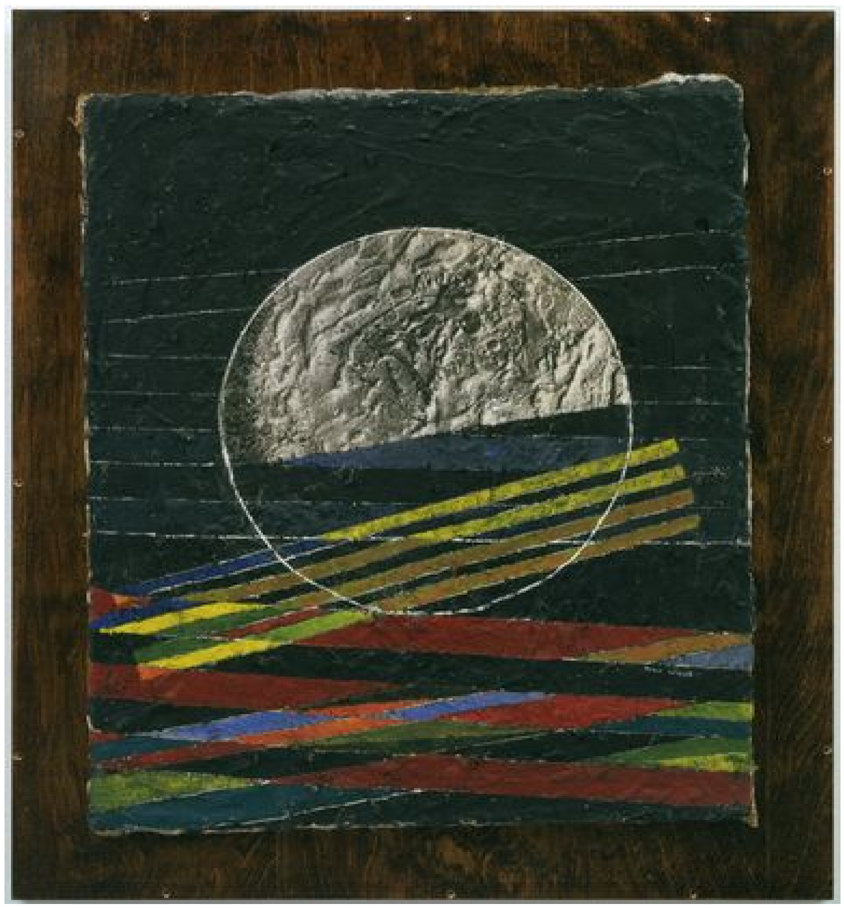
Chapter 4

Was ist feierlicher als zwei Striche im Sand, zwei Parallelen? Schau an den fernsten Horizont, und es ist nichts als Unendlichkeit; schau auf das weite Meer, es ist Weite, nun ja, und schau in die Milchstraße empor, es ist Raum, daß dir der Verstand verdampft, unausdenkbar, aber es ist nicht das Unendliche, das sie allein dir zeigen: Zwei Striche im Sand, gelesen mit Geist …

(Max Frisch,
Don Juan oder die Liebe zur Geometrie)



Chapter 4

Cones over projective algebraic manifolds and their canonical resolution

Generalizing the example of the A_1–singularity - which is described by the vanishing of a homogeneous polynomial of degree 2 - we study in this Chapter a class of *algebraic cones* with (possibly) an isolated singularity at the vertex. In particular, we construct a canonical *resolution* of these singularities, using certain holomorphic line bundles on complex projective space.

4.1 From affine subspaces of $\mathbb{C}^n$ to algebraic cones in $\mathbb{C}^{n+1}$

Linear algebra is traditionally regarded as a method to solve *systems* of (nontrivial) affine equations:

$$\ell_k(x) = \sum_{j=1}^{n} a_{jk}x_j + \alpha_k = 0 \, , \quad k = 1, \dots, m \, .$$

In geometrical terms, the solution space

$$X = \{ x \in \mathbb{C}^n : \ell_k(x) = 0 \, , \, k = 1, \dots, m \}$$

is equal to the intersection

$$H_1 \cap \ldots \cap H_m$$

of the hyperplanes

$$H_k = \{ x \in \mathbb{C}^n : \ell_k(x) = 0 \} \, .$$

Such an *affine* subspace X may be empty (for $m \geq 2$). If not, it consists of a d–dimensional continuum of points, where

$$d = n - r \, , \quad r = \operatorname{rank}(a_{jk}) \, .$$

More precisely: If $x_0 \in X$, then there exist new affine coordinates $y_1, \dots, y_n$ in $\mathbb{C}^n$ such that $x_0 = 0$ and

$$X = \{ (y_1, \dots, y_n) : y_{d+1} = \cdots = y_n = 0 \} \, .$$

We observe that the matrix (a_{jk}) coincides with the Jacobi matrix

$$\left(\frac{\partial \ell_k}{\partial x_j} \, (x) \right)$$

which, consequently, has *constant* rank r on $\mathbb{C}^n$.

If we replace the linear functions $\ell_1, \dots, \ell_m$ by *homogeneous polynomials* of arbitrary degree, we can hope to obtain geometrically more fascinating objects.

From now on, we work in $(n+1)$–dimensional complex number space $\mathbb{C}^{n+1}$ with linear coordinates $x = (x_0, \ldots, x_n)$, $n \geq 1$. A nontrivial polynomial $P \in \mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous* of degree d, if it can be written in the form

$$P(x) = \sum_{|\nu|=d} a_\nu x^\nu = \sum_{\nu_0+\cdots+\nu_n=d} a_{\nu_0\ldots\nu_n} x_0^{\nu_0} \cdot \ldots \cdot x_n^{\nu_n} .$$

In other words, if $(x, \lambda) \mapsto \lambda x$ denotes the standard action $\mathbb{C}^{n+1} \times \mathbb{C}^* \to \mathbb{C}^{n+1}$ of the (multiplicative) group $\mathbb{C}^*$ induced by the $\mathbb{C}$–vector space structure of $\mathbb{C}^{n+1}$, then a polynomial $P \neq 0$ in $\mathbb{C}[x_0, \ldots, x_n]$ satisfies

$$P(\lambda x) = \lambda^d P(x) \text{ for all } \lambda \in \mathbb{C}^*, \; x \in \mathbb{C}^{n+1},$$

if and only if P is homogeneous of degree d.

Given finitely many homogeneous polynomials $P_1, \ldots, P_m$ - of degree $d_1, \ldots, d_m$, say - we call the simultaneous zero set

$$C = C(P_1, \ldots, P_m) := \{ x \in \mathbb{C}^{n+1} : P_1(x) = \cdots = P_m(x) = 0 \}$$

the *algebraic cone* in $\mathbb{C}^{n+1}$ defined by $P_1, \ldots, P_m$. Obviously, we have $0 \in C$, and $x \in C$ implies $\lambda x \in C$ for all $\lambda \in \mathbb{C}$. Consequently, C is a cone in the sense of Section 1.6.

This property calls for building up equivalence classes with respect to the given action of $\mathbb{C}^*$.

4.2 Projective space $\mathbb{P}_n$

As a set, *complex projective space* $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$ consists of all *lines* through the origin in $\mathbb{C}^{n+1}$. Such a line is determined by a vector $x = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, and two vectors x' and x'' define the same line, if and only if $x'' = \lambda x'$ with a complex number $\lambda \in \mathbb{C}^*$. So, $\mathbb{P}_n$ is equal to the *quotient space*

$$(\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$$

of all *equivalence classes*

$$[x] = \{ x' \in \mathbb{C}^{n+1} \setminus \{0\} : \lambda x = x' \text{ for some } \lambda \in \mathbb{C}^* \}$$

which are precisely the orbits for the $\mathbb{C}^*$–action

$$\begin{cases} (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}^* \longrightarrow \mathbb{C}^{n+1} \setminus \{0\} \\ \qquad (x \;,\; \lambda) \longmapsto \qquad \lambda x \end{cases} .$$

For historical reasons, we sometimes write $[x_0 : \cdots : x_n]$ instead of $[x]$, when $x = (x_0, \ldots, x_n)$. $x_0, \ldots, x_n$ are then called *homogeneous coordinates* of the point $[x] \in \mathbb{P}_n$. By abuse of notation, we also write x instead of $[x]$.

The canonical projection $\varphi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n$ sending x to $[x]$ will be a continuous map, when $\mathbb{P}_n$ is equipped with the quotient topology in which $U \subset \mathbb{P}_n$ is open, if and only if $\varphi^{-1}(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. It is easily checked that in this topology the $n+1$ sets

$$U_j = \{ x = [x_0 : \cdots : x_n] : x_j \neq 0 \}, \quad j = 0, \ldots, n,$$

are open and dense in $\mathbb{P}_n$ and that the maps

$$\chi_j : U_j \longrightarrow \mathbb{C}^n$$

given by

$$(t_1^{(j)}, \ldots, t_n^{(j)}) = \left(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j} \right)$$

are (well–defined) homeomorphisms. Moreover, since two different lines in $\mathbb{C}^{n+1}$ through the origin can be separated in $\mathbb{C}^{n+1} \setminus \{0\}$ by disjoint $\mathbb{C}^*$–invariant open sets, $\mathbb{P}_n$ is a Hausdorff space.

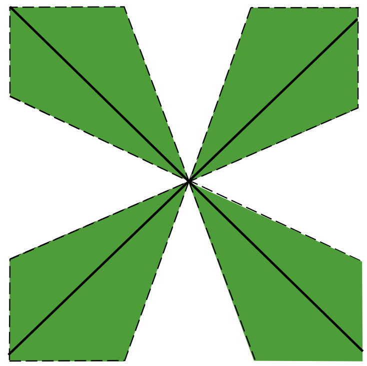


Figure 4.1

Finally, $\mathbb{P}_n$ is compact as the image under the continuous mapping φ of the (compact) sphere

$$S^{2n+1} = \{ x \in \mathbb{C}^{n+1} : |x_0|^2 + \cdots + |x_n|^2 = 1 \} .$$

Summing up, we can say that $\mathbb{P}_n$ is a *compact topological manifold* of (complex) dimension n.

