Chapter 3

Übrigens war diese Zusammenschau ihrer, der Spiegelköpfe, Werk und Lehrbehauptung; sie waren nach ihrer eigenen Aussage sehr stark im Zusammenschauen und darin, alle möglichen Gau- und Ortsbeschirmer dem Atum-Rê-Horachte von On gleich zu achten, der seinerseits schon eine Zusammenschau und Sternbildfigur ursprünglich eigenständiger Numina war. Aus mehrerem eins zu machen, war ihr Vorzugsbetreiben, ja, wenn man sie hörte, gab es im Grunde nur zwei große Götter: einen der Lebenden, das war Hor im Lichtberge, Atum-Rê; und einen Totenherrn, Usir, das thronende Auge. Das Auge aber war auch Atum-Rê, nämlich das Sonnenrund, und so ergab sich bei zugespitztem Denken, daß Usir der Herr der Nachtbarke war, in welche, wie jedermann wußte, Rê nach Untergang umstieg, um von Westen nach Osten zu fahren und den unteren zu leuchten. Mit anderen Worten: auch diese beiden großen Götter waren genau genommen ein und derselbe. Wenn aber der Scharfsinn solcher Zusammenschau zu bewundern war, so war es nicht minder die Kunst der Lehrer, niemanden dabei zu kränken und ungeachtet ihres identifizierenden Betreibens die tatsächliche Vielheit der Götter Ägyptens unangetastet zu lassen. Das gelang ihnen vermittels der Wissenschaft vom Dreieck [...], der schönen Figur der Zusammenschau.

(Thomas Mann, *Joseph in Ägypten*)


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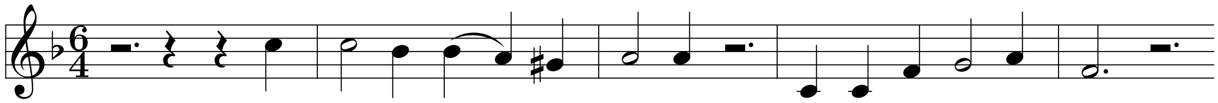

Chapter 3

The Preparation Theorem of Weierstraß

In the present Chapter we discuss several equivalent formulations of the Preparation and Division Theorems in the analytic, formal and differentiable context and derive from them some standard results on the local algebras we are interested in this text as well as a final exact definition and characterization of what we mean by a "singularity". The proof of these theorems will be given at the end of the Chapter via the Polynomial Preparation Theorem in the analytic case; the differentiable situation will be considered in the Appendix.

3.1 The Division and the Preparation Theorem

For any polynomial $G \in \mathbb{K}[t]$ with $\deg G = b$ one has the well known phenomenon of *division with remainder*: for each $F \in \mathbb{K}[t]$ there exist (uniquely determined) polynomials Q, R with

$$F = Q\,G + R\,, \quad \deg R < b\,.$$

Let now R_n, $n \geq 1$, be either of the rings $\mathcal{E}_{n,0}$, $\mathcal{O}_{n,0}$, $\widetilde{\mathcal{O}}_{n,0}$ with distinguished variable x_n. Then, denoting by R_{n-1} the corresponding ring with respect to the variables $x_1, \ldots, x_{n-1}$, we have obviously an inclusion of rings

$$R_{n-1}[\,x_n\,] \subset R_n\,;$$

let $\deg$ always denote in the sequel the x_n–degree of a polynomial in $R_{n-1}[\,x_n\,]$. We want to study such a *division problem* also in the rings R_n: For which $g \in R_n$, $g \neq 0$, is the *Division Theorem* true in the following form ?

There exists a number $a = a(g)$ such that for all $f \in R_n$ there exist elements $q \in R_n$, $r \in R_{n-1}[\,x_n\,]$, $\deg r < a$, with

$$f = q\,g + r\,.$$

Definition. If g satisfies the above condition we say that the *Division Theorem* or the Weierstrass *formula* holds for g in degree a.

We first want to deduce a *necessary* condition for such elements g. Let $f = x_n^a$ and write

$$x_n^a = q\,g + r\,, \quad \deg r < a\,.$$

If we put $\widetilde{h}(x_n) := h(0, \ldots, 0, x_n) \in R_1$ for arbitrary $h \in R_n$, we have in particular $\widetilde{h} \in \mathbb{K}[\,x_n\,]$ for $h \in R_{n-1}[\,x_n\,]$. Hence, in the equation above,

$$\widetilde{q} \cdot \widetilde{g} = x_n^a - \widetilde{r} \in \mathbb{K}[\,x_n\,]$$

is a polynomial of degree a; in particular, $\widetilde{g} \neq 0$, and via a–fold differentiation one concludes by the Leibniz rule that not all derivatives of $\widetilde{g}$ of order $\leq a$ can vanish at the origin. Therefore, $\widetilde{g}$ possesses a *finite* vanishing order $b \leq a$ such that

$$\widetilde{g}(x_n) = g(0, x_n) = \text{unit} \cdot x_n^b\,.$$

In the analytic and formal situation this can be established much faster by a power series argument.

Definition. An element $g \in R_n$, $n \geq 1$, is called x_n*–generic* (or x_n*–regular*, x_n*–distinguished*) of degree b if

$$\widetilde{g}\,(x_n) = g\,(0,\, x_n) = c\, x_n^b + \text{term of higher order}\,, \quad c \neq 0\,,$$

i.e. if $\widetilde{g}$ has a *finite* vanishing order (viz. b) in 0.

It is clear that if g is x_n–generic of degree 0 then $g\,(0) \neq 0$ and thus g is a unit. But for a unit one always has $f = (f\, g^{-1})\, g + 0$. We therefore can assume in the following that $b \geq 1$.

Remark. In the real– and complex analytic case it is easy to see that *each* element $g \neq 0$ in the maximal ideal of R_n is, after a so called *transvection* (i.e. a linear transformation of type $x_j \mapsto x_j$, $j = 1, \ldots, n-1$, $x_n \mapsto x_n + \sum_{j=1}^{n-1} c_j\, x_j$), generic with respect to the variable x_n. (Otherwise, $g = 0$ due to the Identity Theorem). However, in the differentiable situation this is not correct. One can obviously make a germ g generic only if it has a finite vanishing order, i.e. if it has a non vanishing Taylor series. But this is also sufficient: Under this assumption, there exists a smallest integer b with $g \in \mathfrak{m}_n^b$, $g \notin \mathfrak{m}_n^{b+1}$. Write, using Taylor's formula,

$$g = \varphi + \psi\,,$$

where φ is a (nonvanishing) homogeneous polynomial of degree b and $\psi \in \mathfrak{m}_n^{b+1}$. Choose an element $a \in \mathbb{K}^n$ with $\varphi\,(a) \neq 0$ and a linear isomorphism L of $\mathbb{K}^n$ such that $L\,(0, \ldots, 0,\, 1) = a$. Then,

$$(g \circ L)\,(0, \ldots, 0,\, x_n) = g\,(x_n a_1, \ldots, x_n a_n) = x_n^b\, \varphi\,(a) + \psi\,(x_n a)$$

with $\psi\,(x_n a) \in \mathfrak{m}_1^{b+1}$. – This argument, of course, also works in the analytic case.

The aim of the present Chapter is to clarify the logical connections between the *Division Theorem* which we state next and some other central results. The *proof* of the Division Theorem (and consequently of all forms which are shown to be equivalent in this Chapter) will be given later.

Theorem 3.1 (Division Theorem) *If $g \in R_n$, $n \geq 1$, is x_n–generic of order $b \geq 1$ then the Division Theorem holds for g in degree b. In the analytic case, one has uniqueness of the Weierstraß–decomposition.*

Remark. For the formal power series algebra $R_n = \widetilde{\mathcal{O}}_{n,0}$ this statement is not difficult to prove. The analytic case is classical (see, in particular with respect to uniqueness, the last Section), the differentiable case, however, was only successfully treated during the middle of the last century. The first proofs in this context are due to Malgrange and Mather (see the Appendix).

We first show that the *Division Theorem* implies the *Preparation Theorem*.

Theorem 3.2 (Preparation Theorem) *If $g \in R_n$, $n \geq 1$, is x_n–generic of degree $b \geq 1$ then there exists a polynomial*

$$\omega = x_n^b + a_1(x')\, x_n^{b-1} + \cdots + a_b(x') \in R_{n-1}[\, x_n\,]\,, \quad x' = (x_1, \ldots, x_{n-1})\,,$$

and a unit $e \in R_n$ satisfying $g = e\, \omega$.

Remarks. 1. We call ω a Weierstrass *polynomial* of degree b. Necessarily, $a_1(0) = \cdots = a_b(0) = 0$, $0 \in \mathbb{K}^{n-1}$, since

$$x_n^b + a_1(0)\, x_n^{b-1} + \cdots + a_b(0) = \omega\,(0,\, x_n) = c \cdot x_n^b + \text{terms of higher order}\,,\ c \neq 0\,.$$

2. For *uniqueness* of the Weierstraß decomposition in the analytic category c.f. Theorem 57.

The *proof* of the Preparation Theorem via the Division Theorem is most simple. Let g be x_n–generic of degree b and write $x_n^b = q\, g + r$, $\deg r < b$, $q\, g = x_n^b - r =: \omega$. Then $\widetilde{q}\, \widetilde{g} = \widetilde{\omega} = x_n^b - \widetilde{r}$, but without loss of generality $\widetilde{g} = x_n^b +$ terms of higher order. Then, $\widetilde{q}$ cannot vanish at 0, i.e. $\widetilde{q}$ and thus q also are units. $\square$

3.2 The Excellence Theorem

Surprisingly, there are many more, at first glance completely different, manifestations of the Division Theorem. To obtain them, we have to introduce some rather general algebraic notions whose geometric importance shall be worked out later.

Let A, B be local $\mathbb{K}$–algebras with a local homomorphism $\varphi : A \to B$. If M is a given B–module, we can define with the help of φ via $a\,m := \varphi(a)\,m$, $a \in A$, $m \in M$, the structure of an A–module on M; we sometimes write in this situation M_A instead of M. We are interested in conditions on the homomorphism φ which imply that any finitely generated module M on B - eventually assuming some mild properties of M_A - is also finitely generated on A. Recall the following definition in Chapter 2.

Definition. A module M on B is called *finitely generated* (or, sometimes, a *finite B–module* for short) if there exist elements $m_1, \ldots, m_r \in M$ such that each element $m \in M$ can be written in the form

$$m = \sum_{j=1}^{r} b_j\, m_j\ , \quad b_1, \ldots, b_r \in B\ .$$

We call a homomorphism $\varphi : A \to B$ *finite* if B is a finite A–module via φ.

Remarks. 1. Every surjective homomorphism is finite.

2. The canonical inclusion $R_{n-1} \hookrightarrow R_n$ is not finite.

The following is a trivial criterion.

Lemma 3.3 *If $\varphi : A \to B$ is a* finite *homomorphism then every finitely generated B–module M is finitely generated on A. In particular, compositions of finite homomorphisms are again finite.*

Proof. If $m_1, \ldots, m_r$ is a system of generators of M over B and $b_1, \ldots, b_s$ are generators for B on A then the finite system $\{\, b_k\, m_j : j = 1, \ldots, r,\ k = 1, \ldots, s \,\}$ generates the module M_A. □

Remark. If φ is an arbitrary local homomorphism and M_A is finitely generated then so is M.

Definition. The B–module M is called *quasi–finite* if the $B/\mathfrak{m}_B = \mathbb{K}$–vector space $M/\mathfrak{m}_B M$ is finite dimensional. A homomorphism $\varphi : A \to B$ is called quasi–finite if B is a quasi–finite A–module via φ, i.e. if $B/\mathfrak{m}_A B$ is a finite dimensional $\mathbb{K}$–vector space. M is called quasi–finite over A if M_A is a quasi–finite A–module.

Remarks. 1. Recall that for an ideal $\mathfrak{b} \subset B$ and a submodule $N \subset M$ the symbol $\mathfrak{b}\,N$ denotes the smallest submodule of M containing all products $b\,n$, $b \in \mathfrak{b}$, $n \in N$. It consists precisely of all finite sums of such products $b\,n$.

2. If M is finitely generated over B then $M/\mathfrak{m}_B M$ is a finite dimensional $\mathbb{K}$–vector space, i.e. M is quasi–finite. In particular, every finite homomorphism is quasi–finite. The converse is not true in general.

3. It is easily seen that

$$M_A/\mathfrak{m}_A M_A \cong M/(\mathfrak{m}_A B)\,M =: M/\mathfrak{m}_A M$$

as vector spaces on $A/\mathfrak{m}_A \cong \mathbb{K}$.

Lemma 3.4 *If the B–module M is quasi–finite over A it is so on B.*

Proof. Since φ is a local homomorphism, we have $(\mathfrak{m}_A B)\,M \subset \mathfrak{m}_B M$. Consequently, there is a canonical epimorphism from $M/\mathfrak{m}_A M$ onto $M/\mathfrak{m}_B M$. □

Definition. The local homomorphism $\varphi : A \to B$ is called *excellent* if every finitely generated B–module M which is quasi–finite over A is actually finite on A; i.e. if the following implication is valid:

$$\dim_{\mathbb{K}} M/\mathfrak{m}_A M < \infty \Longrightarrow M_A \text{ is finitely generated over } A\ .$$

Lemma 3.5 *Let $\varphi : A \to B$ be excellent and quasi–finite. Then φ is finite.*

Proof. Put $M = B$. □

For our considerations the next theorem is extremely useful.

Theorem 3.6 (Special Excellence Theorem) *The substitution homomorphisms*

$$\varphi : R_m \longrightarrow R_n$$

are excellent (in the analytic and differentiable category).

Remark. This is in the differentiable context the Division Theorem à la MATHER.

The Special Excellence Theorem is indeed *equivalent* to the Division Theorem as we will see at the end of a lengthy chain of arguments. We shall prove it with the help of the Division Theorem, even in a stronger form for which we need still another notion.

Definition. A local homomorphism $\varphi : A = R_m/\mathfrak{a} \to R_n/\mathfrak{b} = B$ is called a *substitution homomorphism* if there exists such a homomorphism from R_m to R_n, called ψ, making the diagram

$$\begin{tikzcd} R_m \arrow[r, "\psi"] \arrow[d, "\pi_m"'] & R_n \arrow[d, "\pi_n"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

commutative. We call φ sometimes also an *analytic* homomorphism (even in the differentiable context); ψ is called a *lifting* of φ.

Theorem 3.7 *In the case of convergent power series each local homomorphism $\varphi : A = R_m/\mathfrak{a} \to R_n/\mathfrak{b} = B$ is a substitution homomorphism.*

Proof. Let $\mathfrak{m}_m = (y_1, \ldots, y_m)\, R_m$ and $f_j \in \mathfrak{m}\,(R_n)$ a π_n–preimage of $\varphi\,(\pi_m(y_j)) \in B$. Then, by letting $\psi\,(y_j) := f_j$, we have a substitution homomorphism $\psi : R_m \to R_n$ for which $\pi_n \circ \psi\,(y_j) = (\varphi \circ \pi_m)\,(y_j)$ for all j and thus (due to Theorem 2.23) $\pi_n \circ \psi = \varphi \circ \pi_m$. Here, one has to use the fact that the rings R_m and B are *noetherian* (see Theorem 17). □

The general *Excellence Theorem* clearly should read as follows.

