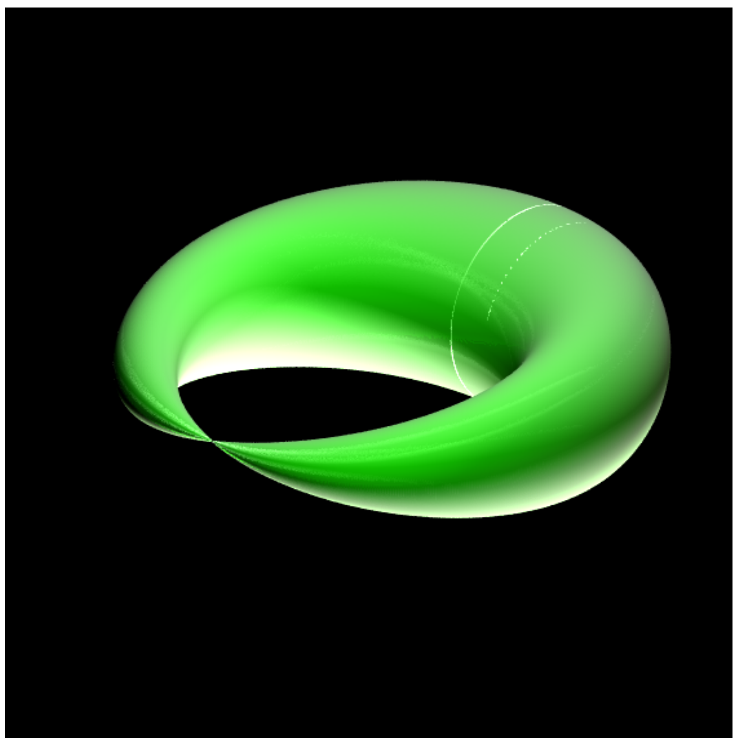
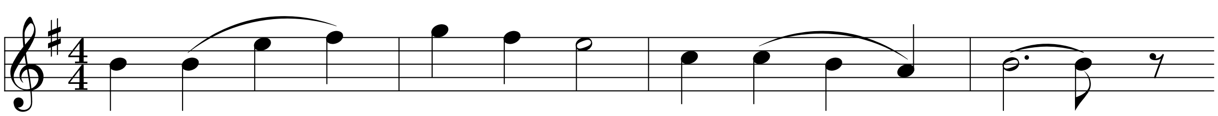
Chapter 2

Es ist nützlich, wenn wir lernen, uns über die richtigen Dinge zu wundern. Oft wundern wir uns über das Erstaunlichste nicht, weil es uns seit langem bekannt ist und darum selbstverständlich scheint.

(Carl Friedrich von Weizsäcker,
Aufbau der Physik)





Chapter 2

Local rings and finitely determined germs

In the present Chapter we develop the local algebra needed for understanding the local theory of *differentiable* and *analytic manifolds* and *spaces*.

2.1 Algebras of germs of functions

Let M be a manifold, or, more generally, a topological space, and let $x^{(0)} \in M$ a fixed point. We introduce an *equivalence relation* between functions with values in a field $\mathbb{K}$ [1] which are defined in a neighborhood U of $x^{(0)}$. This is the appropriate language for encoding *local* properties of functions.

Definition. Let f_1, f_2 be functions that are defined in a neighborhood of U_1 and U_2, resp. of $x^{(0)} \in M$. We say that f_1 and f_2 define the same *germ* in $x^{(0)}$ (and write $f_1 \sim_{x^{(0)}} f_2$) if there exists a neighborhood $U \subset U_1 \cap U_2$ of $x^{(0)}$ such that $f_{1|U} = f_{2|U}$.

It is a trivial exercise to show that $\sim_{x^{(0)}}$ defines indeed an *equivalence relation*. For the equivalence class of a function f at the point $x^{(0)}$ we write $f_{x^{(0)}}$, deleting quite often the point of reference $x^{(0)}$ if this can not cause any confusion (such that we identify without saying the *germ* $f_{x^{(0)}}$ with its *representative* f, allowing to shrink the domain of definition $U = U(x^{(0)})$ of f, if necessary).