Recall that a *complex analytic structure* on a topological manifold M of dimension n is given by a *holomorphic atlas*

$$\mathfrak{A} = \{ (U_\iota, \chi_\iota, V_\iota) : \iota \in I \} ,$$

where U_ι is open in M, V_ι is open in $\mathbb{C}^n$, $\chi_\iota : U_\iota \to V_\iota$ is a homeomorphism, the system $\{ U_\iota : \iota \in I \}$ covers M, i.e.

$$M = \bigcup_{\iota \in I} U_\iota ,$$

and where for all $\iota, \kappa \in I$ the map

$$\chi_\kappa \circ \chi_\iota^{-1}{}_{|\chi_\iota(U_\iota \cap U_\kappa)} : \chi_\iota(U_\iota \cap U_\kappa) \longrightarrow \chi_\kappa(U_\kappa \cap U_\iota)$$

is biholomorphic.

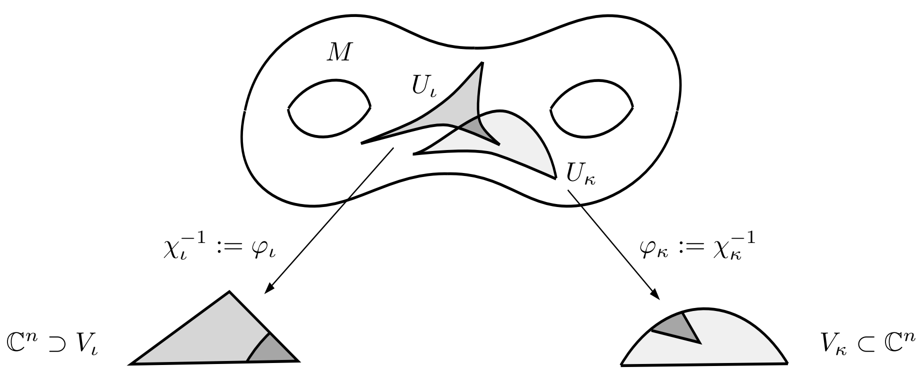


Figure 4.2

The pair $(M, \mathfrak{A})$ - or M for short - is then called a *complex (analytic) manifold*. The concepts of holomorphy for functions on and for maps between such manifolds are introduced without any difficulty via the *coordinate charts* $(U_\iota, \chi_\iota, V_\iota)$. It is also immediate from the definition that in the category of complex manifolds cartesian products exist.

Returning to projective space $\mathbb{P}_n$ with its canonical covering $\mathfrak{U}$ of open sets U_j, $j = 0, \ldots, n$, we want to calculate the *transition maps* $\chi_k \circ \chi_j^{-1}$. First, we notice that

$$\chi_j(U_j \cap U_k) = \{ (t_1^{(j)}, \ldots, t_n^{(j)}) \in \mathbb{C}^n : t_{\ell(j,k)}^{(j)} \neq 0 \}$$

with $\ell\,(j,\,k)\,=\,k+1$, if $k\,\le\,j-1$, and $\ell\,(j,\,k)\,=\,k$, if $k\,\ge\,j+1$, and then, we easily find that (for e.g. $j\,<\,k$) $\chi_k\circ\chi_j^{-1}$ is given on $\chi_j(U_j\cap U_k)$ by holomorphic terms (in fact, the transition functions are *rational*):

$$t_l^{(k)} \,=\, \begin{cases} \dfrac{t_\ell^{(j)}}{t_k^{(j)}}\,, & \ell\,=\,1,\ldots,j\,,\;k\,+\,1,\ldots,n\,, \\ \dfrac{1}{t_k^{(j)}}\,, & \ell\,=\,j\,+\,1\,, \\ \dfrac{t_{\ell-1}^{(j)}}{t_k^{(j)}}\,, & \ell\,=\,j\,+\,2,\ldots,k\,. \end{cases}$$

This implies that $\mathbb{P}_n$ is a *complex manifold* of dimension n.

4.3 The base of an algebraic cone

By definition, any cone $C\subset\mathbb{C}^{n+1}$ is completely determined by its image

$$\underline{C}\,=\,\varphi\,(C\setminus\{\,0\,\})\subset\mathbb{P}_n$$

under the canonical (holomorphic) map

$$\varphi:\,\mathbb{C}^{n+1}\setminus\{\,0\,\}\,\longrightarrow\,\mathbb{P}_n$$

(cf. Section 2) because of the identity

$$C\,=\,\varphi^{-1}(\underline{C})\cup\{\,0\,\}\,.$$

If $C\,=\,C\,(P_1,\ldots,P_m)$ is an *algebraic* cone as in Section 1, then $\underline{C}$ has the structure of a *projective algebraic* set, as we will prove in a moment. Therefore, an algebraic cone C is sometimes termed an *affine cone over the projective algebraic variety* $\underline{C}$.

Theorem 4.1 *For any algebraic cone $C\,=\,C\,(P_1,\ldots,P_m)\subset\mathbb{C}^{n+1}$, the base $\underline{C}\subset\mathbb{P}_n$ is locally given by the vanishing of m holomorphic functions $g_1,\ldots,g_m$ such that the ranks of the Jacobi matrices of $(P_1,\ldots,P_m)$ at a point $x\in C\setminus\{\,0\,\}$ and of $(g_1,\ldots,g_m)$ at the corresponding point $[\,x\,]\in\underline{C}$ coincide. More precisely, if $\{\,(U_j,\chi_j,\mathbb{C}^n):\,j\,=\,0,\ldots,n\,\}$ denotes the standard atlas of $\mathbb{P}_n$, then*

$$\chi_j(\underline{C}\cap U_j)$$

is an algebraic *set in $\mathbb{C}^n$ defined by the vanishing of m polynomials.*

Proof. We start with the last assertion, restricting our calculations to the open set $U_0\,=\,\{\,x_0\,\neq\,0\,\}$. We write t_j instead of the coordinate function $t_j^{(0)}$ on $\mathbb{C}^n$. Then $t\,=\,\chi_0([\,x\,])\,=\,(x_1x_0^{-1},\ldots,x_nx_0^{-1})$, and consequently

$$P_k(x)\,=\,\sum_{|\nu|=d_k}a_\nu^{(k)}\,x^\nu\,=\,x_0^{d_k}\cdot\sum_{\nu_0+\cdots+\nu_n=d_k}a_{\nu_0\ldots\nu_n}^{(k)}\left(\frac{x_1}{x_0}\right)^{\nu_1}\cdot\ldots\cdot\left(\frac{x_n}{x_0}\right)^{\nu_n}\,=\,0\,,$$

if and only if

$$Q_k(t)\,=\,Q_k(t_1,\ldots,t_n)\,=\,\sum_{\nu_0+\cdots+\nu_n=d_k}a_{\nu_0\ldots\nu_n}^{(k)}\,t_1^{\nu_1}\cdot\ldots\cdot t_n^{\nu_n}\,=\,0\,.$$

This yields

$$\chi_0(\underline{C}\cap U_0)\,=\,\{\,t\in\mathbb{C}^n:\,Q_1(t)\,=\cdots=\,Q_m(t)\,=\,0\,\}\,.$$

It remains to prove that $(\partial P_k / \partial x_j)$ and $(\partial Q_k / \partial t_j)$ have the same rank at corresponding points x and t. Now, by *Euler's relation* for homogeneous polynomials, we have for $x \in C$:

$$\sum_{j=0}^{n} x_j \, \frac{\partial P_k}{\partial x_j}\,(x) \,=\, d_k \, P_k(x) \,=\, 0 \,,$$

and this leads (for points $x \in C$ with $x_0 \neq 0$) to

$$\operatorname{rank}\left(\frac{\partial P_k}{\partial x_j}\,(x)\right)_{\substack{k=1,\ldots,m\\ j=0,\ldots,n}} \,=\, \operatorname{rank}\left(\frac{\partial P_k}{\partial x_j}\,(x)\right)_{\substack{k=1,\ldots,m\\ j=1,\ldots,n}} \,.$$

Finally application of the chain rule gives for $j \neq 0$, $t \,=\, (\chi_0 \circ \varphi)\,(x)$:

$$\frac{\partial P_k}{\partial x_j}\,(x) \,=\, x_0^{d_k} \sum_{\ell=1}^{n} \frac{\partial Q_k}{\partial t_\ell}\,(t) \cdot \frac{\partial t_\ell}{\partial x_j}\,(x) \,=\, x_0^{d_k - 1}\, \frac{\partial Q_k}{\partial t_j}\,(t) \,. \qquad \square$$

Intuitively, it seems reasonable that a cone C is smooth outside its vertex 0 precisely, when its base $\underline{C}$ consists only of regular points. For instance, the cone

$$\{\, x \,=\, (x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : \sum_{j=0}^{n} x_j^2 \,=\, 0 \,\}$$

has an isolated singularity at the vertex, and its base, the *quadric*

$$\{\, [\, x\,] \,=\, [\, x_0 : \cdots : x_n \,] \in \mathbb{P}_n : \sum_{j=0}^{n} x_j^2 \,=\, 0 \,\} \,,$$

is seen to be smooth by a straightforward application of the Implicit Function Theorem. In the next Section we make this remark much more precise.