Theorem 3.8 (Excellence Theorem) *The local substitution homomorphisms*

$$\varphi : A = R_m/\mathfrak{a} \longrightarrow R_n/\mathfrak{b} = B$$

are excellent.

We are able to reduce the *proof* of this theorem to the special case of the canonical inclusions $R_m \hookrightarrow R_{m+1}$.

Lemma 3.9 *The following assumptions are equivalent*:

i) *all inclusions $R_m \hookrightarrow R_{m+1}$ are excellent*;

ii) *the* Special Excellence Theorem *holds*;

iii) *the* Excellence Theorem *holds.*

Proof. The implications iii) $\Longrightarrow$ ii) $\Longrightarrow$ i) are trivial. So assume i) and let $\varphi : A \to B$ be an analytic homomorphism and M a finite B–module which is quasi–finite over A. Then it is finite over R_n and quasi–finite over R_m. As in the proof of Theorem 7 we see at once that we can restrict ourselves to the case $A = R_m$ and $B = R_n$. Now, we construct substitution homomorphisms (R_{m+n} denotes the ring of functions in the variables $y_1, \ldots, y_m$, $x_1, \ldots, x_n$, R_n that in the variables $x_1, \ldots, x_n$):

$$i : \begin{cases} R_m \longrightarrow R_{m+n} \\ y_j \longmapsto y_j \end{cases} , \quad \pi : \begin{cases} R_{m+n} \longrightarrow R_n \\ y_j \longmapsto f_j \\ x_k \longmapsto x_k \end{cases}$$

where $f_j = \varphi(y_j)$. Then, the diagram of substitution homomorphisms

$$\begin{tikzcd} R_{m+n} \arrow[r, "\pi"] & R_n \\ R_m \arrow[u, "i"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes and π is surjective by construction. Therefore, we are reduced to the inclusions $i : R_m \hookrightarrow R_{m+n}$. But every (finite) R_{m+n}–module M which is quasi–finite on R_m is quasi–finite over any ring R_{m+k} with $R_m \subset R_{m+k} \subset R_{m+n}$ due to Lemma 4. Therefore, one can go down stepwise from $m+n$ to m or, in other words, it suffices to treat the case $n = 1$. $\square$

3.3 The Quasi - finiteness Theorem

Taking $M := B$ in the Excellence Theorem implies the (general) *Quasi–finiteness Theorem* which states the following.

Theorem 3.10 (Quasi - finiteness Theorem) *Each quasi–finite local substitution homomorphism* $\varphi : A \to B$ *is finite.*

Remark. It is possible to prove the *Quasi–finiteness Theorem* directly with analytic methods such that it may also be used as the *"central" Preparation Theorem* in the complex analytic theory. (Cf. e.g.: R. NARASIMHAN: *Introduction to the Theory of Analytic Spaces*. Lecture Notes in Mathematics 25. Berlin–Heidelberg–New York: Springer 1966).

Corollary 3.11 *A substitution homomorphism* $\varphi : A = R_m/\mathfrak{a} \to R_n/\mathfrak{b} = B$ *is finite if and only if there exists an integer* r *such that* $\mathfrak{m}_B^r \subset \varphi(\mathfrak{m}_A)\, B$.

Proof. This is an immediate consequence of Theorem 8 and Theorem 2.13. $\square$

It is not difficult to *prove* that the *Quasi–finiteness Theorem* (even in a weaker form) in turn implies the *Division Theorem*:

Let $g \in R_n$ be x_n–generic, $n \geq 1$. Look at the composite homomorphism

$$R_{n-1} \hookrightarrow R_n \longrightarrow R_n/\, g\, R_n =: A ,$$

which we denote by φ. If we calculate modulo the maximal ideal of R_{n-1}, we get due to the assumption of x_n–genericity of g:

$$A/\, \varphi(\mathfrak{m}_{n-1})\, A \cong R_1/\, x_n^b R_1 ,$$

R_1 the algebra of the function germs in the variable x_n. This is, without saying, a finite dimensional $\mathbb{K}$–vector space since it is generated by the residue classes of the powers $1, x_n, \ldots, x_n^{b-1}$. Therefore, φ is quasi–finite and hence finite due to our assumption. From the NAKAYAMA Lemma it follows that also the residue classes of $1, x_n, \ldots, x_n^{b-1}$ in A generate the R_{n-1}–module $A = R_n / g R_n$. Consequently, each element $f \in R_n$ can be written in the form $f = q g + r$, $r \in R_{n-1}[x_n]$, $\deg r < b$. □

3.4 Equivalence of the theorems

We are now going to close the circle for obtaining the following Theorem.

Theorem 3.12 *The Division Theorem, the Excellence Theorems and the Quasi–finiteness Theorem are equivalent.*

Due to Lemma 9 we only have to show that the *(very) Special Excellence Theorem* for the canonical inclusions $R_m \hookrightarrow R_{m+1}$ can be extracted from the Division Theorem. For doing so, we need a simple, but quite useful Lemma which has further implications (cf. [01 - 02], p.218).

Lemma 3.13 (Dedekind) *Let $R \subset S$ commutative rings with the same unit element 1, let N be an S–module, finitely generated on R, $\mathfrak{a} \subset R$ an ideal and $s \in S$ an element satisfying $s N \subset \mathfrak{a} N$. Then there exist elements $a_j \in \mathfrak{a}^j$, $1 \leq j \leq r = \mathrm{cg}_R N$, such that we have:*

$$s^r + a_1 s^{r-1} + \cdots + a_r \in \mathrm{An}_S N = \{ x \in S : x n = 0 \text{ for all } n \in N \} .$$

Remark. $\mathrm{An}_S N$ is easily seen to be an ideal in S. It is called by obvious reasons the *annihilating ideal* or *annulator* of N. If $\mathfrak{b} \subset \mathrm{An}_S N$ is an arbitrary ideal then N carries obviously the structure of an $S/\mathfrak{b}$–module, too.

Proof. Let $n_1, \ldots, n_r$ be a system of generators of N over R. Since $s n_j \in \mathfrak{a} N$ there are elements $a_{jk} \in \mathfrak{a}$ with

$$s n_j = \sum_{k=1}^{r} a_{jk} n_k ,$$

hence

$$\sum_{k=1}^{r} (s \delta_{jk} - a_{jk}) n_k = 0 , \quad j = 1, \ldots, r .$$

Put $d := \det A$, $A := (s \delta_{jk} - a_{jk})$. Expanding the determinant, we easily see that d is a monic polynomial in s of degree r with coefficients a_j in the ideal $\mathfrak{a}^j$. Denoting by $\widetilde{A}$ the matrix of *minors* of A (provided with the correct signs), one has by the LAPLACE Theorem for determinants

$$^t\widetilde{A} A = \det A \cdot E .$$

Since $A n = 0$ for the column vector $^t(n_1, \ldots, n_k)$ it follows $d n_j = 0$ for all j, i.e. $d N = 0$. □

Remarks. 1. The DEDEKIND Lemma implies the NAKAYAMA Lemma: Given a local ring A with maximal ideal $\mathfrak{m}$ and a finitely generated A–module N satisfying $N \subset \mathfrak{m} N$, put $A = R = S$, $s = 1$ and $\mathfrak{a} = \mathfrak{m}$. Then there exists an element $a \in \mathfrak{m}$ such that the unit $1 + a \in \mathrm{An}_A N$. Hence, $1 \in \mathrm{An}_A N$ and $n = 1 n = 0$ for all $n \in N$.

2. The DEDEKIND Lemma is equivalent to the following Theorem which is a stronger form of the classical Theorem of CAYLEY and HAMILTON for endomorphisms of finite dimensional vector spaces (take $\mathfrak{a} = R$).

Theorem 3.14 (Cayley - Hamilton) *If N is a finitely generated R–module and $\varphi : N \to N$ an R–endomorphism with $\varphi(N) \subset \mathfrak{a} N$ for an ideal $\mathfrak{a} \subset R$, then there exist elements $a_j \in \mathfrak{a}^j$, $j = 1, \ldots, r := \mathrm{cg}_R N$, such that*

$$\varphi^r + a_1 \varphi^{r-1} + \cdots + a_r \,\mathrm{id} = 0$$

as an endomorphism of N.

Proof. To prove this theorem take $S := R[x]$ and let x act on N via $x \cdot n := \varphi(n)$. The rest is an immediate consequence of Dedekind's Lemma. Conversely, one has to apply the result to the R–module endomorphism $\varphi(n) := s\,n$, $n \in N$. □

We are now in the position to *prove* the above mentioned implication *Division Theorem* $\Longrightarrow$ *(very) Special Excellence Theorem* for the natural inclusion $R_m \hookrightarrow R_{m+1}$. By replacing the index m with n, we have to consider a finite R_{n+1}–module M which is quasi–finite over $R_n \subset R_{n+1}$. We then apply the Dedekind Lemma first in the case

$$R = \mathbb{K}\,,\ S = R_{n+1} \text{ with variables } x_1, \ldots, x_n, x_{n+1},$$

and

$$s = x_{n+1}\,,\ \mathfrak{a} = R = \mathbb{K}\,,\ \mathfrak{m}_n = \mathfrak{m}(R_n)\,,\ N = M/\mathfrak{m}_n M\,.$$

Due to our assumptions, N is a finite R–module, and we have $s\,N \subset \mathfrak{a}\,N$. Thus, there exist $c_1, \ldots, c_t \in \mathbb{K}$ with

$$f := x_{n+1}^t + c_1\,x_{n+1}^{t-1} + \cdots + c_t \in \mathrm{An}_{R_{n+1}} M/\mathfrak{m}_n M\,,$$

i.e. $f\,M \subset \mathfrak{m}_n M$.

Invoking the same Lemma once more with $R = S = R_{n+1}$, $N = M$, $s = f$, $\mathfrak{a} = \mathfrak{m}_n R_{n+1}$ gives elements $h_1, \ldots, h_r \in \mathfrak{m}_n R_{n+1}$ such that

$$g = f^r + h_1\,f^{r-1} + \cdots + h_r \in \mathrm{An}_{R_{n+1}} M\,.$$

Since $g(0, \ldots, 0, x_{n+1}) = f^r$, the element g is x_{n+1}–generic. Then, by the Division Theorem, $R_{n+1}/g\,R_{n+1}$ is a finite R_n–module. But M is, because of $g \in \mathrm{An}_{R_{n+1}} M$, a finite $R_{n+1}/g\,R_{n+1}$–module and finally finite over R_n according to Lemma 3. □

3.5 Analytically generating systems

We first draw some conclusions from the Quasi–finiteness Theorem.

Theorem 3.15 *Let $\varphi : A \to B$ be a substitution homomorphism. The elements $g_1, \ldots, g_t \in B$ generate the ring B as an A–module via φ if and only if their residue classes $\overline{g}_1, \ldots, \overline{g}_t$ in $B' = B/\varphi(\mathfrak{m}_A)\,B$ generate the $\mathbb{K}$–vector space B'.*

If, in particular, $\mathfrak{m}_B^r \subset \varphi(\mathfrak{m}_A)\,B$ for some $r \in \mathbb{N}$ and if $g_1, \ldots, g_s$ is a system of generators for $\mathfrak{m}_B$ then φ is finite and the monomials

$$g_1^{\sigma_1} \cdot \ldots \cdot g_s^{\sigma_s}\,, \quad \sigma_1 + \cdots + \sigma_s < r$$

generate B as an A–module.

φ is an epimorphism if and only if $\mathfrak{m}_B = \varphi(\mathfrak{m}_A)\,B$.

Proof. a) Only one of the directions of the claim has to be shown. If $\overline{g}_1, \ldots, \overline{g}_t$ generate the $\mathbb{K}$–vector space B' then the map $\varphi : A \to B$ is quasi–finite and thus finite. Let $g_1, \ldots, g_t$ in B generate the A–module B''. Then, obviously, $B = B'' + \varphi(\mathfrak{m}_A)\,B$, and the Nakayama Lemma implies $B = B''$.

b) The residue classes of these monomials generate $B/\mathfrak{m}_B^r$ and consequently also B'.

c) For an epimorphism φ one has $\mathfrak{m}_B = \varphi(\mathfrak{m}_A) \subset \varphi(\mathfrak{m}_A)\,B \subset \mathfrak{m}_B$, and, conversely, from $\mathfrak{m}_B = \varphi(\mathfrak{m}_A)$ follows surjectivity of φ. The weaker condition $\mathfrak{m}_B = \varphi(\mathfrak{m}_A)\,B$ suffices because in b) one can choose $r = 1$ and conclude that $1 \in B$ is a generator of B over A. □

Remark. An exponent r as above exists if there is a number $b \geq 1$ such that $g_1^b, \ldots, g_s^b \in \varphi(\mathfrak{m}_A)\,B$. Just take $r = s\,b$.

Definition. The elements $g_1, \ldots, g_t$ in $\mathfrak{m}_A$, $A = R_n/\mathfrak{a}$, form a *system of analytic generators* for A if the substitution homomorphism γ in

$$\begin{tikzcd} R_t \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & R_n \arrow[d, "\varepsilon"] \\ & A \end{tikzcd}$$

with $\alpha(y_\tau) := \widetilde{g}_\tau$, $\varepsilon(\widetilde{g}_\tau) = g_\tau$, is surjective. This, of course, means that each function in $\mathfrak{m}_n$ can be written modulo $\mathfrak{a}$ in the form

$$h(g_1(x), \dots, g_t(x)), \quad h \in \mathfrak{m}_t .$$

The previous Theorem says that this notion is in fact redundant.

Corollary 3.16 *The elements $g_1, \dots, g_t \in \mathfrak{m}_A$ form a system of analytic generators if and only if they generate $\mathfrak{m}_A$ as an ideal.*

Proof. a) If γ as above is surjective then $\mathfrak{m}_A = \gamma(\mathfrak{m}_t) = \gamma((y_1, \dots, y_t) R_t) = (g_1, \dots, g_t) A$.

b) Conversely, if $\mathfrak{m}_A$ is generated by $g_1, \dots, g_t$ and if γ is constructed as above then

$$\mathfrak{m}_A = (g_1, \dots, g_t) A = (\gamma(y_1), \dots, \gamma(y_t)) A = \gamma(\mathfrak{m}_t) A ,$$

and due to the preceding Theorem, γ is surjective. □

Remark. It is a consequence of the Corollary that the definition of an analytic system of generators for A is independent of the representation $A = R_n / \mathfrak{a}$.

3.6 Algebraic properties of analytic algebras

Before we derive in Section 14 the Weierstraß formula and the Division Theorem we deduce from them some standard properties for the rings R_n in the analytic and formal case. Therefore, in the following, R_n is equal to $\mathcal{O}_{n,0}$ or $\widetilde{\mathcal{O}}_{n,0}$.