Regarding exclusively $\mathcal{C}^\infty$–functions on the *differentiable* manifold M, we denote by

$$\mathcal{C}^\infty_{M,x^{(0)}} \text{ or } \mathcal{E}_{M,x^{(0)}}$$

the set of the equivalence classes which obviously carries in a natural way the structure of a *commutative* $\mathbb{K}$–*algebra with unit*. In the case $M = \mathbb{K}^n$ we write simply $\mathcal{E}_{n,x^{(0)}}$ or $\mathcal{E}_{x^{(0)}}$ etc. In the (real or complex) analytic case we use correspondingly the symbols $\mathcal{O}_{M,x^{(0)}}$ or $\mathcal{O}_{n,x^{(0)}}$ or $\mathcal{O}_{x^{(0)}}$. In order to distinguish the real and the complex case, we add to these notions, if necessary, upper indices $\mathbb{R}$ and $\mathbb{C}$, resp. In particular, if $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the symbol

$$\mathcal{O}_{n,0} = \mathcal{O}^{\mathbb{K}}_{n,0} =: \mathbb{K}\langle x_1, \ldots, x_n \rangle ,$$

in the literature usually written as $\mathbb{K}\{x_1, \ldots, x_n\}$, denotes the ring of *convergent power series* in n variables with coefficients in the field $\mathbb{K}$. Plainly, we have $\mathcal{O}_{M,x^{(0)}} \cong \mathcal{O}_{n,0}$ if $\dim_{\mathbb{K}} M = n$.

Very often it is necessary to investigate *formal* power series: one is, e.g., first trying to solve an analytic problem formally and proves, a posteriori, that these solutions, or at least one of them, are convergent. These formal power series build a commutative $\mathbb{K}$–algebra with unit 1 which we denote by

$$\widetilde{\mathcal{O}}_{n,0} \text{ or } \mathbb{K}\{x_1, \ldots, x_n\}$$

(in the literature mostly denoted by $\mathbb{K}\{\{x_1, \ldots, x_n\}\}$). The "tilde" stands here for the operation of "completing" in a suitable sense (namely with respect to the so called $\mathfrak{m}$–*adic* topology where $\mathfrak{m}$ denotes the *maximal* ideal - see the next Section). Some authors use for this completion a "roof"–symbol which we, however, reserve à la Grauert–Remmert for the "normalization".

[1] We write $\mathbb{K}$ instead of K or just k in order to indicate that in most of our applications we are dealing with the field $\mathbb{C}$ of *complex numbers* and for a while also with the field $\mathbb{R}$ of *real numbers*.

2.2 Maximal ideals and local rings

Let now R_n be one of these $\mathbb{K}$–algebras $\mathcal{C}^{\infty}_{n,0}$, $\mathcal{O}_{n,0}$ resp. $\widetilde{\mathcal{O}}_{n,0}$. It is obvious that one can associate to each element $f \in R_n$ a "value" $f(0) \in \mathbb{K}$ (this is the value at 0 of any *representative* of f resp., in the cases $\mathcal{O}_{n,0}$ and $\widetilde{\mathcal{O}}_{n,0}$, the constant term $c_{0,\ldots,0}$ in the formal power series expansion $f = \sum_{\nu \in \mathbb{N}^n} c_\nu x^\nu$, $x^\nu := x_1^{\nu_1} \cdot \ldots \cdot x_n^{\nu_n}$, $\nu = (\nu_1, \ldots, \nu_n)$). It is clear that the map

$$\varepsilon : R_n \longrightarrow \mathbb{K}, \quad f \longmapsto f(0),$$

is a $\mathbb{K}$–algebra–homomorphism. Since the field of constants $\mathbb{K}$ can be embedded into R_n via

$$\mathbb{K} \ni c \longmapsto c \cdot 1 \in R_n$$

and the composition $\mathbb{K} \hookrightarrow R_n \to \mathbb{K}$ is is the identity map, ε is *surjective*.

In particular the set of elements $f \in R_n$ with value $f(0) = 0$ is equal to the *kernel* $\varepsilon^{-1}(0)$ of the homomorphism ε, hence an *ideal* which we write as

$$\mathfrak{m} = \mathfrak{m}_n = \mathfrak{m}(R_n);$$

it satisfies the congruence $R_n / \mathfrak{m}_n \cong \mathbb{K}$. Clearly, the symbol $\mathfrak{m}$ stands for *maximal* ideal.