4.4 Cones and the Rank Theorem

Although an algebraic cone $C \subset \mathbb{C}^{n+1}$ - of dimension $d + 1$, say - which is smooth outside the origin, can be locally described by $n - d$ independent equations (this follows from the trivial direction of the Implicit Mapping Theorem), it is in general impossible to find $n - d$ *polynomials* $P_1, \ldots, P_{n-d}$ with $C \,=\, C\,(P_1, \ldots, P_{n-d})$. We take, for *Example*, the cone C given by the three equations

$$\left\{ \begin{array}{c} P_1 \,=\, x_0 x_2 \,-\, x_1^2 \,=\, 0 \quad P_2 \,=\, x_1 x_3 \,-\, x_2^2 \,=\, 0 \\ P_3 \,=\, x_0 x_3 \,-\, x_1 x_2 \,=\, 0 \end{array} \right.$$

in $\mathbb{C}^4$. If $x \,=\, (x_0,\, x_1,\, x_2,\, x_3) \in C \setminus \{\, 0 \,\} \,=\, C'$, then necessarily $x_0 \neq 0$ or $x_3 \neq 0$. By symmetry, we need only study the first case, in which C' is (locally) parametrized by the equations

$$x_0 \,=\, t_0 \neq 0 \,, \quad x_1 \,=\, t_1 \,, \quad x_2 \,=\, \frac{t_1^2}{t_0} \,, \quad x_3 \,=\, \frac{t_1^3}{t_0^2} \,.$$

Hence, C is - outside the origin - a *two–dimensional* manifold but not one–dimensional as the number of equations might suggest. The reason for this, however, does not lie in an artificial inflation of the number of equations. If we take, for instance, the cone

$$C_1 \,=\, C\,(P_2,\, P_3) \,,$$

then the plane $\{\, x_2 \,=\, x_3 \,=\, 0 \,\}$ is contained in C_1, but not in C. For the cone $C_3 \,=\, C\,(P_1,\, P_2)$, the same reasoning works with the plane $\{\, x_1 \,=\, x_2 \,=\, 0 \,\}$. Of course, from these observations we cannot conclude the impossibility to write C in the form $C\,(Q_1, Q_2)$ with two polynomials Q_1 and Q_2 ; this, in fact, will follow later quite easily once we have some tools at our disposal that will enable us to analyze more carefully the singularity of C at the vertex.

This example indicates - as we already know - that the Implicit Mapping Theorem is not always strong enough to unveil the manifold structure of an analytic set. A closer look to it, however, points to a better criterion: the *Rank Theorem*. In our example, the Jacobi matrix $(\partial P_k / \partial x_j)$ must be of rank ≤ 2 on C' ; in other words: there must exist a nontrivial linear relation between the differentials dP_1 , dP_2 and dP_3 on C' . In fact, the obvious relation

$$x_2\, P_1 \,+\, x_0\, P_2 \,-\, x_1\, P_3 \,=\, 0$$

implies on $C' \cap \{\, x_0 \neq 0 \,\}$ the relation

$$x_2\, dP_1 \,+\, x_0\, dP_2 \,-\, x_1\, dP_3 \,=\, 0\,,$$

and for $C' \cap \{\, x_3 \neq 0 \,\}$ we can use similarly the identity

$$x_3\, P_1 \,+\, x_1\, P_2 \,-\, x_2\, P_3 \,=\, 0\,.$$

On the other hand, the Jacobi matrix

$$\begin{pmatrix} x_2 & -2x_1 & x_0 & 0 \\ 0 & x_3 & -2x_2 & x_1 \\ x_3 & -x_2 & -x_1 & x_0 \end{pmatrix}$$

does contain 2×2 matrices with determinants x_0^2 and x_3^2 , respectively, which have no common zeros on C' . Therefore, the Jacobi matrix has *constant* rank 2 along C' , and we are at least “infinitesimally close” to the linear situation of Section 1. - Using Theorem 1 the *Rank Theorem* (see Chapter 3) implies the following much stronger result.

Theorem 4.2 *Let* $C \,=\, C\,(P_1, \ldots, P_m) \subset \mathbb{C}^{n+1}$ *be an algebraic cone, and suppose that, near a point* $x \in C' \,=\, C \setminus \{\, 0 \,\}$ *, the Jacobi matrix*

$$\left(\frac{\partial P_k}{\partial x_j} \right)_{\substack{1 \leq k \leq m \\ 0 \leq j \leq n}}$$

has constant rank r *along* C' *. Then the base* $\underline{C}$ *is smooth of dimension* $n \,-\, r$ *at* $[\,x\,]$ *, and* C *is smooth of dimension* $n \,+\, 1 \,-\, r$ *at all points* $\lambda\, x$ *,* $\lambda \in \mathbb{C}^*$ *.*

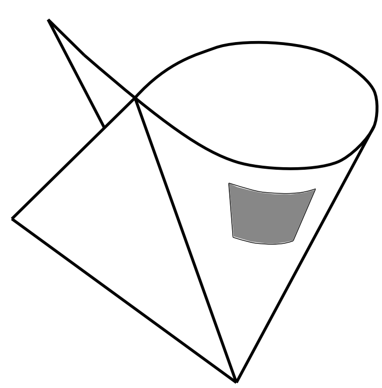


Figure 4.3

4.5 The tautological bundle on $\mathbb{P}_n$

We now attach to each point $[x] \in \mathbb{P}_n$ the line ℓ_x in $\mathbb{C}^{n+1}$ through x and the origin. This construction confronts us with the first example of a (holomorphic) *line bundle*, i.e. a family of holomorphically varying complex lines parametrized by the points of a complex manifold (or a more general analytic object).

More precisely, we study the set

$$L \subset \mathbb{C}^{n+1} \times \mathbb{P}_n$$

consisting of all points $(\xi, [x]) \in \mathbb{C}^{n+1} \times \mathbb{P}_n$ with $\xi = 0$ or $[\xi] = [x]$. In other terms,

$$L = \{ (\xi, [x]) \in \mathbb{C}^{n+1} \times \mathbb{P}_n : \text{ there exists } \lambda \in \mathbb{C} \text{ such that } \xi_j = \lambda x_j , \; j = 0, \ldots, n \} ,$$

or, even more elegantly,

$$L = \{ (\xi, [x]) \in \mathbb{C}^{n+1} \times \mathbb{P}_n : \xi_j x_k - \xi_k x_j = 0 , \; 0 \le j < k \le n \} .$$

Of course, if π denotes the restriction of the projection $\mathbb{C}^{n+1} \times \mathbb{P}_n \to \mathbb{P}_n$ to the subspace L, then π is surjective and

$$\pi^{-1}([x]) = \ell_x .$$

Moreover, on each member U_j of the standard open covering on $\mathbb{P}_n$, n equations suffice to describe $L_j := \pi^{-1}(U_j)$, namely

$$\xi_k = \frac{x_k}{x_j} \, \xi_j , \quad k \neq j ,$$

and this implies at once that L_j is a $(n + 1)$–dimensional complex submanifold of $\mathbb{C}^{n+1} \times U_j$. Furthermore, we have a commutative diagram

$$\begin{tikzcd} L_j \arrow[r, "\overline{\chi}_j"] \arrow[d, "\pi"'] & \mathbb{C} \times \mathbb{C}^n \arrow[d, "p"] \\ U_j \arrow[r, "\chi_j"'] & \mathbb{C}^n \end{tikzcd}$$

where $\overline{\chi}_j$ is defined by

$$\overline{\chi}_j((\xi, [x])) = (\xi_j, t^{(j)}) , \quad t^{(j)} = \chi_j([x]) ,$$

and p denotes the projection to the second factor. An easy exercise shows that $\overline{\chi}_j$ is invertible and linear as a map on the fibers

$$\ell_x = \pi^{-1}([x]) \longrightarrow p^{-1}(\chi_j([x])) = \mathbb{C} .$$

In conclusion, we infer that in the triple $(L, \pi, \mathbb{P}_n)$ the set L is a complex manifold of dimension $n + 1$, and π is a holomorphic map having one–dimensional linear fibers which can (locally with respect to $\mathbb{P}_n$) be trivialized by a simple model of type $(\mathbb{C} \times U, p, U)$, where p denotes projection. We call $(L, \pi, \mathbb{P}_n)$ - or L for short - the *tautological (line–) bundle* on $\mathbb{P}_n$.

4.6 Holomorphic line bundles on complex manifolds

Generalizing the example in the previous Section to a universal concept, we call a triple (L, π, M) a holomorphic *line bundle* on M, if the following conditions are satisfied:

i) L, M are complex manifolds,

ii) $\pi : L \to M$ is a holomorphic map whose fibers $L_x = \pi^{-1}(x)$, $x \in M$, have a linear structure,

iii) for each point $x_0 \in M$ there exists a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "g"] \arrow[dr, "\pi_U"'] & & \mathbb{C} \times U \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

where U is an open neighborhood of x_0, p is projection and g is a biholomorphic map that respects the linear structure on $\pi^{-1}(x) = L_x$ and $p^{-1}(x) = \mathbb{C}$.

For such a line bundle, we sometimes write $L \overset{\pi}{\to} M$, or $L \to M$, or even L, if no confusion can arise. We call two line bundles (L_1, π_1, M) and (L_2, π_2, M) *isomorphic*, if there exists a commutative diagram

$$\begin{tikzcd} L_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & L_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

with a biholomorphic map φ preserving the linear structures on $L_{1,x}$ and $L_{2,x}$ for all $x \in M$. In all our considerations, we can replace a holomorphic line bundle by an isomorphic one; so, we are concerned in reality with *isomorphism classes* of such bundles without mentioning this fact all the time.