Theorem 3.17 *Let $R_n = \mathbb{K}\langle x_1, \dots, x_n \rangle$ or $\mathbb{K}\{x_1, \dots, x_n\}$, $\mathbb{K} = \mathbb{R}$ oder $\mathbb{C}$. Then R_n and each analytic $\mathbb{K}$–algebra $A = R_n / \mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{m}$, is noetherian.*

Proof. a) If B is noetherian and $\varphi : B \to A$ denotes a ring epimorphism then A is noetherian, too. For: if $\mathfrak{a} \subset A$ is an ideal and $\mathfrak{b} = \varphi^{-1}(\mathfrak{a}) = (b_1, \dots, b_r) B$, then $\mathfrak{a} = \varphi(\varphi^{-1}(\mathfrak{a})) = \varphi(\mathfrak{b})$ is generated by the elements $a_j = \varphi(b_j)$, $j = 1, \dots, r$.

b) If A is a local ring and $A / f A$ is noetherian for all $f \in \mathfrak{m} = \mathfrak{m}(A)$, $f \neq 0$, then A is noetherian, too. For: let $\mathfrak{a} \subset A$ be a proper ideal and, without loss of generality, $\mathfrak{a} \neq 0$; then there exists an element $f_0 \in \mathfrak{a}$, $f_0 \neq 0$. Let $B = A / f_0 A$ and $\varphi : A \to B$ the canonical projection. Then, due to Lemma 2.2,

$$\varphi^{-1}(\varphi(\mathfrak{a})) = \mathfrak{a} + \ker \varphi = \mathfrak{a} + A f_0 = \mathfrak{a} .$$

By assumption, $\mathfrak{b} = \varphi(\mathfrak{a})$ is finitely generated: $\mathfrak{b} = (g_1, \dots, g_\ell) B$. Let $f_1, \dots, f_\ell$ be preimages of the elements $g_1, \dots, g_\ell$. If now $f \in \mathfrak{a}$ then

$$\varphi(f) = \sum_{j=1}^{\ell} b_j g_j , \quad b_j \in B ,$$

hence we achieve with $\varphi(a_j) = b_j$:

$$\varphi(f - \sum_{j=1}^{\ell} a_j f_j) = 0$$

and finally

$$f - \sum_{j=1}^{\ell} a_j\, f_j \in \ker\, \varphi \,=\, A\, f_0\,;$$

i.e.: $\mathfrak{a}$ will be generated $f_0,\, f_1, \ldots,\, f_\ell$.

c) We perform induction with respect to n . $R_0 \,=\, \mathbb{K}$ is a field, hence a noetherian ring. Let the statement be proven for R_{n-1} , $n \,\geq\, 1$, and take $f \in R_n$, $n \,\geq\, 1$, $f \,\neq\, 0$. Then, as we have seen before, there exists a (linear and invertible) substitution homomorphism σ of R_n such that $\sigma\,(f)$ is x_n–generic of a certain degree b . Let $\sigma\,(f) \,=\, e\,\omega$, ω a WEIERSTRASS polynomial of degree b , e a unit. It follows from the Division Theorem

$$R_n/\, f\, R_n \,\cong\, R_n/\, \sigma\,(f)\, R_n \,=\, R_n/\, \omega\, R_n \,=\, R_{n-1}[\, x_n\,]/\, \omega\, R_{n-1}[\, x_n\,]\;.$$

By induction hypothesis, R_{n-1} is noetherian and, due to the HILBERT *Basis Theorem*, the polynomial ring $R_{n-1}[\, x_n\,]$ is noetherian. Because of a) this is also true for the quotient modulo ω . b) implies our claim. □

Remark. This proof does not work in the $\mathcal{C}^\infty$–theory since, as we have seen earlier, the *flat* function germs $f \in \mathfrak{m}_n^\infty$ cannot be made x_n–generic.

R_n is, moreover, a *factorial* ring, i.e. the Theorem on *Unique Prime Factor Decomposition* is satisfied.

Definition. Let A be a commutative ring with unit element 1 .

a) An element $f \in A$ is called a *prime element* if $f|g_1\, g_2$ implies $f|g_1$ or $f|g_2$ (the symbol $f|g$ means: there exists an element $h \in A$ such that $g \,=\, f\, h$, or in other words: f *divides* g in A). This is equivalent to the statement that the ideal $f\, A \subset A$ is a *prime ideal*; here, an ideal $\mathfrak{a} \subset A$ is called *prime* if and only if

$$f\, g \in \mathfrak{a}\,,\; g \notin \mathfrak{a} \Longrightarrow f \in \mathfrak{a}\;.$$

This, in turn, is equivalent to the fact that the ring $A/\,\mathfrak{a}$ is an *integral domain*, i.e. a ring without *zero divisors*. In particular, maximal ideals are prime.

b) An element $f \in A$ is called *indecomposable* if $f \,=\, f_1\, f_2$ implies that f_1 or f_2 is a unit.

c) The ring A is called *factorial* if A is an integral domain and if each element $f \,\neq\, 0$ possesses a unique (up to units and order) prime factor decomposition.

Remark. In a *factorial* ring the notions of *prime* and *indecomposable* elements do coincide.

Theorem 3.18 *The rings* $R_n \,=\, \mathbb{K}\,\langle\, x_1, \ldots, x_n\,\rangle$ *and* $\mathbb{K}\,\{\, x_1, \ldots, x_n\,\}$, $\mathbb{K} \,=\, \mathbb{R}$ *or* $\mathbb{C}$, *are factorial.*

Proof. That the ring R_n is an integral domain follows from the Identity Theorem. For the factoriality we perform induction with respect to n where in the case $n \,=\, 0$ nothing has to be proved because of $R_0 \,=\, \mathbb{K}$. Suppose that we know already that R_{n-1} is factorial. If $f \,\neq\, 0$ and $f \,=\, e\,\omega$, ω a WEIERSTRASS–polynomial, e a unit, then due to the Preparation Theorem

$$R_{n-1}[\, x_n\,]/\, \omega\, R_{n-1}[\, x_n\,] \,\cong\, R_n/\, \omega\, R_n \,\cong\, R_n/\, f\, R_n\;,$$

hence f is prime in R_n if and only if ω has the same property in $R_{n-1}[\, x_n\,]$. Due to the well known GAUSS *Lemma* from algebra the monic polynomial ω in $R_{n-1}[\, x_n\,]$ has a unique decomposition into prime factors. □

Remark. It is easily seen that $\mathcal{C}^\infty_{1,0}$ is <u>not</u> an integral domain. This then also applies to all rings $\mathcal{C}^\infty_{n,0}$.

Each (commutative) integral domain A (with unit 1) can canonically be embedded in its *quotient field* $Q\,(A)$.

Definition. An integral domain A is called *normal* if A is *algebraically closed* in its quotient field $Q\,(A)$; in other words: if $h \in Q\,(A)$ is *algebraic* over A , i.e. if there exist $a_1, \ldots, a_r \in A$ with

$$(+) \qquad h^r + a_1 h^{r-1} + \cdots + a_r = 0\,, \quad r \geq 1\,,$$

then necessarily $h \in A$.

Theorem 3.19 *The rings* $\mathcal{O}_{n,0}$ *and* $\widetilde{\mathcal{O}}_{n,0}$ *are normal.*

The *proof* immediately follows from the more general

Theorem 3.20 *Any factorial integral domain* A *is normal.*

Proof. Let $h = f/g \in Q(A)$ satisfy an equation of type (+). Without loss of generality we may assume that f and g are relatively prime. Then, from

$$f^r + a_1 f^{r-1} g + \cdots + a_r g^r = 0\,, \quad f, g, a_1, \ldots, a_r \in A\,,$$

i.e. $f^r = g(-a_1 f^{r-1} - \cdots - a_r g^{r-1})$, we deduce that $g|f^r$ and, because of factoriality, $g|f$ in A, i.e. $h \in A$. □

Remark. There are more general notions of *normality* of local rings not using integrality. However, if A is noetherian then A is necessarily an integral domain ([01 - 02], p. 219).

We want to deepen the investigation of this situation a little further.

Definition. Let $\varphi : A \to B$ be an analytic homomorphism. An element $g \in B$ is called *algebraic* (or *integral*) over A (with respect to φ) if there exist $a_0, \ldots, a_r \in A$ such that

$$g^r + \varphi(a_0)\, g^{r-1} + \cdots + \varphi(a_r) = 0\,.$$

Theorem 3.21 *Let* $\varphi : A \to B$ *be an analytic homomorphism and* $g \in \mathfrak{m}_B$ *algebraic over* A*; let* $\omega \in A[Y]$ *be a monic polynomial of smallest degree annihilating* g*:* $\omega(g) = 0$*. Then* ω *is a* WEIERSTRASS*–polynomial, i.e.*

$$\omega = Y^b + a_1 Y^{b-1} + \cdots + a_b\,, \quad a_\beta \in \mathfrak{m}_A\,.$$

Proof. Without loss of generality let $A \subset B$ and $\varphi = \mathrm{id}$. Since $g \in \mathfrak{m}_B$, we have

$$-a_b = g(g^{b-1} + a_1 g^{b-2} + \cdots + a_{b-1}) \in \mathfrak{m}_B \cap A \subset \mathfrak{m}_A\,.$$

Let d be the largest index with $a_d \notin \mathfrak{m}_A$; in particular, $1 \leq d < b$. Denote by $\rho : A[Y] \to \mathbb{K}[Y]$ the epimorphism induced by taking residue classes $A \to A/\mathfrak{m}_A = \mathbb{K}$, then

$$\rho(\omega) = Y^{b-d}(Y^d + \rho(a_1) Y^{d-1} + \cdots + \rho(a_d)) \in \mathbb{K}[Y]$$

is because of $\rho(a_d) \neq 0$ a relatively prime decomposition of $\rho(\omega)$ in $\mathbb{K}[Y]$. Suppose that there are *monic* polynomials

$$\omega_1, \omega_2 \in A[Y]$$

with $\rho(\omega_1) = Y^{b-d}$, $\rho(\omega_2) = Y^d + \cdots + \rho(a_d)$ and $\omega = \omega_1 \omega_2$. If we would have $\omega_2(g) \in \mathfrak{m}_B$ then, because of $g \in \mathfrak{m}_B$, the constant term of ω_2 would belong to $\mathfrak{m}_A$ and $\rho(\omega_2)$ would be divisible by Y. Contradiction! Therefore, $\omega_2(g)$ is a unit in B, and from $\omega(g) = 0$ it follows that $\omega_1(g) = 0$. Since ω_1 is of degree $b - d < b$ in Y, we arrive at a contradiction to the assumption that ω is of smallest degree. Hence, $a_1, \ldots, a_b \in \mathfrak{m}_A$. □

The proof is not complete since we have to justify the assumption we made above. However, this is a consequence of the following famous Lemma.

***Theorem 3.22 (Hensel's Lemma)** *The algebras* $A = R_n/\mathfrak{a}$, $R_n = \mathcal{O}_{n,0}$ *resp.* $\widetilde{\mathcal{O}}_{n,0}$*, are* henselian*, i.e. if* $\rho : A \to A/\mathfrak{m}_A = \mathbb{K}$ *is the natural residue class epimorphism and* $\rho : A[Y] \to \mathbb{K}[Y]$ *the continuation defined via* $\rho(Y) := Y$*, if moreover* $P \in A[Y]$ *is a monic polynomial and*

$$\rho(P) = Q_1 \cdot \ldots \cdot Q_t$$

with pairwise relatively prime monic polynomials $Q_\tau \in \mathbb{K}[Y]$ *then there exist monic polynomials* $P_\tau \in A[Y]$ *with* $\rho(P_\tau) = Q_\tau$ *and*

$$P = P_1 \cdot \ldots \cdot P_t\,.$$

Proof. C.f. [01 - 02], pp. 49 ff. The proof uses only the WEIERSTRASS formula. □

We now get another *Finiteness Theorem*. Remark that if $\varphi : A \to B$ is finite then each element $g \in B$ is algebraic over A. This easily follows from DEDEKIND's Lemma: $N = B$ is finite on A and with $\mathfrak{a} = A$, $s = g$ we conclude that there exist elements $a_1, \dots, a_r \in \mathfrak{a} = A$ satisfying

$$g^r + a_1 g^{r-1} + \cdots + a_r \in \mathrm{An}_B B = \{ 0 \} .$$

Theorem 3.23 *Let $\varphi : A \to B$ be an analytic homomorphism such that each element of a system of generators in $\mathfrak{m}_B$ is algebraic over A. Then φ is finite.*

Proof. Let $\mathfrak{m}_B$ be generated by $g_1, \dots, g_s$. To each $j = 1, \dots, s$, we find a Weierstraß polynomial $\omega_j \in A[Y]$ with $\omega_j(g_j) = 0$ such that $g_j^{r_j} \in \varphi(\mathfrak{m}_A) B$. With $b = r_1 \cdot \ldots \cdot r_s$ we see $g_j^b \in \varphi(\mathfrak{m}_A) B$ and hence, due to the Remark after Theorem 15, φ is finite. □

3.7 Weierstraß hypersurfaces

In order to understand *hypersurfaces* locally, we are reduced by the Preparation Theorem to study *Weierstraß hypersurfaces*, i.e. zero sets of type

$$X = \{ \omega(x, t) = 0 \} \subset G \times \mathbb{C} , \quad G \subset \mathbb{C}^n \text{ a domain} ,$$

where ω is a *monic* polynomial

$$\omega(x, t) = t^b + a_1(x) t^{b-1} + \cdots + a_b(x) , \quad a_1, \dots, a_b \in \mathcal{O}(G) .$$

To begin with, let us first concentrate on the *topological* properties of the map $\pi : X \to G$ induced by the *projection* $G \times \mathbb{C} \to G$. By the very definition it is plain that π has *finite fibers*. Moreover, due to the *Theorem on Continuity of Roots* (Lemma 60), the map π is *locally bounded* with respect to G. From this remark it is easily deduced that π is a *proper map* which means that preimages of compact sets in G under π are compact.

Definition. A continuous map $f : X \to Y$ between topological Hausdorff spaces X and Y is called *finite* if it has finite fibers and is *closed* in the sense that it maps closed sets onto closed sets.

Finiteness of continuous maps can also be characterized in the following way.

***Lemma 3.24** *If the spaces X and Y are* locally compact*, finiteness of a continuous map $f : X \to Y$ is equivalent to the following properties*:

i) *f is proper*;

ii) *each point $x \in X$ is* isolated *in the fiber $f^{-1}(f(x))$* .