Definition. An ideal $\mathfrak{m}$ in a commutative ring A with unit 1 is called *maximal* if the following holds true:

i) $\mathfrak{m} \subsetneqq A$,

ii) if $\mathfrak{a} \subset A$ is an ideal with $\mathfrak{m} \subsetneqq \mathfrak{a}$ then $\mathfrak{a} = A$.

Indeed, we have the following

Theorem 2.1 *For the "geometric" rings R_n considered above, the ideal $\mathfrak{m}_n$ is maximal.*

This immediately follows from the (elementary) property of any field $\mathbb{K}$ to possess only the two (trivial) ideals 0 and $\mathbb{K}$, the surjectivity of the homomorphism ε and the following

Lemma 2.2 *Let $\varepsilon : A \to B$ be a surjective homomorphism of rings and $\mathfrak{a}_0 = \varepsilon^{-1}(0)$. Then, there is a canonical bijection*

$$\{\, \mathfrak{a} \subset A \text{ ideal with } \mathfrak{a} \supset \mathfrak{a}_0 \,\} \longleftrightarrow \{\, \mathfrak{b} \subset B \text{ ideal} \,\}$$

where $\mathfrak{a}_0$ corresponds to the null ideal *in B.*

Proof. For any ideal $\mathfrak{a} \subset A$ satisfying $\mathfrak{a}_0 \subset \mathfrak{a}$ the image $\varepsilon(\mathfrak{a}) \subset B$ is an ideal. If $\mathfrak{b} \subset B$ is an ideal then also $\varepsilon^{-1}(\mathfrak{b}) \subset A$, and we have $\mathfrak{a}_0 \subset \varepsilon^{-1}(\mathfrak{b})$. Since ε is surjective it immediately follows that $\varepsilon(\varepsilon^{-1}(\mathfrak{b})) = \mathfrak{b}$. If conversely $\mathfrak{a}_0 \subset \mathfrak{a}$ we get $\varepsilon^{-1}(\varepsilon(\mathfrak{a})) = \mathfrak{a} + \ker \varepsilon = \mathfrak{a} + \mathfrak{a}_0 = \mathfrak{a}$. □

In all cases considered up to now, $A = \mathbb{K} \oplus \mathfrak{m}$ as $\mathbb{K}$–vector spaces, and an element $f \in A$ has a *(multiplicative) inverse* in A if and only if $f(0) = \varepsilon(f) \neq 0$, i.e. if $f \notin \mathfrak{m}$. Then, one says that f is *invertible* or a *(multiplicative) unit* in A. This is equivalent to saying that the *principal ideal*

$$fA := \{\, h \in A : \text{there exists } g \in A \text{ with } h = fg \,\}$$

generated by f coincides with A. (Otherwise, the element f is called a *nonunit*). – It is easy to draw from this the following consequence:

Corollary 2.3 *For the rings R_n as above, $\mathfrak{m}_n$ is the* unique *maximal ideal.*

Indeed, each ideal different from $\mathfrak{m}_n$ is either contained in $\mathfrak{m}_n$, or it contains a unit, hence coincides with R_n. $\square$

This phenomenon occurs so often that it deserves a special notion.

Definition. A commutative ring A with 1 is called a *local* ring if it has precisely one maximal ideal $\mathfrak{m}_A$. We then also say that the *pair* $(A, \mathfrak{m}_A)$ is a local ring.

Remark. $\mathbb{C}[x_1, \ldots, x_n]$ is *not* a local ring for $n \geq 1$. The polynomial ring possesses in fact exactly as many maximal ideals as we have points in $\mathbb{C}^n$, i.e. the ideals

$$\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)\,\mathbb{C}[x_1, \ldots, x_n], \quad a = (a_1, \ldots, a_n) \in \mathbb{C}^n .$$

Remark. Whenever an ideal $\mathfrak{a}$ in a ring A is generated by the elements $f_1, \ldots, f_k$, i.e. whenever

$$\mathfrak{a} = \{ f \in A : \text{there exists } g_1, \ldots, g_k \in A \text{ with } f = \sum_{j=1}^{k} f_j g_j \},$$

we write

$$\mathfrak{a} = (f_1, \ldots, f_k)\,A \text{ or } \mathfrak{a} = \sum_{j=1}^{k} f_j A = \sum_{j=1}^{k} A f_j .$$

If there is no risk of confusion, we sometimes write also $(f_1, \ldots, f_k)$ as shorthand instead of $(f_1, \ldots, f_k)\,A$. Similar notations are used by us for finitely generated (sub–) modules on A.