Like complex manifolds themselves, we can construct line bundles by a scissoring and patching procedure. Given a line bundle $L \overset{\pi}{\to} M$, we have an open covering $\mathfrak{U} = \{ U_j \}$ of M with trivializations

$$\begin{tikzcd} L_{|U_j} = \pi^{-1}(U_j) \arrow[rr, "g_j"] \arrow[dr, "\pi_j = \pi_{|U_j}"'] & & \mathbb{C} \times U_j \arrow[dl, "p"] \\ & U_j & \end{tikzcd}$$

On an intersection $U_{jk} = U_j \cap U_k$, the composite map

$$\mathbb{C} \times U_{jk} \xrightarrow{g_j^{-1}} \pi_j^{-1}(U_j \cap U_k) = \pi_k^{-1}(U_k \cap U_j) \xrightarrow{g_k} \mathbb{C} \times U_{kj}$$

is biholomorphic, preserves the product structure and is a linear bijection on each fiber. Therefore, if (v_j, x) denotes points in $\mathbb{C} \times U_j$, $g_k \circ g_j^{-1}$ is of the form

$$g_k \circ g_j^{-1}(v_j, x) = (f_{kj}(x) \cdot v_j, x) ,$$

where f_{kj} is a nowhere vanishing holomorphic function on $U_j \cap U_k$. The system (f_{kj}), $f_{kj} \in \mathcal{O}_M^*(U_j \cap U_k)$, satisfies the following conditions

i) $f_{jj} = 1$ for all j,

ii) $f_{jk} = \dfrac{1}{f_{kj}}$ for all j, k with $U_j \cap U_k \neq \emptyset$,

iii) $f_{\ell k} f_{kj} = f_{\ell j}$ for all j, k, ℓ with $U_j \cap U_k \cap U_\ell \neq \emptyset$.

We call such a system a *holomorphic cocycle* with respect to the open covering $\mathfrak{U}$. The totality of all such cocycles is denoted by

$$Z^1(\mathfrak{U}, \mathcal{O}_M^*) .$$

On the other hand, given a cocycle (f_{kj}), then by patching

$$\mathbb{C} \times U_j \text{ and } \mathbb{C} \times U_k$$

along $\mathbb{C} \times U_{jk}$ via identifying

$$(v_j,\, x_j) \text{ and } (v_k,\, x_k)$$

if and only if $x_k = x_j = x$ and $v_k = f_{kj}(x)\, v_j$, we get a well–defined line bundle on M - which is isomorphic to L, if we construct it with the help of any system (f_{kj}) that is induced by a trivializing covering $\mathfrak{U}$ for L.

It is easily checked using the explicit trivializations given in the previous Section that the tautological bundle L on $\mathbb{P}_n$ belongs to the cocycle

$$f_{kj} = \frac{x_k}{x_j} \in \mathcal{O}^*_{\mathbb{P}_n}(U_j \cap U_k) ,$$

where $[\, x_0 : \cdots : x_n \,]$ denote homogeneous coordinates on $\mathbb{P}_n$.

We leave it as an exercise to the reader to develop the definition of a holomorphic *vector bundle* $(V,\, \pi,\, M)$ of rank r on M which is supposed to have r–dimensional vector spaces as fibers $V_x = \pi^{-1}(x)$, and to build up such bundles via systems (f_{kj}) of *invertible* $r \times r$*–matrices* of holomorphic functions on $U_j \cap U_k$ with respect to an open covering $\mathfrak{U} = \{\, U_j \,\}$ of M, satisfying the matrix relations

$$f_{\ell k}(x) \cdot f_{kj}(x) = f_{\ell j}(x) , \quad x \in U_j \cap U_k \cap U_\ell .$$

We always refer to the matrices (f_{kj}) as to the *transition matrices* for the vector bundle. In case $r = 1$, we have the old notion of a line bundle, where we call the f_{kj} the *transition functions*.

If we allow the entries of the matrices f_{kj} to be only (real) differentiable (or continuous) functions, we get the notion of a *differentiable* (or a *topological*) vector bundle which even makes sense on differentiable (or topological) manifolds. In particular, each holomorphic vector bundle carries the structure of a differentiable bundle, whereas the opposite may fail to hold true.

4.7 The σ - modification

We return to the tautological bundle L on $\mathbb{P}_n$ and investigate now the behaviour of L under the projection from $\mathbb{C}^{n+1} \times \mathbb{P}_n$ to the *first* factor $\mathbb{C}^{n+1}$, i.e. we study the composite σ of the canonical maps

$$L \overset{i}{\hookrightarrow} \mathbb{C}^{n+1} \times \mathbb{P}_n \xrightarrow{q} \mathbb{C}^{n+1} .$$

From the very definition of holomorphic manifolds and holomorphic maps, we conclude the following facts:

1. If N is a submanifold of M, then the inclusion $i : N \hookrightarrow M$ is a holomorphic map;
2. projections $q : M_1 \times M_2 \to M_1$ are holomorphic;
3. compositions $M_1 \to M_2 \to M_3$ of holomorphic maps are holomorphic.

L being an $(n + 1)$–dimensional submanifold of $\mathbb{C}^{n+1} \times \mathbb{P}_n$, these facts imply that $\sigma : L \to \mathbb{C}^{n+1}$ is a holomorphic map.

Topologically, σ is a *proper map* (i.e. preimages of compact sets $K \subset \mathbb{C}^{n+1}$ are compact, since

$$\sigma^{-1}(K) = L \cap q^{-1}(K) = L \cap (K \times \mathbb{P}_n) ,$$

$\mathbb{P}_n$ is compact and L is closed in $\mathbb{C}^{n+1} \times \mathbb{P}_n$). In particular, the fibers $\sigma^{-1}(\xi)$, $\xi \in \mathbb{C}^{n+1}$, are compact subsets of L.

However, there is a dramatic jump of the dimensions of these fibers: While $\sigma^{-1}(\xi)$ consists of one point only for all $\xi \neq 0$, the fiber over $\xi = 0$ is the n–dimensional projective space $\mathbb{P}_n$ itself. The bijective map

$$\sigma_{|\sigma^{-1}(\mathbb{C}^{n+1}\setminus\{0\})} : \sigma^{-1}(\mathbb{C}^{n+1} \setminus \{\,0\,\}) \longrightarrow \mathbb{C}^{n+1} \setminus \{\,0\,\}$$

is moreover biholomorphic, since its inverse is given by

$$\sigma^{-1}(\xi) \,=\, (\xi,\, \varphi\,(\xi)) \in \mathbb{C}^{n+1} \times \mathbb{P}_n\,, \quad \xi \neq 0\,,$$

where $\varphi : \mathbb{C}^{n+1} \setminus \{\,0\,\} \to \mathbb{P}_n$ denotes the canonical holomorphic quotient map. (Here, we also use the trivial fact that a holomorphic map $M_1 \to M_2$, which factorizes over a complex submanifold $N \subset M_2$, induces a holomorphic map $M_1 \to N$).

To get a precise picture of σ, we compute the *closure* of

$$\sigma^{-1}(\ell_\xi \setminus \{\,0\,\})$$

in L for a line ℓ_ξ through $\xi \in \mathbb{C}^{n+1} \setminus \{\,0\,\}$ and the origin which usually is called the *strict transform* or the *strict preimage* of ℓ_ξ in contrast to the *total transform*, i.e. the *full preimage* $\sigma^{-1}(\ell_\xi)$. It turns out that

$$\overline{\sigma^{-1}(\ell_\xi \setminus \{\,0\,\})} \,=\, \{\,(\lambda\,\xi,\, [\xi]) \in \mathbb{C}^{n+1} \times \mathbb{P}_n : \,\lambda \in \mathbb{C}\,\}\,.$$

Therefore, in some sense, the map σ *separates* all lines through the origin in $\mathbb{C}^{n+1}$ in that it *modifies* $\mathbb{C}^{n+1}$, replacing the origin by the manifold $\mathbb{P}_n$ whose points represent the complex directions at $0 \in \mathbb{C}^{n+1}$.

A real picture in which also the line bundle structure can be seen is drawn in the next figure.

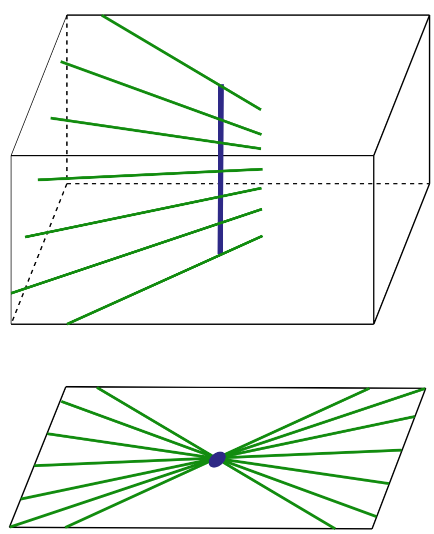


Figure 4.4

More about the σ–modification and general modifications is found in the next Chapter and in Chapters 5 and 6.

4.8 Canonical resolution of cones with a smooth base

We are going to apply the diagram

$$\begin{tikzcd} & L \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}_n & & \mathbb{C}^{n+1} \end{tikzcd}$$

to a *submanifold* $\underline{C} \subset \mathbb{P}_n$ (of dimension d). Because of the local product structure of π, it is clear that

$$\widetilde{C} := L_{|\underline{C}} := \pi^{-1}(\underline{C}) = \bigcup_{[x] \in C} (\ell_x \times [\, x\,])$$

is a $(d + 1)$–dimensional submanifold of L which contains a copy of $\underline{C}$ as a submanifold via the inclusion

$$\underline{C} \ni [\, x\,] \longmapsto (0, [\, x\,]) \in \widetilde{C}\, .$$

The image of $\widetilde{C}$ under σ is obviously equal to the affine cone $C \subset \mathbb{C}^{n+1}$ over $\underline{C}$:

$$\sigma\,(L_{|\underline{C}}) = \bigcup_{[x] \in \underline{C}} \ell_x = C\, ,$$

and we have

$$\sigma\,(\widetilde{C} \setminus \underline{C}) = C \setminus \{\, 0\,\}\, .$$

Since $\sigma_{|L \setminus \sigma^{-1}(0)}$ is biholomorphic onto $\mathbb{C}^{n+1} \setminus \{\, 0\,\}$, we finally have proved (by using σ for both, the original map and its restriction to $\widetilde{C}$):

Theorem 4.3 *Let $\underline{C} \subset \mathbb{P}_n$ be a complex submanifold of dimension d, and let $C \subset \mathbb{C}^{n+1}$ denote the corresponding affine cone. Then one can construct in a canonical way a $(d + 1)$–dimensional complex manifold $\widetilde{C}$ together with a proper holomorphic map*

$$\sigma : \widetilde{C} \longrightarrow C$$

such that

i) *$\sigma^{-1}(0)$ is a submanifold of $\widetilde{C}$ which is biholomorphic to $\underline{C}$,*

ii) *the restriction of σ to $\widetilde{C} \setminus \sigma^{-1}(0)$ is a biholomorphic map onto $C' = C \setminus \{\, 0\,\}$.*

In particular, C' is smooth of dimension $d + 1$ (in fact, it is a complex submanifold of $\mathbb{C}^{n+1} \setminus \{\, 0\,\}$).