The (easy) *proof* is left to the reader. □

As a consequence, we immediately note

Corollary 3.25 *For a Weierstraß hypersurface $X \subset G \times \mathbb{C}$ the canonical projection $\pi : X \to G$ is a finite holomorphic map.*

Let us study the map $\pi : X \to G$ in more detail. For fixed $x^{(0)} \in G$ the polynomial $\omega(x^{(0)}, t) \in \mathbb{C}[t]$ can be decomposed into *linear* factors

$$\omega(x^{(0)}, t) = (t - t_1) \cdot \ldots \cdot (t - t_b)$$

where $t_1 = t_1(x^{(0)}), \dots, t_b = t_b(x^{(0)})$ are uniquely determined by $x^{(0)}$ up to permutation. By the VIETA formulas, $a_\beta(x^{(0)})$ is, up to sign, equal to the β–th *elementary symmetric function* of $t_1, \dots, t_b$; in particular,

$$a_1(x^{(0)}) = -(t_1 + \cdots + t_b) , \quad a_b(x^{(0)}) = (-1)^b t_1 \cdot \ldots \cdot t_b .$$

Define the *discriminant* of $\omega\,(x^{(0)},\,t)$ to be

$$\Delta\,(x^{(0)}) \;=\; \Delta_\omega(x^{(0)}) \;:=\; \prod_{1\le j<k\le b} (t_j \,-\, t_k)\;.$$

As a symmetric function of the t_j, $\Delta\,(x^{(0)})$ is a nontrivial polynomial with coefficients in $\mathbb{C}$ (independently of ω) in the functions $a_1(x^{(0)}),\ldots,a_b(x^{(0)})$. E.g., if $b\,=\,2$, then (up to a constant factor)

$$\Delta\,(x^{(0)}) \;=\; a_1(x^{(0)})^2 \,-\, 4a_2(x^{(0)})\;.$$

This is a consequence of the *Main Theorem on Symmetric Polynomials*. Recall that the *elementary symmetric* functions $\sigma_j \,=\, \sigma_j(x_1,\ldots,x_b)$, $j\,=\,1,\ldots,b$, are defined by

$$\prod_{j=1}^{b}(t\,-\,x_j) \;=\; t^b \,+\, \sum_{j=1}^{b}(-1)^j\,\sigma_j(x)\,t^{b-j}\;.$$

They are invariant under the obvious action of the symmetric group $\mathfrak{S}_b$, i.e. if $\pi\in\mathfrak{S}_b$ is a permutation then

$$\sigma_j(x_{\pi(1)},\ldots,x_{\pi(b)}) \;=\; \sigma_j(x_1,\ldots,x_b)\;.$$

One has, as is well known, the following.

***Theorem 3.26 (Main Theorem on Symmetric Polynomials)** *Let $P\,\in\,\mathbb{K}\,[\,x_1,\ldots,x_b\,]$ be invariant under $\mathfrak{S}_b$, then there is a polynomial Q such that*

$$P\,(x_1,\ldots,x_b) \;=\; Q\,(\sigma_1(x_1,\ldots,x_b),\ldots,\sigma_b(x_1,\ldots,x_b))\;.$$

So, we can view the discriminant $\Delta\,=\,\Delta_\omega$ as a *holomorphic* function $\Delta:\,G\,\to\,\mathbb{C}$. It is clear that $\Delta\,(x^{(0)})\,=\,0$ if and only if the polynomial $\omega\,(x^{(0)},\,t)$ has *multiple* zeros.

Whereas $\mathcal{O}\,(G)$ is in general no factorial ring, it is easily deduced from factoriality of $\mathcal{O}_{n,0}$ that the rings $\mathcal{O}\,(G)\,[\,t\,]$ satisfy unique prime factorization for *monic* polynomials and we can speak, in particular, of *multiple factors* of such polynomials $\omega\in\mathcal{O}\,(G)\,[\,t\,]$. This is easily deduced from unique prime factorization for monic polynomials in $Q\,[\,t\,]$ where Q denotes the quotient field of the integral domain $A\,:=\,\mathcal{O}\,(G)$ and the following Lemma which is a simple consequence of factoriality of all rings $\mathcal{O}_{n,x}$, $x\in G$.

***Lemma 3.27** *If $P_1,\,P_2$ are monic polynomials in $Q\,[\,t\,]$ with $P_1\,P_2\in A\,[\,t\,]$, then $P_1,\,P_2\in A\,[\,t\,]$.*

We also need the notion of an *unbranched covering*.

Definition. Let $f\,:\,X\,\to\,Y$ be a continuous map between topological Hausdorff spaces. f is called a *(finite) unbranched covering* if to each $y^{(0)}\in Y$ there exists a neighborhood $V\subset Y$ of $y^{(0)}$ and pairwise distinct neighborhoods U_j of the (finitely many) preimages $x_j^{(0)}$ of $y^{(0)}$ such that the restrictions $f_{|U_j}\,:\,U_j\,\to\,V$ are homeomorphisms.

We can now state our main result of the present Section.

Theorem 3.28 *For a monic polynomial $\omega\in\mathcal{O}\,(G)\,[\,t\,]$ and the corresponding Weierstraß hypersurface $X\,=\,\{\,\omega\,=\,0\,\}$ the following assertions are equivalent*:

i) *ω has no multiple factors*;

ii) *no germ of ω at any point of $G\times\mathbb{C}$ has multiple factors*;

iii) $\Delta_\omega\,\not\equiv\,0$;

iv) *there exists a proper analytic subset $D\,\subset\,G$ such that the restriction of π to the preimage $X_0\,:=\,\pi^{-1}(G_0)$, $G_0\,:=\,G\setminus D$, is a holomorphic finite unbranched covering.*

Under these assumptions, in particular, X_0 *has no singularities.*

Proof. If i) is not satisfied then ii) is not true, either. If ii) is not satisfied, ω has a multiple factor at a certain point $a \in \mathbb{C}^{n+1}$ as an element of $\mathcal{O}_{n+1,a}$ and thus as an element of $\mathcal{O}_{n,a'}[\,t\,]$. Therefore, $\Delta \equiv 0$ in a neighborhood of a' and thus $\Delta = 0$ on G. Clearly, iv) implies i). So, it remains to show that i) $\Longrightarrow$ iv). Due to the Lemma stated above ω has no multiple factors in the Euclidean ring $Q\,[\,t\,]$, either. Therefore, ω and its derivative ω' with respect to t are relatively prime in this ring such that there exist nontrivial elements $B_1, B_2, C_1, C_2 \in A\,[\,t\,]$ with

$$\frac{B_1}{C_1}\,\omega + \frac{B_2}{C_2}\,\omega' = 1\,,$$

i.e. $B_1\,C_2\,\omega + B_2\,C_1\,\omega' = C_1\,C_2 =: C$. Outside the proper analytic subset $D := \{\,x \in G : C\,(x) = 0\,\}$ the fibers $\pi^{-1}(x)$ consist of b points all counted with multiplicity 1. That the restriction of π to X_0 is locally biholomorphic is an immediate consequence of Hensel's Lemma. $\square$

Definition. An analytic algebra A is called *reduced* if it does not possess any *nilpotent* elements.

Example. If $A = R_n/\,f\,R_n$, A is reduced if and only if f has no multiple factors. Use the factoriality of R_n!

By the preceding Theorem, if a hypersurface equation $f = 0$ in a region $G \subset \mathbb{C}^n$ has no multiple factors at a point $a \in X = \{\,x \in G : f\,(x) = 0\,\}$ no germ f_x has multiple factors at any point x in a neighborhood of a. We call such an equation a *reduced* equation of X.

Theorem 3.29 *For hypersurfaces* $X = \{\,x \in G \subset \mathbb{C}^n : f\,(x) = 0\,\}$, *reduced equations exist locally. With regard to those equations the singular locus* $S\,(X)$ *of* X *is lower dimensional. For all points* $a \in X \setminus S\,(X)$, *the equation* $f\,(x) = 0$ *itself becomes, after a suitable change of variables near* a, $(x_n - a_n)^b = 0$, *the reduced equation, of course, being* $(x_n - a_n) = 0$.

3.8 Chevalley dimension of analytic algebras

There are many different approaches to introduce the *dimension* of an analytic algebra. One of the (possible) definitions is the following that goes back to Chevalley:

Definition. $\dim A := \min\{\,n :$ it exists a finite morphism $R_n \longrightarrow A\,\}$.

From this definition, however, it is by no means easy to deduce that $\dim R_n = n$. We will discuss other definitions later in Chapter 6 and show that they are all equivalent. In particular, we have to convince the reader that this definition is the same as the one presented in Appendix A to Chapter 1 that gives R_n obviously the correct dimension n.

Clearly, $\dim A = 0$ by definition, if and only if A is an *artinian* ring, i.e. of finite dimension as a vector space over $\mathbb{K}$. This makes sense due to the Rückert *Nullstellensatz*, since this condition is in turn equivalent to (see Theorem 2.13)

$$\mathfrak{m}_n^k \subset \mathfrak{a}\,,\ \text{for some}\ k\,,\quad \text{if}\ A = R_n/\,\mathfrak{a}\,,$$

and $N\,(\mathfrak{a}) = \{\,0\,\} = N\,(\mathfrak{m}_n) \Longleftrightarrow \mathfrak{m}_n^k \subset \mathfrak{a}$.

In linear algebra one learns that a linear subspace of $\mathbb{K}^n$ has dimension d if and only if d suitable chosen linear forms cut down the subspace to a point, but no fewer forms suffice. This, in fact, is the geometric idea behind our introduction of the dimension.

Definition and *Remarks*. 1. A system of elements $f_1, \ldots, f_r \in \mathfrak{m}_A$ is called a *weak parameter system*, if the quotient algebra $\overline{A} = A/(f_1, \ldots, f_r)\,A$ is artinian.

2. If $A = R_n/\mathfrak{a}$, $\mathfrak{a} = (g_1, \ldots, g_m)$ and $F_1, \ldots, F_r$ are preimages of the elements $f_1, \ldots, f_r$, then $\overline{A} \cong R_n/(g_1, \ldots, g_m,\, F_1, \ldots, F_r)$, then

$$N\,(\mathfrak{a}) \cap N\,(f_1) \cap \ldots \cap N\,(f_r) = N\,(g_1, \ldots, g_m,\, F_1, \ldots, F_r) = \{\,0\,\}\,.$$

Hence, the r *hypersurfaces* $N(f_1), \ldots, N(f_r)$ cut down $N(\mathfrak{a})$ to a point.

3. A system of elements $f_1, \ldots, f_r \in \mathfrak{m}_A$ is by definition a weak parameter system if and only if the substitution homomorphism given by sending the analytic generators $y_1, \ldots, y_r$ of R_r to $f_1, \ldots, f_r$, resp., is quasi–finite and hence finite.

4. A weak parameter system of *minimal length* r is called a *parameter system*. Clearly, each weak parameter system contains a parameter system.

These remarks imply immediately:

Lemma 3.30 *For each analytic algebra* A *there exist parameter systems* $f_1, \ldots, f_r \in \mathfrak{m}_A$. *They all have the same length*

$$r = \dim A .$$

As an immediate consequence of the definition we note:

Lemma 3.31 *If* $A \longrightarrow B$ *is finite then* $\dim B \le \dim A$.

Now, every (finite) homomorphism $\varphi : A \to B$ factorizes over an *epimorphism* $A \to A/\ker\varphi$ and a (finite) *monomorphism* $A/\ker\varphi \hookrightarrow B$. So, the question arises how much the dimension drops, if at all, in either case. As a hint for a satisfying answer to the question in the "injective" case we note the following easy Lemma.

Lemma 3.32 *If* $d = \dim A$ *and* $\varphi : R_d \to A$ *is finite then* φ *is injective.*

Proof. Suppose $f \in \ker\varphi$ is not trivial. Then, without loss of generality, we may assume that f is x_d–generic. It is plain that φ induces a finite homomorphism

$$R_d / f\, R_d \longrightarrow A$$

which composed with the Weierstraß homomorphism $R_{d-1} \hookrightarrow R_d / f\, R_d$ yields a finite homomorphism $R_{d-1} \to A$ in contradiction to the assumption $d = \dim A$. □

Therefore, we expect the following to be true.

Theorem 3.33 *If* $\varphi : A \hookrightarrow B$ *is finite and injective, then* $\dim B = \dim A$.

If we assume that $\dim R_d = d$, the last statement implies a (more precise) *converse* to Lemma 32:

Theorem 3.34 *If* $\varphi : R_d \hookrightarrow A$ *is finite and injective, then* $\dim A = d$.

Remark. Such a finite, injective homomorphism $R_d \hookrightarrow A$ is called a *Noether normalization* of A. The last Theorem will be proven in Chapter 6 and shall serve us as the anchoring statement for the interplay between dimension and finite homomorphisms of analytic algebras. Taking it for granted, it implies, e. g., immediately:

Theorem 3.35

$$\dim R_d = d .$$

Moreover it is strong enough to deduce Theorem 33: Let d be the dimension of A and take a finite monomorphism $R_d \hookrightarrow A$. Composed with the finite monomorphism $A \hookrightarrow B$ this yields a finite monomorphism $R_d \hookrightarrow B$ such that $\dim B = d = \dim A$. □

In the case of an *epimorphism* $A \to B = A/\mathfrak{a}$ the main result is the following:

Theorem 3.36 *If* $\mathfrak{a}$ *is generated by* $f_1, \ldots, f_r \in \mathfrak{m}_A$ *, then*

$$\dim B + r \geq \dim A .$$

Equality holds if and only if the system $f_1, \ldots, f_r$ *is necessarily minimal and can be extended to a parameter system of* A*.*

In particular,

$$\dim A/\mathfrak{a} \geq \dim A - \mathrm{cg}\, \mathfrak{a} .$$

Proof. Let $\overline{d}$ be the dimension of B, and let $\overline{f}_{r+1}, \ldots, \overline{f}_{r+\overline{d}}$ be a parameter system of B. Choose preimages $f_{r+1}, \ldots, f_{r+\overline{d}}$ of these elements in $\mathfrak{m}_A$. Then,

$$A/\,(f_1, \ldots, f_{r+\overline{d}})\,A \equiv B/\,(\overline{f}_{r+1}, \ldots, \overline{f}_{r+\overline{d}})\,B$$

is a finite dimensional vector space, such that $f_1, \ldots, f_{r+\overline{d}}$ is a weak parameter system of A which implies

$$d = \dim A \leq r + \overline{d} .$$

One has equality if and only if $f_1, \ldots, f_{r+\overline{d}}$ is a parameter system of A. □

Remark. By Theorem 36 we know that $\dim A/f\,A \geq \dim A - 1$ for any $f \in \mathfrak{m}_A$ and it is easy to produce examples with $\dim A/f\,A = \dim A$. In Chapter 6, we clarify the conditions under which an element $f \in \mathfrak{m}_A$ really drops the dimension of A by 1.

3.9 Embedding dimension of analytic algebras and cotangent vector spaces

It goes without saying that each analytic algebra A can be written as a quotient $R_n/\,\mathfrak{a}$ (up to $\mathbb{K}$–algebra isomorphisms) in many different ways. The *minimal* number e with such an isomorphism

$$A \cong R_e/\,\mathfrak{a}$$

is an important invariant attached to A which we call the *embedding dimension* of A, in symbols

$$e = \mathrm{emb}\, A ,$$

since in geometric terms the number e is the smallest one with the property that the complex analytic space germ associated to A can (locally around the origin) be embedded into $\mathbb{K}^e$. By definition,

$$\mathrm{emb}\, A = \min\{\, n \in \mathbb{N} : \exists \text{ epimorphism } R_n \longrightarrow A \,\} \geq \dim A ,$$

and the considerations of the preceding Sections immediately imply that e is the cardinality of any *minimal* system of analytic generators of A and hence equals the corank $\mathrm{cg}\, \mathfrak{m}_A$ of the maximal ideal of A. Therefore, due to Nakayama's Lemma, we have proved the following Lemma.