Definition. Let A be a local commutative $\mathbb{K}$–algebra with 1 and $\mathfrak{m} \subset A$ its unique maximal ideal. A is then called a *local* $\mathbb{K}$*–algebra* if the canonical homomorphism $\mathbb{K} \to A/\mathfrak{m}$ of fields is an *isomorphism*.

Remark. The $\mathbb{K}$–algebras $\mathcal{C}^{\infty}_{n,0}$, $\mathcal{O}_{n,0}$, $\widetilde{\mathcal{O}}_{n,0}$ have this property as well as their quotients by proper ideals. Of special interest to us are the *analytic* $\mathbb{K}$*–algebras*

$$A := \mathcal{O}_{n,0}/\,\mathfrak{a}, \quad \mathfrak{a} \subset \mathfrak{m} = \mathfrak{m}(\mathcal{O}_{n,0}) \text{ an ideal},$$

since they represent analytic singularities in an algebraic way.

In the geometric rings R_n we can characterize the maximal ideal and its powers in still another way. Observe that one can associate to each element $f \in R_n$ not only a value $f(0) \in \mathbb{K}$ (even in the formal case) but also the *partial deratives*

$$D^{\nu} : \begin{cases} R_n \longrightarrow R_n \\ f \longmapsto D^{\nu} f : \dfrac{\partial^{|\nu|} f}{\partial x_1^{\nu_1} \cdot \ldots \cdot \partial x_n^{\nu_n}} \end{cases}$$

and also there values at 0. In particular, we have

$$\mathfrak{m}_n = \{ f \in R_n : f(0) = 0 \} = \{ f \in R_n : D^0 f(0) = 0 \} .$$

More generally, one can prove the Theorem stated below. One should have in mind here that for two ideals $\mathfrak{a}$, $\mathfrak{b}$ in a commutative ring R the *product* $\mathfrak{a}\,\mathfrak{b}$ will be generated, by definition, by all products $f\,g$, $f \in \mathfrak{a}$, $g \in \mathfrak{b}$. The *ideal power* $\mathfrak{a}^r$ is nothing else but the r–fold product of $\mathfrak{a}$ with itself.

Theorem 2.4 *The ideals* $\mathfrak{m}_n^r$, $r \geq 1$, *are generated by the monomials*

$$x^{\nu} = x_1^{\nu_1} \cdot \ldots \cdot x_n^{\nu_n}, \quad |\nu| := \nu_1 + \cdots + \nu_n = r .$$

One has

$$m_n^r = \{ f \in R_n : D^{\nu} f(0) = 0 \text{ for all } |\nu| < r \} .$$

Proof. In the case $r = 1$ we have only to show that $\mathfrak{m}_n$ is generated by the coordinate functions $x_1, \ldots, x_n$. But this has been already proven in Chapter 1 (HADAMARD's Lemma 1.9). From this, the first claim follows by induction on r. The second is an easy consequence of the LEIBNIZ *rule*

$$D^\mu(fg) = \sum_{\lambda \le \mu} \binom{\mu}{\lambda} D^\lambda f \cdot D^{\mu-\lambda} g = \binom{\mu}{\lambda} = \frac{\mu!}{\lambda! \cdot (\mu - \lambda)!}\ , \quad \mu! := \mu_1! \cdot \ldots \cdot \mu_n!\ ,$$

which implies

$$D^\mu x^\nu \notin \mathfrak{m}_n \iff \mu = \nu\ .$$

Therefore, $\mathfrak{m}_n^r \subset \{ f \in R_n : D^\mu f(0) = 0,\ |\mu| < r \}$. For the opposite inclusion, we proceed by induction on r, the case $r = 1$ being already done. So, suppose that $D^\mu f(0) = 0$ for all μ with $|\mu| \le r$. By induction hypothesis, we know that $f \in \mathfrak{m}_n^r$ such that we can write

$$f = \sum_{|\mu| = r} x^\mu g_\mu\ .$$

It suffices to show that all functions g_μ vanish at the origin. Therefore, choose an arbitrary multiindex ν with $|\nu| = r$. Due to our assumption, $D^\nu f(0) = 0$. On the other hand,