In a certain sense, the (possibly) singular vertex of such a cone C *disappears*, if one replaces the origin by the base $\underline{C}$. This is our first example for the process of *resolving* singularities.

The following picture gives an idea of the geometrical content of Theorem 5 (herein, $\underline{C}$ is just the union of two different points in $\mathbb{P}_1$).

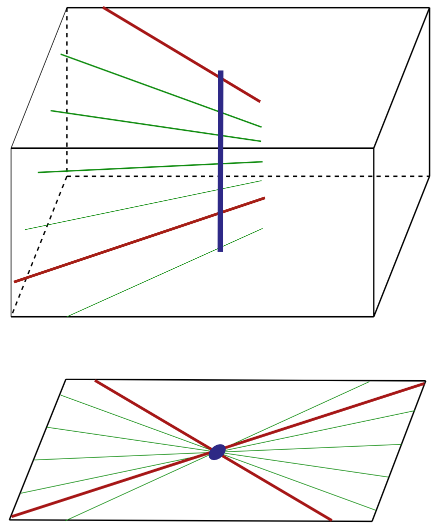


Figure 4.5

We should add here the remark that the a priori more general *complex analytic* cones that are treated in the present Section are nevertheless algebraic in the sense of our definition in Section 1. This statement is equivalent to *Chow's Lemma* (see the Appendix to this Chapter).

4.9 Sections in holomorphic line bundles

In the next three Sections we endeavor to give a description of the resolution $\sigma : \widetilde{C} \to C$ completely in terms of the line bundle L (restricted to the submanifold $\underline{C}$). It turns out that the holomorphic functions on $\widetilde{C} = L_{|\underline{C}}$ that we need in order to form σ can be constructed from *holomorphic sections* in the *dual* bundle of L.

A *holomorphic section* of a holomorphic vector bundle $F \xrightarrow{\pi} M$ on an open set $U \subset M$ is a holomorphic map $s : U \to F$ with $\pi \circ s = \mathrm{id}$. Such a section s can be thought of as being a family of vectors $s(x) \in F_x = \pi^{-1}(x)$, $x \in U$, varying holomorphically with $x \in U$.

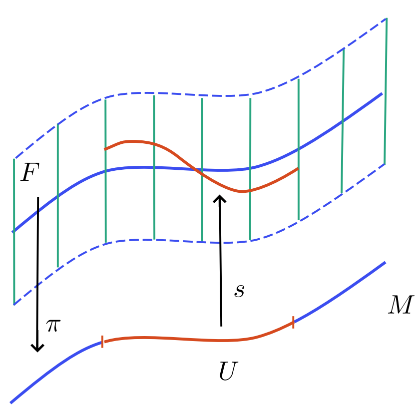


Figure 4.6

If F is given (on U, say) by transition matrices f_{kj} with respect to a covering $\mathfrak{U}$, then $s_j = s_{|U_j}$ can be represented by an r–tuple of holomorphic functions $s_j = (s_j^1, \ldots, s_j^r)$ on U_j in the sense that for the trivialization $F_{|U_j} \xrightarrow{\sim} \mathbb{C}^r \times U_j$ the diagram

$$
\begin{tikzcd}
F_{|U_j} \arrow[rr, "\sim"] \arrow[dr, "s"'] & & \mathbb{C}^r \times U_j \arrow[dl, "x \,\mapsto\, (s_j(x),\, x)"] \\
& U_j &
\end{tikzcd}
$$

commutes, and that on $U_j \cap U_k$ these local representations satisfy the compatibility condition

$$ {}^t s_k(x) \;=\; f_{kj}(x) \cdot {}^t s_j(x) \; . $$

The set of all such sections could be denoted by the somewhat clumsy symbol

$$ \mathcal{O}_M(F)\,(U) \; . $$

We prefer to use from now on the *cohomological* notations

$$ H^0(U,\, \mathcal{O}_M(F)) \;\text{ or }\; H^0(U,\, \mathcal{O}\,(F)) $$

(the zeroth cohomology group of U with values in the sheaf of holomorphic sections in the vector bundle F). We shall have to say more on cohomology groups with values in sheaves later.

As a matter of fact, the set $H^0(U,\, \mathcal{O}\,(F))$ is canonically endowed with a group structure, since sections can be added fiberwise. They can be multiplied by constants $c \in \mathbb{C}$ and, moreover, by holomorphic functions $f \in \mathcal{O}_M(U)$; it is easily checked that $H^0(U,\, \mathcal{O}\,(F))$ is a (unitary) $\mathcal{O}\,(U)$–module with respect to this operation.

In particular, we have always the zero-section in F over any set U, i.e. the zero element of $H^0(U,\, \mathcal{O}\,(F))$. As a map, this section embeds M into F as a submanifold (see the picture on the preceding page).

If we take the *trivial* line bundle $\mathbb{C} \times M \;\rightarrow\; M$, then holomorphic sections are nothing else but holomorphic functions. For this reason, we write

$$ H^0(U,\, \mathcal{O}) \;\text{ instead of }\; H^0(U,\, \mathcal{O}\,(\mathbb{C} \times M)) $$

and identify $\mathcal{O}\,(U)$ with $H^0(U,\, \mathcal{O})$.

Given any line bundle L on a manifold M , we can form new bundles by raising the transition functions f_{kj} of L to a fixed power $\ell \in \mathbb{Z}$. One can show that (the isomorphism classes of) these bundles do not depend on the special choice of a trivializing covering. We denote them by L^ℓ or $L^{\otimes \ell}$; in particular, $L^{\otimes 1}$ is equal to L , and $L^{\otimes 0}$ is the trivial bundle. $L^{\otimes -1}$ is also denoted by L^* , the *dual bundle* of L .

As an *Example*, we consider the dual bundle $H \,=\, L^*$ of the tautological bundle L on $\mathbb{P}_n$ and its powers $H^{\otimes \ell}$. In algebraic geometry, the notions

$$ \mathcal{O}\,(\ell) \;=\; \mathcal{O}\,(H^{\otimes \ell}) \; , \quad \mathcal{O}\,(-\ell) \;=\; \mathcal{O}\,(L^{\otimes \ell}) \; , \quad \ell \,\geq\, 0 \; , $$

are in common use. So we ask: what are the elements in $H^0(\mathbb{P}_n,\, \mathcal{O}\,(\ell))$, $\ell \in \mathbb{Z}$? For $\ell \,\geq\, 0$, we take a homogeneous polynomial of degree ℓ in the variables $x_0, \ldots, x_n$:

$$ P \;=\; \sum_{|\nu| = \ell} a_\nu \, x_0^{\nu_0} \cdot \ldots \cdot x_n^{\nu_n} \; . $$

Such a polynomial does not define a function on $\mathbb{P}_n$, but by dividing out x_j^ℓ we get well–defined holomorphic functions

$$ s_j \;=\; \frac{P}{x_j^\ell} \;=\; \sum_{|\nu| = \ell} a_\nu \left(\frac{x_0}{x_j} \right)^{\nu_0} \cdot \ldots \cdot \left(\frac{x_j}{x_j} \right)^{\nu_j} \cdot \ldots \cdot \left(\frac{x_n}{x_j} \right)^{\nu_n} $$

on U_j for all j which satisfy the relations

$$s_k = \left(\frac{x_j}{x_k}\right)^{\ell} \cdot s_j = f_{kj}^{-\ell} \cdot s_j .$$

Hence, each homogeneous polynomial of degree $\ell \geq 0$ determines an element in $H^0(\mathbb{P}_n, \mathcal{O}(\ell))$. One can show that this correspondence is bijective and respects the $H^0(\mathbb{P}_n, \mathcal{O}) = \mathbb{C}$–module structure:

$H^0(\mathbb{P}_n, \mathcal{O}(\ell))$, $\ell \geq 0$, is canonically isomorphic to the $\mathbb{C}$–vector space of all homogeneous polynomials of degree ℓ in $n+1$ variables. In particular, we have

$$\dim_{\mathbb{C}} H^0(\mathbb{P}_n, \mathcal{O}(\ell)) = \binom{n+\ell}{\ell} .$$

Since in general there exist no distinguished local trivializations of a line bundle, it does not make sense to speak of the *value* of a section at a point x. However, there is an intrinsic meaning for saying that s *vanishes* at x of order k. Especially in case $\ell = 1$, each nontrivial section of H, being determined by a nontrivial linear form, vanishes of precisely first order on a hyperplane in $\mathbb{P}_n$, and each hyperplane is the exact vanishing locus of a suitable section. This is the reason why H is called the *hyperplane bundle* on $\mathbb{P}_n$.

It is easily shown by Taylor expansions on the U_j that one has

$$H^0(\mathbb{P}_n, \mathcal{O}(\ell)) = 0 , \quad \ell < 0 .$$

We usually refer to this fact by saying that $\mathcal{O}(\ell)$ has *no sections* for $\ell < 0$, by what we mean, of course, that it has no *nontrivial holomorphic* sections. There are necessarily nontrivial *meromorphic* sections in $\mathcal{O}(\ell)$, $\ell < 0$, namely the inverses of sections in $\mathcal{O}(-\ell)$.