Lemma 3.37 *For any noetherian analytic algebra, one has the identity*

$$\mathrm{emb}\, A = \dim_{\mathbb{K}} \mathfrak{m}_A/\,\mathfrak{m}_A^2 .$$

Corollary 3.38

$$\mathrm{emb}\, R_n = n .$$

It is natural to interpret the vector space $\mathfrak{m}_n/\,\mathfrak{m}_n^2$ as the vector space of linear functions on the tangent space of $\mathbb{K}^n$ at the origin. We therefore call it the *cotangent (vector) space* of R_n; we write

$$\dot{R}_n := \mathfrak{m}_n/\,\mathfrak{m}_n^2 ,$$

and more generally, we introduce the *cotangent space*

$$\dot{A} := \mathfrak{m}/\mathfrak{m}^2 , \quad \mathfrak{m} = \mathfrak{m}_A$$

for an arbitrary local algebra A. We have $\dot{A} = 0$ if and only if $A = R_0 = \mathbb{K}$, since $\mathfrak{m} = \mathfrak{m}^2$ implies by Nakayama's Lemma $\mathfrak{m} = 0$. The canonical mapping $\delta = \delta_A : \mathfrak{m}_A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ extends to a map $\delta : A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ via $\delta(f) := \delta(f - f(0))$ which is $\mathbb{K}$–linear, but does not respect the *algebra*–structures. Instead, it fulfills the LEIBNITZ–*rule*

$$\delta(f\,g) = f\,\delta(g) + g\,\delta(f)$$

and as such it is by definition a *derivation*.

Any analytic homomorphism $\varphi : A \to B$ induces a $\mathbb{K}$–linear homomorphism, the *differential*,

$$\dot{\varphi} : \dot{A} \longrightarrow \dot{B}$$

making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\delta_A"'] & B \arrow[d, "\delta_B"] \\ \dot{A} \arrow[r, "\dot{\varphi}"'] & \dot{B} \end{tikzcd}$$

Remark. If $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are minimal sets of generators for $\mathfrak{m}_A$ and $\mathfrak{m}_B$, resp., then the differential of any homomorphism $\varphi : A \to B$ will be represented by a certain $(m \times n)$–matrix in the bases $\delta(f_1), \ldots, \delta(f_m)$ and $\delta(g_1), \ldots, \delta(g_n)$, resp., whose rank $\dot{\varphi}$ equals the dimension of the image $\dot{\varphi}(\dot{A}) \subset \dot{B}$. In the special situation $A = R_m = \mathbb{K}\langle y_1, \ldots, y_m \rangle$, $B = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ and φ given by substitution $y_j = f_j(x_1, \ldots, x_n)$, $j = 1, \ldots, m$, this matrix is nothing else but the *Jacobi* matrix

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}\,(0) = \left(\frac{\partial f_j}{\partial x_k}\,(0) \right)_{\substack{j=1\ldots m \\ k=1\ldots n}} .$$

The differential $\dot{\varphi}$ is obviously surjective if φ is an epimorphism. In particular, any epimorphism $R_n \to A = R_n/\mathfrak{a}$ induces a vector space epimorphism $\dot{R}_n \to \dot{A}$ which is an isomorphism precisely if $n = \operatorname{emb} A$.

Theorem 15 and the Nakayama Lemma imply the converse of the preceding remark.

Theorem 3.39 *A homomorphism $\varphi : A \to B$ is surjective if and only if the differential $\dot{\varphi} : \dot{A} \to \dot{B}$ is an epimorphism.*

Proof. Surjectivity of $\dot{\varphi}$ leads to the inclusion $B \subset \varphi(\mathfrak{m}_A) + \mathfrak{m}_B^2$. Nakayama's Lemma, applied to the finitely generated B–modules $\mathfrak{m}_B$ and $\varphi(\mathfrak{m}_A)\,B$ yields $\mathfrak{m}_B = \varphi(\mathfrak{m}_A)\,B$ from which the claim follows because of Theorem 15. □

Corollary 3.40 (Jacobi Criterion) *A homomorphism $\varphi : R_m \to R_n$ is an isomorphism if and only if $\dot{\varphi} : \dot{R}_m \to \dot{R}_n$ is bijective (and hence necessarily $m = n$).*

Proof. The direction $\Longrightarrow$ is obvious. If, on the other hand, $\dot{\varphi} : \dot{R}_m \to \dot{R}_n$ is an isomorphism, then the dimensions of the cotangent spaces as $\mathbb{K}$–vector spaces coincide, i.e. $m = n$. Moreover, by Theorem 39, φ is an epimorphism and R_n is, in particular, a finite R_n–module via φ. Due to Theorem 35 and Lemma 32, we can deduce that φ is injective, too. □

Remark. In case $m = n$, $\dot{\varphi} : \dot{R}_n \to \dot{R}_n$ is an isomorphism if and only if the *functional determinant*

$$\det(D\,F)\,(0) \neq 0$$

(see Remark above). This is the standard formulation of the *classical* Jacobi criterion.

Analytic isomorphisms have, of course, isomorphic differentials. The converse is, by far, not true. Take, for instance, the standard epimorphisms $R_n \to R_n/\mathfrak{m}_n^2$ which are not injective for $n \geq 1$, but have isomorphic derivatives. - This phenomenon cannot occur when the right hand side is some R_n.

Theorem 3.41 *Every analytic homomorphism* $\varphi : A \to R_n$ *with bijective derivative is an isomorphism.*

Proof. By Theorem 39, $\varphi : A \to R_n = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ is surjective. Hence, there exist elements $g_1, \ldots, g_n \in \mathfrak{m}_A$ with $\varphi(g_j) = x_j$, $j = 1, \ldots, n$. Obviously, the elements $\delta_A(g_j)$ are preimages of the elements $\delta_{R_n}(x_j)$ under $\dot{\varphi}$, and consequently form, by our assumption on $\dot{\varphi}$, a basis of $\dot{A}$. Therefore, the elements $g_1, \ldots, g_n$ form a minimal set of analytic generators for A. Denote by γ the analytic homomorphism $R_n \to A$ given by sending x_j to g_j. Since $(\gamma \circ \varphi)(g_j) = g_j$ for all j, we have $\gamma \circ \varphi = \mathrm{id}_A$, and φ has to be injective, too. $\square$

Corollary 3.42 *If for an analytic homomorphism* $\varphi : A \to R_n$ *we have* $\mathrm{emb}\, A = \mathrm{rank}\, \dot{\varphi} = n$*, then* φ *is bijective.*

The *proof* is trivial, since under the given assumptions the derivative $\dot{\varphi}$ ist bijective. $\square$

Corollary 3.43 *Let* $f_1, \ldots, f_r \in \mathfrak{m}_n \subset R_n = \mathbb{K}\langle x_1, \ldots, x_n \rangle$, $1 \leq r \leq n$ *be given. Then, the following are equivalent*:

i) $f_1, \ldots, f_r$ *can be extended to a minimal set of generators for* $\mathfrak{m}_n$;

ii) *the rank of the functional matrix* $\dfrac{\partial(f_1, \ldots, f_r)}{\partial(x_1, \ldots, x_n)}(0)$ *is* r;

iii) *the* r *residue classes* $\delta(f_j) \in \dot{R}_n$ *are linearly independent.*

Proof. The implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) being trivial, let us assume iii). Then, we find elements $f_{r+1}, \ldots, f_n \in \mathfrak{m}_n$ whose δ–images form together with the given $\delta(f_j)$ a basis in $\dot{R}_n$, and the substitution homomorphism $\varphi : R_n \to R_n$ defined by $\varphi(x_j) = f_j$ is an isomorphism due to Theorem 41. Hence, the elements $f_1, \ldots, f_n$ form a minimal set of generators for $\mathfrak{m}_n$. $\square$

The following is a characterization of the embedding dimension which can easily be checked in examples.

Theorem 3.44 *If* $A = R_n/\mathfrak{a}$ *then* $\mathrm{emb}\, A = n$ *if and only if* $\mathfrak{a} \subset \mathfrak{m}_n^2$.

Proof. If $\mathfrak{a} \not\subset \mathfrak{m}_n^2$ then there exists an element $f \in \mathfrak{a} \setminus \mathfrak{m}_n^2$, and after a suitable isomorphism of R_n we may assume that $f = x_n$. Thus, A is the quotient of $R_{n-1} = R_n/x_n R_n$ and $\mathrm{emb}\, A \leq n-1$. If, on the other hand, $\mathfrak{a} \subset \mathfrak{m}_n^2$, take a minimal set $g_1, \ldots, g_e$ of generators for $\mathfrak{m}_A$ and lift the elements g_j to elements $f_j \in \mathfrak{m}_n$. By assumption, $\mathfrak{m}_n \subset (f_1, \ldots, f_e)R_n + \mathfrak{m}_n^2$, whence by Nakayama's Lemma $\mathfrak{m}_n = (f_1, \ldots, f_e)$ and $e \geq n$. $\square$

Remark. See also Remark 2 in Chapter 2.10.

Examples. 1. The plane curve singularities $x^2 - y^{k+1} = 0$, $k \geq 1$, have embedding dimension 2 at the origin.

2. The *cones over the rational normal curve* of arbitrary degree are given by quadratic equations that can be written in a short form as follows:

$$\mathrm{rank} \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} < 2 .$$

Hence their *embedding dimension* at the vertex is $n+1$. On the other hand, these are *two–dimensional* objects such that the singularities they create at the origin are never *hypersurface singularities* unless $n = 2$ and the singularity is our good friend A_1. For more details, see the next Chapter 4.

One can always "arrange" matters such that the situation is exactly as in the preceding Theorem.

Theorem 3.45 *Let* $\varphi : R_n \to A$ *be an epimorphism and* $e = \text{emb}\, A$. *Then, after a suitable change of variables,* $R_n \cong \mathbb{K}\langle x_1, \ldots, x_n \rangle \to A$ *has the property that* $x_{e+1}, \ldots, x_n \in \ker \varphi$ *and the induced homomorphism* $\psi : R_e = \mathbb{K}\langle x_1, \ldots, x_e \rangle \cong R_n/(x_{e+1}, \ldots, x_n)R_n \to A$ *is still surjective.*

Proof. Let $R_n = \mathbb{K}\langle y_1, \ldots, y_n \rangle$. By definition, $\delta(\ker \varphi)$ is a vector subspace of $\dot{R}_n$ of dimension

$$\text{jg} \ker \varphi = n - \text{emb}\, A = n - e\,.$$

The system of preimages $f_{e+1}, \ldots, f_n$ of any basis of this vector space can by Corollary 43 extended to a minimal set $f_1, \ldots, f_n$ of generators for $\mathfrak{m}_n$. In the coordinates $x_j = f_j$, $j = 1, \ldots, n$, the kernel of φ is contained in the ideal generated by $x_{e+1}, \ldots, x_n$, and with the canonical projection $\pi : R_n \to R_e = R_n/(x_{e+1}, \ldots, x_n)R_n$ we have a factorization $\varphi = \psi \circ \pi$. ψ is automatically surjective. □

3.10 Jacobi rank of ideals and regularity criteria

We are now in the position to clarify the notion of a *singularity* or a *singular point* of a (germ of an) analytic subset as opposed to the notion of a *regular point*. Set–theoretically the zero–set of an ideal $\mathfrak{a} \subset R_n$ is a d–dimensional submanifold of $\mathbb{K}^n$ if after a suitable change of coordinates $N(\mathfrak{a}) = N(x_{d+1}, \ldots, x_n)$ such that by Rückert's Nullstellensatz $\text{rad}\, \mathfrak{a} = \text{rad}\,(x_{d+1}, \ldots, x_n)$. Now, $R_n/(x_{d+1}, \ldots, x_n) \cong R_d$ has no zero divisors such that $\text{rad}\,(x_{d+1}, \ldots, x_n) = (x_{d+1}, \ldots, x_n)$ and $\text{rad}\, \mathfrak{a} = (x_{d+1}, \ldots, x_n)$. Consequently, for the analytic algebra $A := R_n/\mathfrak{a}$, we have $\text{red}\, A = R_n/\text{rad}\, \mathfrak{a} \cong R_d$. The consideration of the powers of the maximal ideal $\mathfrak{m}_n$ indicate that we should also view *nonreduced* algebras as *not regular*, i.e. *singular*. Therefore, we give the following

Definition. An analytic algebra A is called *regular*, if $A \cong R_d$ for some d.

There is a wonderful characterization of *regular* algebras (whose proof, however, uses in one direction Theorem 35 for which the reader still has to wait until Chapter 6).

Theorem 3.46 (Regularity criterion) *An analytic algebra* A *is regular if and only if* $\dim A = \text{emb}\, A$.

Proof. If $A = R_d$, then $\dim A = d = \text{emb}\, A$. If, on the other hand, $\dim A = \text{emb}\, A = d$, there exists a *surjective* homomorphism $R_d \to A$ which by Lemma 32 is also injective. □

The aim of the rest of the present Section is to elucidate the notion of *regularity* for analytic algebras $A = R_n/\mathfrak{a}$ in terms of the ideal $\mathfrak{a} \subset \mathfrak{m}_n$. We need first a better understanding how the embedding dimension depends on $\mathfrak{a}$. As R_n–modules, $\mathfrak{m} := \mathfrak{m}_A \cong \mathfrak{m}_n/\mathfrak{a}$, and there is an exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{m}_n \longrightarrow \mathfrak{m} \longrightarrow 0\,.$$

Tensoring with $R_n/\mathfrak{m}_n$ yields an epimorphism

$$\mathfrak{m}_n/\mathfrak{m}_n^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2\,,$$

whose kernel is isomorphic to

$$\mathfrak{a}/\mathfrak{a} \cap \mathfrak{m}_n^2 \cong (\mathfrak{a} + \mathfrak{m}_n^2)/\mathfrak{m}_n^2$$

(for more details see, e.g., [01 - 02], Anhang, §2.6).

Definition. The dimension of this kernel is sometimes called the *Jacobi rank* of $\mathfrak{a}$, in symbols: $\text{jg}\, \mathfrak{a}$.