$$D^\nu \Big(\sum_{|\mu|=r} x^\mu g_\mu \Big) = \sum_{|\mu|=r} D^\nu(x^\mu g_\mu) = \sum_{|\mu|=r} \sum_{\lambda \le \nu} \binom{\nu}{\lambda} D^\lambda x^\mu \cdot D^{\nu-\lambda} g_\mu\ ,$$

and each term $D^\lambda x^\mu$ on the right hand side vanishes at the origin unless $\lambda = \mu = \nu$. Consequently, $g_\nu(0) = 0$. □

The property of an arbitrary ring to be *local* is in fact *equivalent* to the assumption that the invertible elements form the complement of an *ideal*. To prove this we need the validity of ZORN's *Lemma* from which one can deduce the following result.

***Theorem 2.5** *Every proper ideal* $\mathfrak{a}$ *in a (commutative) ring* A *with unit is contained in a maximal ideal* $\mathfrak{m}$ *of* A.

Without *proof*.

Before we formulate and prove the above mentioned equivalence we show another useful general result.

Theorem 2.6 *Let* A *be a commutative ring with* 1*, and let* I *be the set of* nonunits *of* A*, i.e.* $f \in I,\ g \in A \Longrightarrow fg \ne 1$*. Then, we have*

$$I = \bigcup_{\mathfrak{a} \ne A} \mathfrak{a}\ ,$$

where $\mathfrak{a}$ *runs through the set of all (proper)* ideals *in* A.

Proof. If $f \in I$, then the principal ideal $\mathfrak{a} = fA \ne A$ and hence I is contained in the right hand side. If, conversely, $f \notin I$, thus $fg = 1$ for some g, so we have $fA = A$ and consequently $\mathfrak{a} = A$ for each ideal $\mathfrak{a}$ which contains f. □

Theorem 2.7 *Let* A *be a commutative ring with* 1, I *the set of its nonunits. Then, the following statements are equivalent* :

i) I *is an ideal* ;

ii) *the set of proper ideals* $\mathfrak{a} \subset A$ *has a greatest element (with respect to inclusion)* ;

iii) A *possesses exactly one maximal ideal.*

In the cases ii) *and* iii)*, the* distinguished *ideal is equal to* I *.*

Proof. i) $\Longrightarrow$ ii) Due to the preceding Theorem.

ii) $\Longrightarrow$ iii) Let $\mathfrak{a}_0$ be such a greatest element. Then,

$$\mathfrak{a}_0 = \bigcup_{\mathfrak{a} \neq A} \mathfrak{a}$$

is the unique maximal ideal and, because of Theorem 6, $\mathfrak{a}_0 = I$.

iii) $\Longrightarrow$ i) Let $\mathfrak{m}$ be the maximal ideal. Then we have, according to the preceding Theorem, $\mathfrak{m} \subset I$. If $f \in I$, then $\mathfrak{a} = fA \neq A$ and $\mathfrak{a}$ must be contained in a maximal ideal, hence in $\mathfrak{m}$; therefore $f \in \mathfrak{m}$.
$\square$

With the help of this Theorem we are able to characterize *maximal* ideals in general.

Corollary 2.8 *The ideal* $\mathfrak{m} \subset A$ *is maximal if and only if the residue class ring* $A/\mathfrak{m}$ *is a field.*

Proof. Because of the preceding Theorems the equivalence of the following statements is guaranteed:

a) $\mathfrak{m}$ is maximal;

b) $\overline{A} = A/\mathfrak{m}$ possesses only the ideals (0) and $\overline{A}$;

c) $\overline{A}$ is a local ring with (0) as maximal ideal;

d) each element in $\overline{A} \setminus \{ 0 \}$ is a unit;

e) $\overline{A}$ is a field. $\square$

2.3 The Nakayama Lemma

The following Theorem is rather trivial and, therefore, only deserves to be named a “Lemma”. Nevertheless, we label it as a “Theorem” since, as we will see soon, it has extremely deep consequences.