4.10 Very ample line bundles

Let now L denote again an arbitrary holomorphic line bundle on a compact complex manifold M. Then we have, as in the case $M = \mathbb{P}_n$, that $H^0(M, \mathcal{O}) \cong \mathbb{C}$ by applying the *Maximum Principle* for holomorphic functions. Moreover, the following *Finiteness Theorem* holds true:

***Theorem 4.4** *For a holomorphic line bundle L on a compact complex manifold M, the $\mathbb{C}$–vector space $H^0(M, \mathcal{O}(L))$ is finite dimensional.*

We assume that $H^0(M, \mathcal{O}(L)) \neq 0$, and choose a basis $s_0, \ldots, s_N$ of that finite dimensional vector space. As we mentioned already in the previous Section, the vector $(s_0(x), \ldots, s_N(x)) \in \mathbb{C}^{N+1}$ has a meaning only after we have chosen a trivialization of L near x. But taking another trivialization causes only the multiplication of $(s_0(x), \ldots, s_N(x))$ by a nonzero factor $\gamma(x) \in \mathbb{C}$. Therefore, we can state:

In the situation above, we get a well–defined holomorphic map

$$s : M \setminus Z \longrightarrow \mathbb{P}_N , \quad Z = \{ x \in M : s_0(x) = \cdots = s_N(x) = 0 \} ,$$

by sending x to $[\, s_0(x) : \cdots : s_N(x) \,]$ (in any local trivialization of L near x).

It is clear that the zero set Z does only depend on the line bundle L rather than on the special choice of a basis for $H^0(M, \mathcal{O}(L))$, and that the map $s : M \setminus Z \to \mathbb{P}_N$ is uniquely determined by L up to a projective linear automorphism of $\mathbb{P}_N$ (i.e. an automorphism of $\mathbb{P}_n$ which is induced from a linear automorphism of $\mathbb{C}^{n+1}$).

The points $x \in Z$ are classically called *base points* of L. If $Z = \emptyset$, L is said to be *base point free*. A holomorphic line bundle L on M without base points is called *very ample*, if the holomorphic map

$$s : M \longrightarrow \mathbb{P}_N$$

is injective and has everywhere constant rank equal to the dimension of M. By the definition of complex submanifolds, this implies that M can be identified with the complex analytic submanifold $s(M)$ in $\mathbb{P}_N$; that is: M is necessarily a projective algebraic manifold. (Notice that this argument yields a priori only the insight that $s(M)$ is *locally* a complex manifold. But since M is compact, its image $s(M)$ is closed in $\mathbb{P}_N$).

It is easily shown that the positive powers of the hyperplane bundle H on $\mathbb{P}_n$ are all very ample. The sections of $H^{\otimes \ell}$ embed $\mathbb{P}_n$ into $\mathbb{P}_N$, where

$$N = \binom{n+\ell}{\ell} - 1 \, ;$$

the image of $\mathbb{P}_n$ in $\mathbb{P}_N$ is called the *Veronese* or *Veronese embedding*.

4.11 The resolution of cones with smooth base, revisited

Taking up our former symbols, we denote by $\underline{C}$ an arbitrary compact complex manifold and by L a holomorphic line bundle on $\underline{C}$, whose *dual* L^* is now supposed to be very ample. Moreover, let $s : \underline{C} \to \mathbb{P}_n$ be the embedding supplied by the sections of L^*. It is clear that each vector bundle F on a manifold N can be lifted under holomorphic maps $f : M \to N$ in a unique manner to a bundle f^*F on M (we just lift the transition matrices). Since the coordinate "functions" x_j on $\mathbb{P}_n$ lift to s_j on $\underline{C}$ via the map s, we can easily prove that $L^* = s^*H$, where H denotes the hyperplane bundle on $\mathbb{P}_n$. Or, in other words: If we identify $\underline{C}$ with its image $s(\underline{C}) \in \mathbb{P}_n$, then L^* is (isomorphic to) the *restriction* $H_{|\underline{C}}$. (For a submanifold $M \subset N$ and a vector bundle F on N, the restriction $F_{|M}$ is the same as the bundle i^*F, where $i : M \hookrightarrow N$ denotes inclusion).

We can now formulate Theorem 3 in a somewhat more precise manner:

Theorem 4.5 *Let $L \to \underline{C}$ be a holomorphic line bundle on a compact complex manifold $\underline{C}$ such that L^* is very ample. Let $(s_0, \ldots, s_n)$ be a basis of $H^0(\underline{C}, \mathcal{O}(L^*))$, and denote by $s : \underline{C} \to \mathbb{P}_n$ the corresponding holomorphic embedding. Then the affine cone C over $s(\underline{C})$ can be resolved by L (viewed as a manifold) via the map $L \to C$ which is induced by the canonical map*

$$\sigma = (s_0, \ldots, s_n) : L \longrightarrow \mathbb{C}^{n+1}$$

(where sections in $H^0(\underline{C}, \mathcal{O}(L^*))$ are interpreted as holomorphic functions on L which are linear on the fibers of L).

Only the last sentence needs an explanation: If L is given by a cocycle (f_{kj}), then a section s of L^* is represented by holomorphic functions $s^{(j)} \in \mathcal{O}(U_j)$ such that

$$s^{(k)} = f_{kj}^{-1} s^{(j)} \text{ on } U_j \cap U_k \, .$$

The coordinates v_j with respect to a trivialization $\mathbb{C} \times U_j$ of L satisfy the conditions

$$v_k = f_{kj} \, v_j \, .$$

Hence, $s := s(x, v_j) = s^{(j)}(x) \, v_j$, $x \in U_j$, is well-defined on L, holomorphic and linear along the fibers.

4.12 The cone over the rational normal curve of degree n

According to our low dimensional aims, we are mainly interested in cones over complex *curves*. The simplest abstract compact complex manifold of dimension one *(Riemann surface)* is projective space $\mathbb{P}_1$, the *Riemann sphere*. We can realize it as a submanifold of $\mathbb{P}_n$ via the n–th power of the hyperplane bundle $H \to \mathbb{P}_1$. Our goal in this Section is to compute explicitly the resolution of the corresponding algebraic cone.

If u_j denotes the coordinate of $U_j \cong \mathbb{C}$ with respect to the standard covering $\mathfrak{U} = \{ U_0, U_1 \}$ of $\mathbb{P}_1$, then the total space of $H^{\otimes n}$ can be constructed directly by patching two copies of $\mathbb{C}^2$ with coordinates (u_0, w_0) and (u_1, w_1) by the rule

$$\begin{cases} u_0 = \dfrac{1}{u_1} \\ w_0 = \dfrac{w_1}{u_1^n} \end{cases} \qquad (u_1 \neq 0) \, .$$

The $n + 1$ independent sections in $H^{\otimes n}$ are given in local coordinates by

$$s_j = (u_0^j \, , \, u_1^{n-j}) \, , \quad j = 0, \ldots, n \, ,$$

sucht that the induced map $s : \mathbb{P}_1 \to \mathbb{P}_n$ is described on $U_0 \cong \mathbb{C}$ by

$$U_0 \ni u_0 \longmapsto (u_0, u_0^2, \ldots, u_0^n) \in \mathbb{C}^n \cong U_0 \subset \mathbb{P}_n \, .$$

Using homogeneous coordinates $[\, \xi_0 : \xi_1 \,]$ on $\mathbb{P}_1$, the map s can also be written as

$$[\, \xi_0 : \xi_1 \,] \longmapsto [\, \xi_0^n : \xi_0^{n-1} \xi_1 : \cdots : \xi_1^n \,] \, .$$

From these descriptions it is deduced very easily that s is injective and of constant rank 1, as we claimed in Section 12.

It is customary to call the image $s(\mathbb{P}_1)$ the *rational normal curve* of degree n in $\mathbb{P}_n$. Here, the term "rational" refers to the fact that $s(\mathbb{P}_1)$ is biholomorphic to the *rational curve* $\mathbb{P}_1$; $s(\mathbb{P}_1)$ is called "of degree n", since almost all hyperplanes in $\mathbb{P}_n$ cut $s(\mathbb{P}_1)$ in exactly n points. We shall prove in Chapter 6 that the cone C over $s(\mathbb{P}_1)$ has at the vertex an isolated singularity which is a quotient of $\mathbb{C}^2$ by the action of a cyclic group of order n, whence a *normal* singularity. C is obviously realized in $\mathbb{C}^{n+1}$ as the locus of common zeros of the polynomials

$$x_j x_k - x_{j+1} x_{k-1} \, , \quad 0 \leq j \, , k \leq n \, , \quad j \leq k - 2 \, .$$

In particular, for $n = 2$, we have the A_1–singularity

$$x_0 x_2 - x_1^2 = 0 \, .$$

For $n = 3$, we meet again the example given in Section 6.

The dual bundle of $H^{\otimes n}$ whose total space provides us with a resolution $\widetilde{C}$ of the affine cone emerges by patching

$$\mathbb{C}^2 \ni (u_0, v_0) \sim (u_1, v_1) \in \mathbb{C}^2$$

if and only if

$$\begin{cases} u_0 = u_1^{-1} \\ v_0 = u_1^n \, v_1 \end{cases} \qquad (u_1 \neq 0) \, .$$

On this manifold, the $n + 1$ functions of Theorem 5 are the following:

$$s_0 = u_0^n v_0 = v_1 \, , \; s_1 = u_0^{n-1} v_0 = u_1 v_1 \, , \; \ldots \, , \; s_n = v_0 = u_1^n v_1 \, .$$

It is obvious that $s = (s_0, \ldots, s_n) : \widetilde{C} \to \mathbb{C}^{n+1}$ factorizes over C and that it pushes the set

$$\mathbb{P}_1 \cong \{ v_0 = v_1 = 0 \} \subset \widetilde{C}$$

down to the origin.