Theorem 3.47 *For an analytic algebra* $A = R_n/\mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{m}_n$, *one has the following identity*:

$$\text{emb}\, A = n - \text{jg}\, \mathfrak{a}\,.$$

Remark. If the ideal $\mathfrak{a} \subset R_n$ is generated by $f_1, \ldots, f_r$, the vector space $(\mathfrak{a} + \mathfrak{m}_n^2)/\mathfrak{m}_n^2$ will obviously be generated by the 1-jets of these elements. Hence, the Jacobi rank of the ideal equals the rank of the matrix

$$\left(\frac{\partial f_j}{\partial x_k} (0) \right)_{\substack{j=1,\ldots,r \\ k=1,\ldots,n}} ,$$

and the last number is independently of the generators well defined. In particular, we have always

$$\mathrm{jg}\, \mathfrak{a} \leq \mathrm{cg}\, \mathfrak{a} .$$

Since the dimension of an analytic algebra is never larger than its embedding dimension, one readily deduces from the last Theorem:

Corollary 3.48 *For an analytic algebra* $A = R_n/\mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{m}_n$, *one has the following inequality*:

$$\dim A + \mathrm{jg}\, \mathfrak{a} \leq n .$$

Taking the Regularity Criterion Theorem 46 for granted, we immediately get the following.

Theorem 3.49 *An analytic algebra* $A = R_n/\mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{m}_n$, *is* regular *if and only if*

$$\dim A + \mathrm{jg}\, \mathfrak{a} = n .$$

More precisely, we can prove the following Theorem.

Theorem 3.50 *For an analytic algebra* $A = R_n/\mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{m}_n$, *the following statements are equivalent*:

i) $A \cong R_d$;

ii) $\mathrm{cg}\, \mathfrak{a} = \mathrm{jg}\, \mathfrak{a}\, (= n - d)$;

iii) *after a convenient choice of coordinates* $x_1, \ldots, x_n$, *the ideal* $\mathfrak{a}$ *is generated by* $x_{d+1}, \ldots, x_n$.

Proof. The implication iii) $\Rightarrow$ ii) is obvious. If ii) is satisfied, we get by former general results

$$\dim A \geq n - \mathrm{cg}\, \mathfrak{a} = n - \mathrm{jg}\, \mathfrak{a} = \mathrm{emb}\, A \geq \dim A .$$

Hence A is isomorphic to R_d with $d = n - \mathrm{cg}\, \mathfrak{a}$ due to Theorem 46. It remains to show i) $\Rightarrow$ iii). Since d is the embedding dimension of $A \cong R_d$ we may assume according to Theorem 45 without loss of generality that $x_{d+1}, \ldots, x_n \in \mathfrak{a}$. From this, we deduce a commutative diagram

$$\begin{tikzcd} & R_n \arrow[dl, "\psi"'] \arrow[dr, "\pi"] & \\ R_d \cong R_n/(x_{d+1}, \ldots, x_n)R_n \arrow[rr, "\varphi"'] & & A \cong R_n/\mathfrak{a} \end{tikzcd}$$

in which φ is surjective. Consequently, φ is an isomorphism (Corollary 42) and $\mathfrak{a} = \ker \pi = \ker \psi = (x_{d+1}, \ldots, x_n)\, R_n$. $\square$

3.11 The Rank Theorem and analyticity of the singular locus

Let us now look more "globally" to an analytic subset $X \subset U \subset \mathbb{C}^n$, given, say, by a (fixed) set of holomorphic functions $g_1, \ldots, g_s$ on U. These functions define an ideal $\mathcal{I} \subset \mathcal{O}(U)$ and thus the ideals $\mathcal{I}_x \subset \mathcal{O}_{\mathbb{C}^n,x}$, $x \in U$, that are generated by the germs $(g_1)_x, \ldots, (g_s)_x$. If, at a point $x^{(0)}$, the ideal $\mathcal{I}_{x^{(0)}}$ is also generated by the germs of the functions $f_1, \ldots, f_r$ which are defined in a neighborhood V of $x^{(0)}$, then the whole ideal $\mathcal{I}$ is generated by these functions in (a perhaps smaller neighborhood) V. In particular, we have according to the *permanence principle* (see the "*important note*" at the end of Chapter 2.8 and the Supplement)

$$\mathcal{I}_x = ((f_1)_x, \ldots, (f_r)_x)\, \mathcal{O}_{\mathbb{C}^n,x} \text{ for all } x \in V .$$

These remarks have a trivial consequence, namely the following

Lemma 3.51 *The function* $U \ni x \mapsto \operatorname{cg} \mathcal{I}_x$ *is* upper semi-continuous. *In other words*:

$$\operatorname{cg} \mathcal{I}_x \le \operatorname{cg} \mathcal{I}_{x^{(0)}}$$

for all $x \in U$ *near* $x^{(0)}$.

Remark. By definition, $\operatorname{cg} \mathcal{I}_x \geq 0$ and $\operatorname{cg} \mathcal{I}_x = 0$, if and only if $\mathcal{I}_x = 0$. Hence, $\operatorname{cg} \mathcal{I}_x = 0$ is an *open* condition which means that the set of elements $x \in U$ with that condition is an *open* subset of U. Moreover, as we remarked already in Chapter 1, this set *coincides* with U if the latter is *connected*.

A much stronger result can be drawn from Theorem 50. Note, that an automorphism of $\mathcal{O}_{\mathbb{C}^n,x}$ can be extended because of the *Jacobi criterion* to an actual biholomorphic map on a neighbourhood of x thus introducing *new holomorphic coordinates* near x. Therefore we can state:

Theorem 3.52 *If* $\mathcal{I}_{x^{(0)}} \subset \mathcal{O}_{\mathbb{C}^n,x^{(0)}}$ *satisfies one of the equivalent conditions of* Theorem 50*, then the ideal* $\mathcal{I}$ *is after a suitable change of holomorphic variables generated near* $x^{(0)} = 0$ *by the functions* $x_{d+1}, \ldots, x_n$.

Remarks. 1. Since $\mathcal{O}_{\mathbb{C}^n,0}$ modulo the ideal $\mathcal{I}_0$ generated by $x_{d+1}, \ldots, x_n$ is isomorphic to $\mathcal{O}_{\mathbb{C}^d,0}$ and therefore an *integral domain*, the ideal itself is a *prime ideal* and thus coincides with its *radical*. As a consequence, we not only have *set–theoretically* in a neighbourhood V of such a *regular* or *smooth* point $0 \in X = N(\mathcal{I})$ after holomorphic coordinate change:

$$X \cap V = \{ x \in V : x_{d+1} = \cdots = x_n = 0 \} ,$$

but also that the ideal of all holomorphic functions vanishing on X is generated by the coordinate functions $x_{d+1}, \ldots, x_n$.

2. If these conditions are satisfied for all $x \in X = N(\mathcal{I})$ with fixed d, we call X a d–dimensional *submanifold* of U.

Since the Jacobi rank of an ideal is independent of given generators and of holomorphic coordinate changes (because of the chain rule) and since finally

$$\operatorname{rank} \frac{\partial(x_{d+1}, \ldots, x_n)}{\partial(x_1, \ldots, x_n)} = n - d ,$$

we immediately derive from Theorem 50 the following important insight.

Theorem 3.53 *If* $\mathcal{I}_{x^{(0)}} \subset \mathcal{O}_{\mathbb{C}^n,x^{(0)}}$ *satisfies one of the equivalent conditions of* Theorem 50*, then the ideal* $\mathcal{I}$ *has locally constant Jacobi rank around* $x^{(0)}$:

$$\operatorname{jg} \mathcal{I}_x = n - d , \quad x \text{ near } x^{(0)} .$$

The true generalization of linear algebra to complex analytic geometry consists in the converse to this result that is usually called the *Rank Theorem*:

***Theorem 3.54** *Let $X = N(f_1, \ldots, f_m)$ be an analytic subset of the open set $U \subset \mathbb{C}^n$, and let*

$$r(x) = \operatorname{rank}\left(\frac{\partial f_j}{\partial x_k}(x)\right)_{\substack{1\le j\le m\\ 1\le k\le n}}$$

be constant in a neighbourhood of X : $r(x) = r$ for all $x \in V = V(X)$. Then X is an $(n-r)$–dimensional submanifold of U.

The *proof* of this statement is standard and follows the arguments in the *differentiable* case. □

A special case of the *Rank Theorem* is the *Implicit Mapping Theorem* which obviously generalizes the *Implicit Function Theorem*:

***Theorem 3.55** *Let $x^{(0)}$ be a point of an analytic subset $X \subset U \subset \mathbb{C}^n$. $x^{(0)}$ is a regular point of X (of dimension d), if and only if there exist holomorphic functions $f_{d+1}, \ldots, f_n$ in a neighborhood V of $x^{(0)}$ such that*

$$X \cap V = \{ x \in V : f_{d+1}(x) = \cdots = f_n(x) = 0 \}$$

and

$$\operatorname{rank}\left(\frac{\partial f_j}{\partial x_k}(x^{(0)})\right)_{\substack{j=d+1,\ldots,n\\ k=1,\ldots,n}} = n-d\,.$$

Remark. Near a smooth point of dimension d, the analytic set X is also *parametrized* by a d–dimensional continuum. This follows from a (local) inverse ψ of the coordinate transformation φ by observing that

$$X \cap V = \{ x = (x_1, \ldots, x_n) \in V : x_k = \alpha_k(t_1, \ldots, t_d)\,,\ k = 1, \ldots, n \}$$

with a map α given by

$$x_k = \alpha_k(t_1, \ldots, t_d) := \psi_k(t_1, \ldots, t_d, 0, \ldots, 0)\,, \quad k = 1, \ldots, n\,,$$

whose Jacobi matrix has constant rank d in a neighborhood of 0 in $W' \subset \mathbb{C}^d$. (We also say in this situation that α itself has constant rank). On the other hand, if $X \cap V$ can be represented by

$$X \cap V = \{ x \in V : x_k = \alpha_k(t_1, \ldots, t_d)\,,\ k = 1, \ldots, n\,,\ t = (t_1, \ldots, t_d) \in W' \subset \mathbb{C}^d \}$$

with a holomorphic map $\alpha = (\alpha_1, ..., \alpha_n) : W' \to V$ of maximal rank d, then $\alpha(0)$ is a regular point of X. To see this, we may assume that $\alpha(0) = 0$ and

$$\left(\frac{\partial \alpha_k}{\partial t_j}(t)\right)_{1\le j,k\le d}$$

is invertible for all $t \in W'$. If we denote by β a (local) inverse to $\alpha' = (\alpha_1, \ldots, \alpha_d) : W' \to V' \subset \mathbb{C}^d$, then

$$\begin{cases} y_k = x_k\,, & k = 1, \ldots, d \\ y_k = x_k - \alpha_k(\beta(x_1, \ldots, x_d))\,, & k = d+1, \ldots, n \end{cases}$$

define new holomorphic coordinates near the origin (by the Inverse Mapping Theorem) and X is locally given by the vanishing of the coordinates $y_{d+1}, \ldots, y_n$.

Remark. It should be clear that all considerations above can be made mutatis mutandis for analytic subsets of arbitrary complex manifolds (instead of $\mathbb{C}^n$).

We are finally coming back to a closer inspection of the "singular locus" sing X of an analytic subset $X = N(\mathcal{I}) = N(f_1, \ldots, f_m) \subset U \subset \mathbb{C}^n$ where, of course, sing X denotes the set of *nonsmooth*, i. e.

singular points of X. We know already that this is a closed subset of X since the complement $\operatorname{reg} X$ of *smooth* points is open in X as we have seen above. Of course, it may happen that $\operatorname{sing} X = X$, e. g. if all points $x \in X$ have *nonreduced* structure. We want to show in the following, at least in a special situation, that $\operatorname{sing} X$ itself is *analytic* and give some hints why this is true in general. (Another more conceptual argument shall be presented in the Supplement).

Theorem 3.56 *For an analytic subset* $X = N(f_1, \ldots, f_m) \subset U \subset \mathbb{C}^n$ *the singular locus* $\operatorname{sing} X$ *is analytic, too.*

Sketch of proof. Because $S(X) := \operatorname{sing} X$ is closed in X and hence in U, it suffices to prove analyticity only *locally* at any point $x^{(0)} \in S(X)$. Now, X decomposes in a neighbourhood $V = V(x^{(0)}) \subset U$ into a finite union of "irreducible" (analytic) components:

$$X \cap V = X_1 \cup \cdots \cup X_t ,$$

and the singular locus of $X \cap V$ is just the union of the singular loci of the components *and* the intersections $X_\tau \cap X_\nu$. Moreover, for sufficiently small V, all components are *pure–dimensional*, i. e. $\dim_x X_\tau = \dim_{x^{(0)}} X_\tau$ for all $x \in X_\tau \cap V$. (For more details, c. f. Chapter 6). Therefore, we are reduced to the case that X is *pure–dimensional*, say $\dim_x X = d$ for all $x \in X$ such that due to Theorem 49:

$$S(X) = X \cap \{ x \in U : \operatorname{jg} \mathcal{I}_x < n - d \} .$$

In other words, we have to study the set of points $x \in U$ such that the rank

$$\operatorname{rank} \left(\frac{\partial f_j}{\partial x_k}(x) \right)_{\substack{1 \le j \le m \\ 1 \le k \le n}} < n - d .$$

But this set consists of all points $x \in U$ in which all $(n - d) \times (n - d)$–subdeterminants of the functional matrix $\dfrac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}$ vanish. □

3.12 Uniqueness of the Weierstraß decomposition

In the following Section we will prove the Division Theorem for the rings $R_n = \mathbb{C}\langle x_1, \ldots, x_n \rangle$ and $\mathcal{C}^\infty_{n,0}$ via the *Polynomial Division Theorem* which is treated in Section 14 in the analytic situation and in the Appendix for differentiable germs. In the present Section we discuss the question about *uniqueness* and derive the real analytic case $\mathbb{R}\langle x_1, \ldots, x_n \rangle$ from the complex analytic result. The *formal* case is more or less trivial and will be left to the reader.

As we will see below it is easy to prove uniqueness of the Weierstrass *decomposition*

$$f = q\, g + r$$

in the first (and hence in the real analytic and also in the formal) case. However, for differentiable germs this is wrong as one can see by the following

Example. $g(x, t) := t^2 - x$ is t–generic of order 2. Consider the germ $f = 0$ and write

$$r(x, t) = r(x) := \begin{cases} \exp(-1/x^2) , & x < 0 , \\ 0 , & x \ge 0 , \end{cases}$$

$q(x, t) := -r(x, t)/ g(x, t)$ if $g(x, t) \ne 0$, $q(x, t) := 0$ otherwise. Then, obviously, $q_0 \in \mathcal{E}_{2,0}$, $r_0 \in \mathcal{E}_{1,0} \subset \mathcal{E}_{2,0}$ and

$$q\, g + r = 0 = 0\, g + 0 .$$

Remark. For $n = 1$, however, uniqueness holds for *differentiable* functions, as well.

We now carry out the proof for uniqueness in the complex analytic case. For doing so, we change our notations in order to elucidate the problem. We work in $\mathbb{C}^{n+1}$ with variables $(x_1, \ldots, x_n, t)$ and write $x = (x_1, \ldots, x_n)$ for short and $f(x, t)$ instead of $f(x_1, \ldots, x_n, t)$ etc.

Theorem 3.57 *If $g \in \mathcal{O}_{n+1,0}$ is t–generic of order b then any decomposition*

$$f = q\,g + r\,, \quad \deg_t r < b\,,$$

is unique.