Theorem 2.9 (Nakayama Lemma) *Let* $(A, \mathfrak{m})$ *be a local ring,* M *a finitely generated* A*–module and* $N \subset M$ *a submodule satisfying*

$$(*) \qquad M \subset N + \mathfrak{m} M .$$

Then, $N = M$ *. In particular, it follows that* $M = 0$ *if* $M = \mathfrak{m} M$ *.*

Proof. Together with M the quotient M/N is finitely generated. Since from $(*)$ it follows that $M/N \subset \mathfrak{m}(M/N)$ we can restrict ourselves to the case $N = 0$. Choose then a *minimal* system $m_1, \ldots, m_r$ of generators of M on A , $r \geq 1$; because of $(*)$ we have

$$m_r = \sum_{j=1}^{r} a_j m_j , \quad a_j \in \mathfrak{m} ,$$

hence

$$(1 - a_r) m_r = \sum_{j=1}^{r-1} a_j m_j .$$

Now, $1 - a_r \notin \mathfrak{m}$ since otherwise $1 \in \mathfrak{m}$ and $\mathfrak{m} = A$. Therefore,

$$m_r = \sum_{j=1}^{r-1} b_j m_j , \quad b_j = (1 - a_r)^{-1} a_j ,$$

and, consequently, the chosen system of representatives is not minimal if $r \geq 2$. For $r = 1$, we conclude by the same argument that $m_1 = 0$ and $M = 0$. $\square$

The Nakayama Lemma has a sharper formulation which is quite useful.

Theorem 2.10 *Let A be a local ring with maximal ideal $\mathfrak{m}$, N, N' submodules of a fixed A–module M, N' finitely generated. If then $N' \subset N + \mathfrak{m} N'$ it follows that $N' \subset N$.*

Proof. $N'' := N' \cap N$ implies $N' = N'' + \mathfrak{m} N'$ and therefore, due to the NAKAYAMA Lemma, $N' = N'' \subset N$. □

In applications, the following result is used quite frequently.

Corollary 2.11 *Let M be a finitely generated A–module on the local ring A, and let $\omega : M \to M/\mathfrak{m} M$ be the canonical residue class map. Then the following are equivalent :*

a) *$m_1, \ldots, m_r$ generate (minimally) the module M;*

b) *the residue classes $\omega(m_j) = \overline{m}_j$, $j = 1, \ldots, r$, form a system of generators (a basis) of the $A/\mathfrak{m} A$–vector space $M/\mathfrak{m} M$.*

In particular, from each system of generators of a finitely generated A–module M one can extract a minimal one of uniquely determined length

$$\mathrm{cg}_A M := \dim_{A/\mathfrak{m}A} M/\mathfrak{m} M .$$

Proof. a) $\Longrightarrow$ *b)* is correct by trivial reasons. Conversely, let the elements $m_1, \ldots, m_r$ be given as in b) and $N = A(m_1, \ldots, m_r) \subset M$ the A–submodule of M generated by $m_1, \ldots, m_r$. For $m \in M$, one has $\omega(m) = \sum c_j \, \omega(m_j)$ with $c_j \in A/\mathfrak{m}$. Now, choose representatives $a_j \in A$ of the elements c_j; then $m - \sum a_j m_j \in \ker \omega = \mathfrak{m} M$, hence $M \subset N + \mathfrak{m} M$ and therefore $M \subset N \subset M$, i.e. $M = N$. □

Similarly, one can see the following

Theorem 2.12 *Any homomorphism $M \to M''$ of finitely generated modules on a local ring A is surjective if and only if the associated $A/\mathfrak{m}A$–vector space homomorphism $M/\mathfrak{m}M \to M''/\mathfrak{m} M''$ does.*

2.4 Ideals of finite codimension

We deduce now from the version of the NAKAYAMA Lemma stated above the following Theorem which will be applied several times in these notes.

Theorem 2.13 *Let A be a local $\mathbb{K}$–algebra with finitely generated maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n) A$. Then, the following two statements for an ideal $\mathfrak{a} \subset A$ are equivalent :*

i) *$\mathfrak{a}$ is of finite codimension in A, i.e. $\dim_{\mathbb{K}} A/\mathfrak{a} < \infty$;*

ii) *there exists a power $\mathfrak{m}^\ell \subset \mathfrak{a}$.*

If i) *is satisfied, one can choose $\ell = \dim_{\mathbb{K}} A/\mathfrak{a}$ in* ii).