4.13 Resolution of singularities of determinantal varieties

It is an illuminating exercise to *resolve the singularities* of the hypersurface

$$H = M^{(n-1)}(n \times n, \mathbb{C}) \subset M^{(n)}(n \times n, \mathbb{C}) = \mathbb{C}^{n \times n} , \quad n \geq 2 .$$

Quite naturally, one is tempted to study the space

$$\widetilde{H} = \{ (A, x) \in H \times \mathbb{C}^n : A\, x = 0 \} .$$

Since $\operatorname{rank} A \leq n - 1$ for $A \in H$, the linear space $\{ x \in \mathbb{C}^n : A\, x = 0 \}$ is not trivial; therefore we have a well–defined closed analytic subspace

$$\widetilde{H} = \{ (A, [\, x\,]) \in H \times \mathbb{P}_{n-1} : A\, x = 0 \} \subset H \times \mathbb{P}_{n-1}$$

such that the restriction of the projection $H \times \mathbb{P}_{n-1} \to H$ to $\widetilde{H}$ is necessarily a proper surjective holomorphic map

$$(*) \qquad \pi : \widetilde{H} \longrightarrow H .$$

We leave it as an *Exercise* to the reader to convince himself or herself that $\widetilde{H}$ is in fact a smooth *submanifold* of $M^{(n)}(n \times n, \mathbb{C}) \times \mathbb{P}_{n-1} = \mathbb{C}^{n \times n} \times \mathbb{P}_{n-1}$ of dimension $n^2 - 1$. To be a little bit more precise, the Jacobi matrix of the map $(x, A) \mapsto A\, x$ is of the form

$$J := \left(\; A \;\middle|\; B \; \right) ,$$

where B is the $n \times n^2$–matrix

$$\begin{pmatrix} x_1 & \cdots & x_n & 0 & \cdots & 0 & \cdots\cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_1 & \cdots & x_n & \cdots\cdots & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots\cdots & x_1 & \cdots & x_n \end{pmatrix} .$$

Consequently, $\operatorname{rank} J = n$ for all A and $x \neq 0$ with $A\, x = 0$ (such that necessarily $\operatorname{rank} A \leq n - 1$). Therefore, by the Implicit Function Theorem, the fiber of the map $(\mathbb{C}^n \setminus \{ 0 \}) \times M^{(n)}(n \times n, \mathbb{C}) \ni (x, A) \mapsto A\, x \in \mathbb{C}^n$ over the origin is a submanifold of dimension $n^2 + n - n = n^2$. This immediately implies that $\widetilde{H}$, which is the quotient of this fiber by the natural action of $\mathbb{C}^*$ on the first factor, is a smooth (sub)manifold of dimension $n^2 - 1$.

If $A \notin \operatorname{sing} H$, then $\operatorname{rank} A = n - 1$ and $\{ x \in \mathbb{C}^n : A\, x = 0 \}$ is 1–dimensional, thus defining a *point* in $\mathbb{P}_{n-1}$. Hence,

$$\pi \Big|_{\pi^{-1}(H \setminus \operatorname{sing} H)} : \widetilde{H} \setminus \pi^{-1}(\operatorname{sing} H) \longrightarrow H \setminus \operatorname{sing} H$$

is bijective and moreover, as one can easily prove as in the exercise just proposed, *biholomorphic*. - In conclusion, we can state the following theorem.

Theorem 4.6 *The map $\pi : \widetilde{H} \to H$ is a resolution of the singularities of*

$$H = \{ A \in M\,(n \times n, \mathbb{C}) : \det A = 0 \} .$$

It is interesting to study the manifold $\widetilde{H} \subset M\,(n \times n, \mathbb{C}) \times \mathbb{P}_{n-1}$ under the projection to $\mathbb{P}_{n-1}$ more closely. If $x \in \mathbb{C}^n \setminus \{ 0 \}$ is a representative of $[\, x\,] \in \mathbb{P}_{n-1}$, the matrices $A \in M\,(n \times n, \mathbb{C})$ with $A\, x = 0$ form a vector space of dimension $n^2 - n = n\,(n - 1)$. In fact, the map $\widetilde{H} \to \mathbb{P}_{n-1}$ exhibits $\widetilde{H}$ as a *vector bundle* of rank $n^2 - n$.

We prove the last statement in case $n = 2$, identifying the rank 2–bundle $\widetilde{H} \to \mathbb{P}_1$ in a more specific way. For $x = (x_0, x_1)$, $x_1 \neq 0$, all 2×2–matrices killing x are of the form

$$\begin{pmatrix} \lambda_0 & -\lambda_0 \dfrac{x_0}{x_1} \\ \lambda_1 & -\lambda_1 \dfrac{x_0}{x_1} \end{pmatrix} , \quad (\lambda_0, \lambda_1) \in \mathbb{C}^2 .$$

Correspondingly, we have for $x = (x_0, x_1)$, $x_0 \neq 0$, the matrices

$$\begin{pmatrix} \mu_0 \dfrac{x_1}{x_0} & -\mu_0 \\ \mu_1 \dfrac{x_1}{x_0} & -\mu_1 \end{pmatrix} , \quad (\mu_0, \mu_1) \in \mathbb{C}^2 .$$

This shows that $\widetilde{H} \to \mathbb{P}_1$ can be identified with a vector bundle of rank 2. On the intersection $x_0 x_1 \neq 0$, we necessarily have

$$\lambda_0 = \mu_0 \frac{x_1}{x_0} , \quad \lambda_1 = \mu_1 \frac{x_1}{x_0} .$$

Therefore, we see that

$$\widetilde{H} = \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$$

as a bundle over $\mathbb{P}_1$. Writing a general 2×2–matrix in the form

$$\begin{pmatrix} \xi_1 + \xi_2 & \xi_3 + s \\ \xi_3 - s & \xi_1 - \xi_2 \end{pmatrix}$$

we get a concrete resolution of the 3–dimensional A_1–singularity given by the equation

$$(**) \qquad \xi_1^2 - \xi_2^2 - \xi_3^2 + s^2 = 0 .$$

This can be thought of as being a one–dimensional deformation of the 2–dimensional A_1–singularity

$$\xi_1^2 - \xi_2^2 - \xi_3^2 = 0$$

with the parameter s. In fact, it is a $2:1$–covering of the *versal* deformation which is given by the equation

$$\xi_1^2 - \xi_2^2 - \xi_3^2 + t = 0 ;$$

(c.f. the Appendix to Chapter 5). Moreover, it is easily checked that the fiber of $\widetilde{H}$ over any s is smooth and resolves the singularities of the fiber of $(**)$ over s. The existence for such a *simultaneous resolution* of the versal deformation (after finite base change) is characteristic for the *rational double points* (c.f. Chapter 16). Clearly, restricting our considerations to the special fiber over $s = 0$, i.e. to the subspace $\{ a_{12} = a_{21} \}$ of *symmetric* 2×2–matrices whose determinantal variety has an A_1–singularity in $\mathbb{C}^3$ at the origin, we must put

$$\lambda_1 = -\lambda_0 \frac{x_0}{x_1} .$$

Therefore, we get again a resolution of the A_1–singularity as a *line bundle* over $\mathbb{P}_1$. In local fiber coordinates λ_0, $-\mu_1$, we find

$$-\mu_1 = -\lambda_1 \frac{x_0}{x_1} = \lambda_0 \left(\frac{x_0}{x_1} \right)^2 ,$$

such that this bundle is, as we know, isomorphic to the bundle $\mathcal{O}_{\mathbb{P}_1}(-2)$.

4.A Appendix: Algebraicity criteria

4.A.1 Analytic cones

By a (local) *analytic cone* we understand an analytic set $C \subset \mathbb{C}^n$ (resp. an analytic set defined in a neighborhood of the origin $0 \in \mathbb{C}^n$) satisfying the condition

$$C \ni x \implies t\,x \in C \text{ for all } t \in \mathbb{C} \text{ (resp. with } |\,t\,| \le \varepsilon\,).$$

It is plain that algebraic cones are necessarily analytic. The converse is true but not obvious. We give a more precise statement and an outline of the proof.

Theorem 4.7 *Let $X \subset \mathbb{C}^n$ be a cone. Then the following are equivalent* :

i) X *is algebraic* ;

ii) X *is analytic* ;

iii) X *is locally analytic* ;

iv) X *is locally near* 0 *the zero set of finitely many homogeneous polynomials* ;

v) X *is the zero set of finitely many homogeneous polynomials.*

Proof. It is plain that v) $\implies$ i) $\implies$ ii) $\implies$ iii). To show iv) $\implies$ v) we assume that there are a neighborhood $U = U(0)$ and finitely many *homogeneous* polynomials $P_1, \ldots, P_\ell$ such that

$$X \cap U \,=\, A \cap U\,, \quad A \,=\, \{\, x \in \mathbb{C}^n :\, P_1(x) \,=\, \cdots \,=\, P_\ell(x) \,=\, 0\,\}\,.$$

Clearly, A is an (algebraic) cone. The conclusion then follows from the following general

Lemma 4.8 *Suppose that X, Y are cones in $\mathbb{C}^n$ with $X \cap U \subset Y \cap U$ for a certain neighborhood U of $0 \in \mathbb{C}^n$. Then, $X \subset Y$.*

Proof. Obviously, $0 \in X \cap Y$. So, let $x \in X$ be different from zero. Then, for a certain $t \in \mathbb{C}^*$, $t\,x \in X \cap U \subset Y \cap U \subset Y$, and therefore $x \,=\, t^{-1}(t\,x) \in Y$. $\square$

It remains to show iii) $\implies$ iv). This conclusion follows from the slightly more general

Lemma 4.9 *Let X be an analytic subset of a ball $B \,=\, B(0,\, R)$ with $\mathbb{C}{\times}X \ni (t,\, x) \mapsto t\,x \in X$ for all $t \in \mathbb{C}$ with $|\,t\,| \,<\, \varepsilon$, $\varepsilon \,>\, 0$. Then, there exist homogeneous polynomials $P_1, \ldots, P_\ell \in \mathbb{C}[\,x_1, \ldots, x_n\,]$ such that*

$$X \cap B' \,=\, \{\, x \in B' :\, P_1(x) \,=\, \cdots \,=\, P_\ell(x) \,=\, 0\,\}\,, \quad B' \,=\, B(0,\, R')\,, \quad 0 \,<\, R' \,<\, R$$

sufficiently small.

