}$. - These remarks imply

Theorem 6.44 *The multiplicity m of a d–dimensional Cohen–Macaulay singularity X is the smallest number ℓ such that X can be realized as an ℓ–sheeted finite branched covering of $\mathbb{C}^d$.*

Corollary 6.45 *A Cohen–Macaulay singularity of multiplicity one is regular.*

Notice that these results apply to any normal two–dimensional singularity.

Notes and References

Some of the material, especially the treatment of the Artin–Rees Lemma in Section 6, is taken from [02 - 01].