Proof. It is only to show that $q\,g = r$, $\deg r < b$ implies $q = r = 0$. For this it is sufficient to prove $r = 0$ since due to the fact that $\mathcal{O}_{n+1,0}$ is integer and $g \neq 0$ it follows automatically $q = 0$. Therefore, there is nothing to prove for $b = 0$ (this case is clear anyway since g is then a unit). Thus, we may assume that always $b \geq 1$ in the following.

So, let

$$r = r(x, t) = a_0(x) + a_1(x)\,t + \cdots + a_{b-1}(x)\,t^{b-1}$$

be not identically zero. Then we find a sequence $\xi_j \in \mathbb{C}^n$ with $\lim \xi_j = 0$ and an index $0 \leq k \leq b-1$ such that $a_k(\xi_j) \neq 0$ for all j. The polynomials

$$P_j(t) = r(\xi_j, t) \in \mathbb{C}[t]$$

have therefore only zeros of order at most $b - 1$ in $\mathbb{C}$.

Consider conversely $g(x, t)$. According to our assumption, $g(0, t)$ possesses in $t = 0$ an (isolated) zero of order b. Hence, there is a $\delta_0 > 0$ such that the origin is the only zero of $g(0, t)$ in $\{\,|t| \leq \delta_0\,\}$. Put

$$\varepsilon_0 := \min_{|t|=\delta_0} \{\,|g(0, t)|\,\} > 0\,.$$

For suitable $\sigma > 0$, the compact set $\{\,|x| \leq \sigma\,,\ |t| \leq \delta_0\,\}$ is contained in the region of definition for g and due to continuity of g we get for sufficiently small $\sigma_0 \leq \sigma$ that

$$|g(x, t) - g(0, t)| \leq \frac{\varepsilon_0}{2}\,, \quad |x| \leq \sigma_0\,, \quad |t| = \delta_0\,.$$

Therefore, for such x and t:

$$|g(x, t) - g(0, t)| < |g(0, t)|\,.$$

ROUCHÉ's Theorem then tells us that $g(x, t)$ has for fixed x near 0 exactly as many zeros (with multiplicity) in $|t| < \delta_0$ as $g(0, t)$, namely b of them, and this leads for large j because of

$$q(\xi_j, t)\,g(\xi_j, t) = P_j(t)$$

to a contradiction. $\square$

Remark. If f is a *real analytic* germ at $0 \in \mathbb{R}^n$ then f can be extended *uniquely* to a *holomorphic* germ F at $0 \in \mathbb{C}^n$. From this remark, one easily deduces the following:

a) if g is x_n–generic then the extension G is z_n–generic, $x_n = \mathrm{Re}\, z_n$;

b) the decomposition $f = q\,g + r$, $\deg r < b$, is unique if and only if the extended decomposition $F = Q\,G + R$ is unique;

c) a decomposition $F = Q\,G + R$ induces a decomposition $f = q\,g + r$ with $q(x_1, \ldots, x_n) = \mathrm{Re}\, Q(x_1, \ldots, x_n)$ etc., $x = \mathrm{Re}\, z$. In other words: the real analytic case follows directly from the complex analytic one.

3.13 The Polynomial Division Theorem

The complex analytic and the differentiable Division Theorem is a consequence of the *special* or *polynomial* Division Theorem.

Theorem 3.58 (Polynomial Division Theorem) *Let* $f(x, t)$ *be holomorphic in a neighborhood of* $(0, 0) \in \mathbb{C}^n \times \mathbb{C}$ *(resp. complex valued and* $\mathcal{C}^\infty$ *near* $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$*). Then, for the* generic *polynomial*

$$P_b = P_b(t, \lambda) = t^b + \sum_{j=1}^{b} \lambda_j\, t^{b-j} ,$$

there exists a decomposition

i) $f(x, t) = q(x, t, \lambda)\, P_b(t, \lambda) + r(x, t, \lambda)$ *with*

ii) $r(x, t, \lambda) = \sum_{j=1}^{b} r_j(x, \lambda)\, t^{b-j}$, $r_j(x, \lambda)$ *holomorphic in a neighborhood of* $(0, 0) \in \mathbb{C}^n \times \mathbb{C}^b$ *resp. differentiable near* $0 \in \mathbb{R}^n \times \mathbb{R}^b$.

Before we proof this theorem at the end of the present Section, we demonstrate its usefulness. One should note that it follows, in turn, from the Division Theorem: just view P_b as a holomorphic function in the variables x, t, λ and remark that it is t–generic.

Theorem 3.59 *The Polynomial Division Theorem implies the Preparation Theorem and the general Division Theorem.*

Proof. If the Preparation Theorem is already proven we write for the t–generic germ g:

$$g = q_1\, P$$

with the polynomial $P = P(x, t) = t^b + \sum_{j=1}^{b} u_j(x)\, t^{b-j}$ and a unit q_1. From the Special Division Theorem we have for arbitrary f:

$$f(x, t) = q_2(x, t, \lambda)\, P_b(t, \lambda) + r_2(x, t, \lambda) , \quad \deg_t r_2 < b$$

for all λ near 0. Setting $\lambda_j = u_j(x)$ leads to a decomposition of the desired manner:

$$f = (q_1^{-1} q_2)\, g + r , \quad r = r(x, t) = r_2(x, t, u(x)) .$$

It remains to conclude the Preparation Theorem from the Polynomial Division Theorem. Thus, let $g = g(x, t)$ be t–generic of order $b \geq 1$. Then,

$$(*) \qquad \left\{ \begin{aligned} g(x, t) &= Q(x, t, \lambda)\, P_b(t, \lambda) + R(x, t, \lambda), \\ R(x, t, \lambda) &= \sum_{j=1}^{b} h_j(x, \lambda)\, t^{b-j} , \quad P_b(t, \lambda) = t^b + \sum_{j=1}^{b} \lambda_j\, t^{b-j} . \end{aligned} \right.$$

We are seeking functions $\lambda_j = u_j(x)$ such that $R(x, t, u(x)) \equiv 0$, i.e. $h_j(x, u(x)) \equiv 0$ for all j. Since then $P_b(t, u(x))$ is a Weierstrass polynomial and $g(x, t) = Q(x, t, u(x))\, P_b(t, u(x))$ is the decomposition of the form we want.

Now, $(*)$ implies for $x = 0$, $\lambda = 0$ that

$$Q(0, 0, 0) \neq 0$$

(Q is, in fact, a unit) and $h_j(0, 0) = 0$. Differentiating $(*)$ at $x = 0$, $\lambda = 0$ with respect to λ_k we find

$$0 = t^{b-k}\, Q(0, t, 0) + t^b\, \frac{\partial Q}{\partial \lambda_k}(0, t, 0) + \sum_{j=1}^{b} \frac{\partial h_j}{\partial \lambda_k}(0, 0)\, t^{b-j} .$$

Since $Q(0, 0, 0) \neq 0$, this implies

$$\frac{\partial h_j}{\partial \lambda_k}(0, 0) = 0, \ j > k, \quad \frac{\partial h_k}{\partial \lambda_k}(0, 0) \neq 0.$$

Therefore,

$$\det \left(\frac{\partial h_j}{\partial \lambda_k}(0, 0 \right)_{j,k=1,\ldots,b} \neq 0,$$

and the equations $h = 0$ with

$$h : \begin{cases} \mathbb{K}^n \times \mathbb{K}^b \longrightarrow & \mathbb{K}^b \\ (x \ , \ \lambda) \longmapsto & (h_1(x, \lambda), \ldots, h_b(x, \lambda)) \end{cases}$$

can be, because of $h(0, 0) = 0$, holomorphically ($\mathbb{K} = \mathbb{C}$) resp. differentiably ($\mathbb{K} = \mathbb{R}$) resolved with respect to λ near $\lambda = 0$. $\square$

3.14 Proof of the Polynomial Division Theorem in the analytic case

The Polynomial Division Theorem is for *complex analytic* functions a simple consequence of the Cauchy *Integral Theorem* (in one complex variable). The following is easy to prove.

Lemma 3.60 (Continuity of Roots) *If $|\lambda| \leq \delta = \delta(\varepsilon)$ is sufficiently small then all the roots of*

$$P(t, \lambda) := P_b(t, \lambda)$$

are lying in a fixed disk $\{ |t| \leq \varepsilon \}$.

For the *proof* repeat the arguments of the second part of the proof of Theorem 57. From this we construct to each $\varepsilon > 0$ a $\delta > 0$ such that $P(t, \lambda)$ has exactly b roots in $\{ |t| \leq \varepsilon \}$ for fixed λ with $|\lambda| \leq \delta$. $\square$

We are now in the position to prove the Polynomial Division Theorem in the complex analytic case; Let $D = D_\varepsilon(0) \subset \mathbb{C}$. If one takes into consideration that

$$\frac{P(z, \lambda) - P(t, \lambda)}{z - t} = \rho(z, t, \lambda)$$

is a polynomial in t of degree $< b$ with analytic coefficients, then Cauchy's *Integral Formula* says for small $\varepsilon > 0$ - if one divides the equation above by $P(z, \lambda)$:

$$\begin{aligned} f(x, t) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(x, z)}{z - t} \, dz \\ &= \left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(x, z)}{(z - t) \, P(z, \lambda)} \, dz \right) P(t, \lambda) + \frac{1}{2\pi i} \int_{\partial D} \frac{f(x, z) \, \rho(z, t, \lambda)}{P(z, \lambda)} \, dz . \end{aligned}$$

Since, for small λ and t, the integrals exist and define holomorphic functions in (x, t, λ), the proof is finished. $\square$

3.15 A formal criterion

As a last application of the investigations above we regard the differentiable and analytic category, i.e. R_n equal to $\mathcal{O}_{n,0}$ or $\mathcal{E}_{n,0}$, and their relation to the formal situation, i.e. to $\widetilde{R}_n = \mathbb{K}\{x_1, \ldots, x_n\}$. We have in both cases a canonical homomorphism

$$\varepsilon = \varepsilon_n : R_n \longrightarrow \widetilde{R}_n ,$$

which in the differentiable case is *surjective* and, in the analytic case, *injective*. Each substitution homomorphism

$$\varphi : \begin{cases} R_m \longrightarrow & R_n \\ g \longmapsto g\,(f_1, \ldots, f_m)\,, \end{cases} \quad , \quad f_1, \ldots, f_m \in \mathfrak{m}_n$$

gives rise to a substitution homomorphism

$$\widetilde{\varphi} : \begin{cases} \widetilde{R}_m \longrightarrow & \widetilde{R}_n \\ \widetilde{g} \longmapsto \widetilde{g}\,(\varepsilon(f_1), \ldots, \varepsilon(f_m)) \end{cases} .$$

The corresponding diagram

$$(\times) \qquad \begin{tikzcd} R_m \arrow[r, "\varphi"] \arrow[d, "\varepsilon_m"'] & R_n \arrow[d, "\varepsilon_n"] \\ \widetilde{R}_m \arrow[r, "\widetilde{\varphi}"'] & \widetilde{R}_n \end{tikzcd}$$

is obviously commutative (Theorem 2.23).

The following theorem is, in the $\mathcal{C}^\infty$–category, the version of the *Division Theorem* à la Malgrange.

Theorem 3.61 *For a substitution homomorphism* $\varphi : R_m \to R_n$ *the following statements are equivalent*:

i) φ *is finite*;

ii) φ *is quasi–finite*;

iii) $\widetilde{\varphi}$ *is finite*;

iv) $\widetilde{\varphi}$ *is quasi–finite*.

Proof. We have already shown the equivalence of i) and ii) resp. of iii) and iv). Now, if φ is finite and if $x_1, \ldots, x_n$ form a system of generators for $\mathfrak{m}_n$ then each x_j is algebraic over R_m with respect to φ. Due to $(\times)$ the generators $\widetilde{x}_j := \varepsilon_n(x_j)$ of $\widetilde{\mathfrak{m}}_n = \mathfrak{m}\,(\widetilde{R}_n)$ are algebraic with respect to $\widetilde{\varphi}$ over $\widetilde{R}_m$; hence $\widetilde{\varphi}$ is finite (Theorem 23).

Let conversely $\widetilde{\varphi}$ be quasi–finite. Then, because of $(\times)$,

$$\widetilde{R}_n /\, (\varepsilon_n \circ \varphi)\,(\mathfrak{m}_m)\,\widetilde{R}_n$$

is a finite dimensional $\mathbb{K}$–vector space and thus

$$\dim_{\mathbb{K}}(R_n /\, (\mathfrak{m}_m R_n + \mathfrak{m}_n^\infty)) < \infty\,.$$

Consequently, there exists a number $k \in \mathbb{N}$ with

$$\mathfrak{m}_n^k \subset \mathfrak{m}_n^{k+1} + \mathfrak{m}_m R_n + \mathfrak{m}_n^\infty \subset \mathfrak{m}_m R_n + \mathfrak{m}_n^{k+1}\,,$$

and the Nakayama Lemma gives $\mathfrak{m}_n^k \subset \mathfrak{m}_m R_n$. Hence, Theorem 15 guaranties the finiteness of φ. □

3.A Appendix: The Polynomial Division Theorem in the differentiable case

3.A.1 Proof of the Theorem

Let us here consider the *differentiable* case $f = f\,(x,\,t) : (\mathbb{R}^n \times \mathbb{R},\, 0) \to \mathbb{C}$. As before, put

$$P_b = P_b(z,\,\lambda) = z^b + \sum_{j=1}^{b} \lambda_j\, z^{b-j}\,, \quad (z,\,\lambda) \in \mathbb{C} \times \mathbb{C}^b\,.$$

We try to find suitable $\mathcal{C}^\infty$*–extensions*

$$F = F(x, z, \lambda) : (\mathbb{R}^n \times \mathbb{C} \times \mathbb{C}^b, 0) \longrightarrow \mathbb{C}$$

of f, i.e. $F(x, t, \lambda) = f(x, t)$, $(x, t) \in (\mathbb{R}^n \times \mathbb{R}, 0)$, such that the wanted decomposition follows from the *inhomogeneous* Cauchy *Integral Formula*. The latter reads, as is well known, for suitable $D = D_\varepsilon(0) \subset \mathbb{C}$:

$$f(x, t) = F(x, t, \lambda) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(x, z, \lambda)}{z - t}\, dz + \frac{1}{2\pi i} \int_D \frac{\dfrac{\partial F}{\partial \overline{z}}(x, z, \lambda)}{z - t}\, dz \wedge d\overline{z}\,.$$

With similar considerations as in the complex analytic case it follows that

$$f(x, t) = Q(x, t, \lambda)\, P(t, \lambda) + R(x, t, \lambda)\,,$$

where

$$(2\pi i)\, Q(x, t, \lambda) = \int_{\partial D} \frac{F(x, z, \lambda)}{(z - t)\, P(z, \lambda)}\, dz + \int_D \frac{\dfrac{\partial F}{\partial \overline{z}}(x, z, \lambda)}{(z - t)\, P(z, \lambda)}\, dz \wedge d\overline{z}$$

and

$$(2\pi i)\, R(x, t, \lambda) = \int_{\partial D} \frac{F(x, z, \lambda)\, \rho(z, t, \lambda)}{P(z, \lambda)}\, dz + \int_D \frac{\partial F}{\partial \overline{z}}(x, z, \lambda)\, \frac{\rho(z, t, \lambda)}{P(z, \lambda)}\, dz \wedge d\overline{z}\,.$$

In the *boundary integrals* along ∂D the numerators do not vanish. Therefore, these parts of Q and R are $\mathcal{C}^\infty$–functions resp. polynomials in t of degree smaller than b with $\mathcal{C}^\infty$–coefficients. This is also true for the *domain integrals* in as far as we can prove the following *Extension Lemma* which goes back to Nirenberg.