Proof. After choosing R sufficiently small we may assume that

$$X \,=\, \{\, f_1(x) \,=\, \cdots \,=\, f_r(x) \,=\, 0\,, \quad f_1, \ldots, f_r \in \mathcal{O}(B)\,\}\,.$$

Define $F_j(t,\, x) \,:=\, f_j(t\,x)$ for $|\,t\,| \,<\, \varepsilon \,\le\, 1$ and $x \in B$. Clearly, F_j is a holomorphic function on $D \times B$, $D \,=\, \{\,|\,t\,| \,<\, \varepsilon\,\} \subset \mathbb{C}$. Since $0 \,=\, 0\,x \in X$ for $x \in X$, $F_j(0,\, x) \,=\, 0$ for all $x \in B$. Expanding F_j for fixed $x \in B$ into a power series in t, we get a convergent series

$$F_j(t,\, x) \,=\, \sum_{k=1}^{\infty} f_{jk}(x)\, t^k$$

with holomorphic functions

$$f_{jk} \, = \, \frac{1}{k!} \, \frac{\partial^k F_j}{\partial t^k}\bigg|_{t=0}$$

on B. Expanding f_j near 0 in a convergent power series

$$f_j(x) \, = \, \sum_{k=1}^{\infty} P_{jk}(x) \, , \quad P_{jk}(x) \, = \, \sum_{|\alpha|=k} c_\alpha \, x^\alpha \, ,$$

we have

$$F_j(t, \, x) \, = \, \sum_{k=1}^{\infty} P_{jk}(t\,x) \, = \, \sum_{k=1}^{\infty} P_{jk}(x) \, t^k \, .$$

In particular, $f_{jk}(x) \, = \, P_{jk}(x)$ is a (globally defined) homogeneous polynomial of degree k.

Now, it is easily seen that $f_{jk}(x) \, = \, 0$ for all $j, \, k$ and $x \in X \cap B$: Assume that this is true for all $k \, \leq \, \kappa$; then,

$$f_j(t\,x) \, = \, F_j(t, \, x) \, = \sum_{k=\kappa+1}^{\infty} f_{jk}(x) \, t^k \, = \, t^{\kappa+1} \sum_{k=\kappa+1}^{\infty} f_{jk}(x) \, t^{k-(\kappa+1)} \, = \, 0$$

for all $x \in X \cap B$, $0 \, < \, |\,t\,| \, < \, \varepsilon$, hence

$$\sum_{k=\kappa+1}^{\infty} f_{jk}(x) \, t^{k-(\kappa+1)} \, = \, 0 \, , \quad x \in X \cap B \, , \; 0 \, < \, |\,t\,| \, < \, \varepsilon \, .$$

Forming the limit for $t \, \to \, 0$, we immediately see that

$$f_{j,\kappa+1}(x) \, = \, 0 \, , \quad x \in X \cap B \, .$$

Hence, $X \subset \{\, x \in B \, : \, f_{jk}(x) \, = \, 0 \, , \; j \, = \, 1, \ldots, r \, , \; k \, = \, 1, \ldots, \infty \,\}$. On the other hand, if $x \in B' \, = \, B\,(0, \, R')$ and $f_{jk}(x) \, = \, 0$ for all $j, \, k$, R' sufficiently small, then

$$f_j(x) \, = \sum_{k=1}^{\infty} P_{jk}(x) \, = \sum_{k=1}^{\infty} f_{jk}(x) \, = \, 0 \, .$$

Therefore,

$$X \cap B' \, = \, \{\, x \in B' \, : \, f_{jk}(x) \, = \, 0 \, , \quad j \, = \, 1, \ldots, r \, , \; k \, = \, 1, \ldots, \infty \,\} \, .$$

Finally, the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by the germs of the functions f_{jk} at 0 will be generated by finitely many of them which we call $P_1, \ldots, P_\ell$. Shrinking B' again if necessary, we see that

$$X \cap B' \, = \, \{\, x \in B' \, : \, P_\lambda(x) \, = \, 0 \, , \quad \lambda \, = \, 1, \ldots, \ell \,\} \, . \qquad \square$$

4.A.2 Chow's Lemma

This is one of the most prominent algebraicity results, sometimes also referred to as the *Theorem of Chow*:

***Theorem 4.10** *Each analytic subset $A \subset \mathbb{P}_n$ is algebraic; i.e. it is equal to the base of an algebraic cone $C \subset \mathbb{C}^{n+1}$.*

A *proof* of this result can be given along these lines: Let $\varphi : \, \mathbb{C}^{n+1} \setminus \{\,0\,\} \, \to \, \mathbb{P}_n$ denote the canonical map as usual, and put $C' \, = \, \varphi^{-1}(A)$. Since φ is a holomorphic map, C' is an analytic subset of $\mathbb{C}^{n+1} \setminus \{\,0\,\}$, whose closure C in $\mathbb{C}^{n+1}$ is clearly equal to $C' \cup \{\,0\,\}$. In view of the preceding Section, the claim amounts to maintaining that the closure C is an analytic subset locally near the origin. This is indeed a consequence of the *Extension Theorem for Analytic Sets* which we do not want to formulate at the moment.

4.A.3 Artin's algebraicity criterion

By Chow's Lemma, any analytic cone is *algebraic* in the sense that it can be described near the vertex by the vanishing of polynomials instead of holomorphic functions. An analytic set $A \subset U \subset \mathbb{C}^n$ is called *algebraizable* at a point $x^{(0)} \in A$, if, after an *analytic* change of coordinates near $x^{(0)}$, the set A is *algebraic* in the new coordinates. Examples of such points are smooth ones, but also isolated hypersurface singularities (see Chapter 1.10). Perhaps surprisingly, even the following holds true:

***Theorem 4.11 (M. Artin)** *An arbitrary analytic set is algebraizable at any of its isolated singular points.*

There is no need for applications of this result in our book, though it is present all the time in so far as only polynomials will show up in concrete examples.

Notes and References

In this Chapter we crossed the borderline between the realm of *complex analytic* and that of *(projective) algebraic geometry* for the first but not for the last time. That such a fluctuation in language or style is more or less unavoidable should be absolutely clear from the results we mentioned in the Appendix. There, however, we touched only the tip of an iceberg compound by the results of type GAGA, as thoroughly explained by

[04 - 01] J. P. Serre: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier 6, 1-42 (1956).

For instance, it is no accident that the transition functions for the hyperplane bundle on $\mathbb{P}_n$ are rational. This example only reflects the fact that all holomorphic vector bundles on projective algebraic manifolds (and, more generally, on projective algebraic varieties) are indeed naturally equipped with an algebraic structure.

Due to our education, our treatment will be anchored in the complex analytic category; but yet we shall use methods from Local Algebra, Commutative Algebra, Homological Algebra and Algebraic Geometry, whenever we can simplify arguments in this way. No attempt will be made, however, to cover the corresponding algebraic questions for (algebraically closed) ground fields k of characteristic $p > 0$.

From this point of view, the most helpful accompanying text on *Algebraic Geometry* for the reader of the present treatise should be

[04 - 02] Ph. Griffiths, J. Harris: Principles of Algebraic Geometry. Pure and Applied Mathematics, New York–Chichester–Brisbane–Toronto: John Wiley & Sons 1978

where also most of the material on Complex Analytic Geometry can be found. Occasionally, we will consult the other now standard textbooks on this subject:

[04 - 03] R. Hartshorne: Algebraic Geometry. Graduate Texts in Mathematics 52, New York–Heidelberg–Berlin: Springer 1977,

[04 - 04] D. Mumford: Algebraic Geometry I. Complex Projective Varieties. Grundlehren der mathematischen Wissenschaften 221, Berlin–Heidelberg–New York: Springer 1976.

[04 - 05] I. R. Shafarevich: Basic Algebraic Geometry. Die Grundlehren der mathematischen Wissenschaften 213, Berlin–Heidelberg–New York: Springer 1974.

Theorem 4 is only a special case of the *Finiteness Theorem* of Cartan and Serre which says that *all* cohomology groups with values in a coherent analytic sheaf on a compact complex analytic variety are finite dimensional vector spaces - see e.g. [04 - 03], III. Theorem 5.2 for the projective algebraic case, and for the complex analytic case VIII A. Corollary 10 in [01 - 03] or Chapter 6 in

[04 - 06] H. Grauert, R. Remmert: Theorie der Steinschen Räume. Grundlehren der mathematischen Wissenschaften 227, Berlin–Heidelberg–New York–Tokyo: Springer–Verlag 1977.

We will have the opportunity to come back to such questions in connection with Grauert's much more general *Coherence Theorem for Direct Image Sheaves* in the Supplement.

The outline for the *proof* of Chow's Lemma follows [01 - 01], p.184; see also [01 - 10], pp. 171–172. Artin's algebraicity criterion was proved in

[04 - 07] M. Artin: Algebraic approximation of structures over complete local rings. Publ. Math. IHES 36, 23–58 (1969).

More on analytic cones can be found in

[04 - 08] G. Fischer: Lineare Faserräume und kohärente Modulgarben über komplexen Räumen. Arch. Math. 18, 609–617 (1967).

[04 - 09] D. Prill: Über lineare Faserräume und schwach negative holomorphe Geradenbündel. Math. Zeitschr. 105, 313–326 (1968).

[04 - 10] R. Axelsson, J. Magnusson: Complex analytic cones. Math. Ann. 273, 601–627 (1986).

[04 - 11] R. Axelsson, J. Magnusson: A closure operation on complex analytic cones and torsion. Ann. Fac. Sci. Toulouse, VI. Ser., Math. 7, No. 1, 5–33 (1998).