Lemma 3.62 *Let $f(x, t)$ be a complex valued $\mathcal{C}^\infty$–function on a neighborhood of $0 \in \mathbb{R}^n \times \mathbb{R}$. Then there is an extension $F = F_b$ of f near $0 \in \mathbb{R}^n \times \mathbb{C} \times \mathbb{C}^b$ such that the following holds true* :

(+) $\dfrac{\partial F}{\partial \overline{z}}$ *vanishes of infinite order on the real analytic sets* $\{\operatorname{Im} z = 0\}$ *and* $\{P(z, \lambda) = 0\}$.

This suffices for proving the Polynomial Division Theorem. Indeed, putting

$$g(x, z, t, \lambda) := \frac{\dfrac{\partial F}{\partial \overline{z}}(x, z, \lambda)}{(z - t)\, P(z, \lambda)}$$

for $z \neq t$ and $P(z, \lambda) \neq 0$ resp. $= 0$ otherwise, *each* partial derivative of g at a place where the numerator does not vanish is a finite sum of functions of the form

$$(++) \qquad \frac{G_0(x, z, \lambda)}{[(z - t)\, P(z, \lambda)]^k}\,,$$

where G vanishes of infinite order on $\{\operatorname{Im} z = 0\}$ and on $\{P(z, \lambda) = 0\}$. Remark that the numerator is complex analytic in all variables.

Now, set $\lambda' = (\lambda_1, \ldots, \lambda_{b-1})$ and look at the map

$$(z, \lambda', \lambda_b) \longmapsto (z, \lambda', u)$$

with $u = P(z, \lambda)$. Since $\partial u / \partial \lambda_b \equiv 1$ this is a diffeomorphism under which the set $\{P(z, \lambda) = 0\}$ will be carried to the hyperplane $\{u = 0\}$. Hence, due to our assumption, G_0 can be written for all $\ell \in \mathbb{N}$ in the form

$$G_0(x, z, \lambda) = (\operatorname{Im} z)^\ell\, G_1(x, z, \lambda)$$

and moreover

$$G_1(x,\, z,\, \lambda) \,=\, (\mathrm{Re}\, u)^\ell\, G_2(x,\, z,\, \lambda) \,+\, (\mathrm{Im}\, u)^\ell\, G_3(x,\, z,\, \lambda)\ .$$

Hence, for $\ell > k$, the expression (++) converges together with $(z - t)\, P\,(z,\, \lambda) \to 0$ to zero, and from this one deduces immediately that all partial derivatives of g exist for $z = t$ and $P\,(z,\, \lambda) = 0$ (and vanish). In particular, the parts of Q and R which consist of domain integrals exist and are, as we wanted, $\mathcal{C}^\infty$–functions. □

After all, we must prove the NIRENBERG *Extension Lemma*. For this, we need a Lemma that goes back to E. BOREL.

***Lemma 3.63** *Let* (f_j) *be a sequence of* $\mathcal{C}^\infty$*–functions on a neighborhood* $U = U\,(0) \subset \mathbb{R}^n$. *Then there exists a* $\mathcal{C}^\infty$*–function* $F = F\,(x,\, t)$ *in a neighborhood* $V = V\,(0) \subset \mathbb{R}^n \times \mathbb{R}$ *with* $V \cap \mathbb{R}^n \subset U$ *such that*

$$\frac{\partial^j F}{\partial t_j}\,(x,\, 0) \,=\, f_j(x)\ , \quad j \in \mathbb{N}\ .$$

Remark. For $n = 0$, this is the surjection $\mathcal{C}^\infty_{n,0} \to \mathbb{R}\{\, x_1, \ldots, x_n \,\}$ which we used already in a former Section.

Idea of *proof*. Let $\varphi\,(t)$ be a $\mathcal{C}^\infty$–function on $\mathbb{R}$ with $\varphi\,(t) \equiv 1$, $|\,t\,| \le 1/2$, $\varphi\,(t) \equiv 0$, $|\,t\,| \ge 1$. Then one puts

$$F\,(x,\, t) \,=\, \sum_{j=0}^{\infty} \frac{t^j}{j!}\, \varphi\,(\rho_j t)\, f_j(x)$$

and shows that $F\,(x,\, t)$ has the desired properties if the positive sequence (ρ_j) grows sufficiently fast. (C.f. [01 - 20], pp. 98, 99). □

With the same idea one proves

***Lemma 3.64** *Let* $f : (\mathbb{R}^n \times \mathbb{R},\, 0) \to \mathbb{C}$ *be a differentiable germ. Then there exists a differentiable germ* $F : (\mathbb{R}^n \times \mathbb{C},\, 0) \to \mathbb{C}$ *with* $F_{|\mathbb{R}^n \times \mathbb{R}} = f$ *such that* $\partial F/\,\partial \overline{z} : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$ *vanishes to infinite order on* $\mathbb{R}^n \times \mathbb{R}$.

Idea of *proof*. Let $z = t + s\,i$ and, with the function φ as above and suitable $\rho_j > 0$:

$$F\,(x,\, z) \,=\, \sum_{j=0}^{\infty} \left(i\, \frac{\partial}{\partial t} \right)^j f\,(x,\, t)\, \frac{s^j}{j!}\, \varphi\,(\rho_j s)\ .$$

Then F is arbitrarily often differentiable and $F\,(x,\, t) = f\,(x,\, t)$. Now,

$$2\,\partial/\,\partial\,\overline{z} \,=\, \partial/\,\partial t \,+\, i\,\partial/\,\partial s \,=\, i\,(-i\,\partial/\,\partial t \,+\, \partial/\,\partial s)\ ;$$

this implies

$$\begin{aligned} \frac{2}{i}\, \frac{\partial F}{\partial \overline{z}}\,(x,\, z) \;=\; & \sum_{j=0}^{\infty} \left(i\, \frac{\partial}{\partial t} \right)^{j+1} f\,(x,\, t)\, \frac{s^j}{j!}\, \left[\, \varphi\,(\rho_{j+1} s) \,-\, \varphi\,(\rho_j s) \,\right] \,+ \\ & \sum_{j=0}^{\infty} \left(i\, \frac{\partial}{\partial t} \right)^{j} f\,(x,\, t)\, \frac{s^j}{j!}\, \rho_j\, \varphi'(\rho_j s)\ , \end{aligned}$$

and each term vanishes in a neighborhood of $s = 0$. □

The *proof* of NIRENBERG's *Extension Theorem* is carried out via induction on b. For $b = 0$ the set $\{\, P\,(z,\, \lambda) = 0 \,\}$ is empty and, therefore, our claim is true due to the preceding Lemma. Suppose that the Extension Lemma is already proven for $b - 1$. We make again the transformation

$$(z,\, \lambda',\, \lambda_b) \longmapsto (z,\, \lambda',\, u)\ , \quad u \,=\, P\,(z,\, \lambda) \,=\, P_b(z,\, \lambda)\ ,$$

and compute easily that the operator $\partial/\partial\overline{z}$ is equal to

$$L = \frac{\partial}{\partial \overline{z}} + \overline{P'(z,\lambda)}\,\frac{\partial}{\partial \overline{u}}$$

where, in principle,

$$P' = \frac{\partial P_b}{\partial t}$$

is (up to a factor and a homotethy) equal to $P_{b-1} = P_{b-1}(z, \lambda')$ (the coordinate λ_b does not occur anymore).

After induction hypothesis, there exists a $\mathcal{C}^\infty$–function $F' = F_{b-1} = F_{b-1}(x, z, \lambda')$ with

$$F'(x, t, \lambda') = f(x, t)\,, \quad (x, t, \lambda') \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{C}^{b-1}\,,$$

and $\partial F'/\partial\overline{z}$ vanishes to arbitrary high order on $\{\operatorname{Im} z = 0\}$ and $\{P'(z, \lambda') = 0\}$.

Put now (φ is chosen similarly as above)

$$F(x, z, \lambda) = \sum_{j=0}^{\infty} \left(-\frac{1}{P'}\,\frac{\partial}{\partial\overline{z}}\right)^j F'(x, z, \lambda')\,\frac{\overline{u}^j}{j!}\,\varphi(\rho_j u)$$

where $\rho_0 = 0$ and the ρ_j are increasing so rapidly that one is allowed to differentiate the series term by term (up to arbitrary high order) on $P' \neq 0$. Due to our assumption on F', the function F may be extended via $F(x, z, \lambda) = 0$ for $P'(z, \lambda') = 0$ to a $\mathcal{C}^\infty$–function in a neighborhood of 0 in $\mathbb{R}^n \times \mathbb{C} \times \mathbb{C}^b$.

For $\operatorname{Im} z = 0$, i.e. $z = t \in \mathbb{R}$, all summands vanish to arbitrary high order, besides for the index $j = 0$. Hence, F is an extension of f:

$$F(x, t, \lambda) = F'(x, t, \lambda') = f(x, t)\,.$$

Now, $\partial F'/\partial\overline{u} \equiv 0$, and hence, LF vanishes to arbitrary high order on $\{\operatorname{Im} z = 0\}$. However,

$$\begin{aligned} LF &= \overline{P'}\left(\frac{1}{P'}\,\frac{\partial}{\partial\overline{z}} + \frac{\partial}{\partial\overline{u}}\right) F \\ &= -\overline{P'}\sum_{j=0}^{\infty}\left(-\frac{1}{P'}\,\frac{\partial}{\partial\overline{z}}\right)^{j+1} F'\cdot\frac{\overline{u}^j}{j!}\,\left[\varphi(\rho_j u) - \varphi(\rho_{j+1} u)\right] \\ &\quad +\overline{P'}\sum_{j=0}^{\infty}\left(-\frac{1}{P'}\,\frac{\partial}{\partial\overline{z}}\right)^{j} F'\cdot\frac{\overline{u}^j}{j!}\,\rho_j\,\frac{\partial\varphi}{\partial\overline{u}}\,(\rho_j u)\,, \end{aligned}$$

and here, each summand on the right hand side vanishes locally around $u = 0$, i.e. $\partial F/\partial\overline{z}$ vanishes of infinite order on $\{u = 0\} = \{P_b(z, \lambda) = 0\}$. □

3.A.2 Symmetric germs

In order to close this Chapter we apply the Division Theorem to *symmetric* germs carrying over the main theorem on symmetric polynomials to differentiable and analytic function germs.

Theorem 3.65 (Glaeser) *Each symmetric, i.e. $\mathfrak{S}_n$–invariant germ $f \in R_n$ is of the form $f = g \circ \sigma$, $g \in R_n$, $\sigma = (\sigma_1, \ldots, \sigma_n)$.*

Proof. Putting in the defining equation for the elementary symmetric functions $t = x_k$ leads to

$$x_k^n = \sum_{j=1}^{n} (-1)^{j-1}\,\sigma_j(x)\,x_k^{n-j}\,,$$

whence

$$x_k^n \in \sigma(\mathfrak{m}_n) R_n \text{ and } (\mathfrak{m}_n)^{n^n} \subset \sigma(\mathfrak{m}_n) R_n .$$

Thus, the monomials of degree $< n^n$ generate R_n as a module on the ring of germs $f \circ \sigma$.

Let $\varphi_1, \ldots, \varphi_N$ be the monomials of degree $< n^n$. Further, let f be a symmetric function germ and write

$$f(x) = \sum_{j=1}^{N} g_j(\sigma(x)) \varphi_j(x) .$$

Due to the symmetry of f and $g_j \circ \sigma$, we get

$$f(x) = \sum_{j=1}^{N} g_j(\sigma(x)) \widetilde{\varphi}_j(x)$$

with

$$\widetilde{\varphi}_j(x) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \varphi_j(x_{\pi(1)}, \ldots, x_{\pi(n)}) .$$

Since the polynomials $\widetilde{\varphi}_j$ are symmetric they can be written in the form $P_j \circ \sigma$, $P_j \in \mathbb{K}[x_1, \ldots, x_n]$, and

$$g = \sum_{j=1}^{N} g_j P_j$$

does the job. □

Remark. The symmetric group $\mathfrak{S}_n$ is generated by the *transpositions* $\tau = (j\,k)$. τ operates on R_n via $\tau(x_j) = x_k$, $\tau(x_k) = x_j$, $\tau(x_i) = x_i$, $i \neq j, k$. Hence $\tau^2 = \mathrm{id}$ and τ is a *reflection*. In other words: $\mathfrak{S}_n$ operates on $\mathbb{K}^n$ as a *reflection group*. We have shown above that the *invariant ring*

$$R_n^{\mathfrak{S}_n} := \{ f \in R_n : f \circ \pi = f \text{ for all } \pi \in \mathfrak{S}_n \}$$

is equal to the ring $\{ g \circ \sigma : g \in R_n \}$, thus *isomorphic* to R_n. We shall study such invariant rings R_n^G later in more generality for arbitrary *finite* subgroups $G \subset \mathrm{Aut}\, R_n$. If G is an arbitrary reflection group then always $R_n^G \cong R_n$.

Notes and References

Historically, the – by Weierstrass – so called *Vorbereitungssatz* comes first in the complex analytic case (due to his own statement from „1860 wiederholt in meinen Universitätsvorlesungen vorgetragen", published only after 1886, by Poincaré already in 1879, but also for $n = 2$ by Cauchy 1831 - cf. Bemerkung 2 in Grauert - Remmert, *Analytische Stellenalgebren* [01 - 02], pp. 35–36. This theorem, however, is equivalent to the *Division Theorem* (Stickelberger 1887, Brill 1891, Späth 1929, Rückert 1933). It is even possible to derive from the Division Theorem the Preparation Theorem for general ground fields as has been noted by Stickelberger and Siegel, c.f. loc. cit., pp. 43–44.

Most of the algebraic material in our text can be found in a very similar manner in [01 - 02]. The use and proof of the Polynomial Division Theorem is taken from [01 - 26]. The proof in the differentiable category follows Bröcker [01 - 24] and Bröcker - Lander [01 - 25] with a certain abbreviation which - according to Theodor Bröcker - goes back to Andreas Dress. I thank my colleague Peter Slodowy who called my attention to this variant. For the proof of Hilbert's *Basis Theorem*, the Gauss *Lemma* and further applications of the Cayley–Hamilton *Theorem*, cf. [02 - 01].