Proof. The ideal $\mathfrak{m}^\ell$ can be generated by the monomials $x_1^{\nu_1} \cdot \ldots \cdot x_n^{\nu_n}$, $|\nu| = \nu_1 + \cdots + \nu_n = \ell$. It is easy to see that each element $f \in A$ modulo $\mathfrak{m}^\ell$ may be represented as a polynomial on $\mathbb{K}$ in the variables $x_1, \ldots, x_n$ of total degree $< \ell$ (c.f. the next Lemma). Thus, $A/\mathfrak{m}^\ell$ is a finitely generated $\mathbb{K}$–vector space for all ℓ. If now ii) is satisfied, there exists an epimorphism $A/\mathfrak{m}^\ell \to A/\mathfrak{a}$ of $\mathbb{K}$–vector spaces such that the ideal $\mathfrak{a}$ is of finite codimension. If, on the other hand, i) is satisfied, consider the infinite sequence of inclusions

$$\mathfrak{a} + \mathfrak{m} \supset \mathfrak{a} + \mathfrak{m}^2 \supset \mathfrak{a} + \mathfrak{m}^3 \supset \cdots \supset \mathfrak{a} ,$$

which has the epimorphisms

$$A/\mathfrak{a} \longrightarrow A/(\mathfrak{a} + \mathfrak{m}^{j+1}) \longrightarrow A/(\mathfrak{a} + \mathfrak{m}^j) \longrightarrow \cdots \longrightarrow A/(\mathfrak{a} + \mathfrak{m})$$

as a consequence. Thus, we have for all j:

$$(+)\qquad \dim_{\mathbb{K}} A/(\mathfrak{a} + \mathfrak{m}^j) \le \dim_{\mathbb{K}} A/(\mathfrak{a} + \mathfrak{m}^{j+1}) \le \dim_{\mathbb{K}} A/\mathfrak{a} < \infty\,,$$

and for at least one (in fact for almost all) j the inequality on the left hand side in (+) must become an equality. Therefore, there exists a number ℓ satisfying

$$\mathfrak{m}^\ell \subset \mathfrak{a} + \mathfrak{m}^\ell = \mathfrak{a} + \mathfrak{m}^{\ell+1} = \mathfrak{a} + \mathfrak{m}\cdot\mathfrak{m}^\ell\,.$$

The NAKAYAMA Lemma in the guise of Theorem 10 implies immediately the claim ii). The last assertion follows by the same reasoning from (+). □

We have used a Lemma in the proof of the preceding Theorem which we now wish to state expressis verbis. It is a generalization of TAYLOR's *formula* which holds in the geometric rings $\mathcal{O}_{n,0}$, $\mathcal{C}^\infty_{n,0}$, $\widetilde{\mathcal{O}}_{n,0}$.

Lemma 2.14 *Let A a be a local $\mathbb{K}$–algebra with finitely generated maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)\,A$. Then there exists, to each $f \in A$ and every $\ell \in \mathbb{N}$, a polynomial $P_{f,\ell}$ in the variables $x_1, \dots, x_n$ with coefficients in $\mathbb{K}$ of degree $\le \ell$ such that*

$$f - P_{f,\ell} \in \mathfrak{m}^{\ell+1}\,.$$

Proof. We proceed by induction with respect to ℓ, the case $\ell = 0$ being trivial. Suppose

$$f - P_{f,\ell} = \sum_{|\nu|=\ell+1} f_\nu\, x^\nu\,.$$

Decompose f_ν into $c_\nu + g_\nu$, $c_\nu \in \mathbb{K}$, $g_\nu \in \mathfrak{m}$, and set

$$P_{f,\ell+1} = P_{f,\ell} + \sum_{|\nu|=\ell+1} c_\nu\, x^\nu\,.$$ □

Remark. In the complex analytic context, an ideal $\mathfrak{a} \subset R_n$ is of finite codimension precisely if the zero set of $\mathfrak{a}$ is zero–dimensional (see Section 8 below).

2.5 Noetherian rings and modules

Local *noetherian* rings and algebras have particularly pleasant properties. We collect here some general facts on such rings and modules for the convenience of the reader.

Definition. A commutative ring A is called *noetherian* if each ideal $\mathfrak{a} \subset A$ is finitely generated.

Remark. We will prove later (Theorem 3.17) that the rings $\mathbb{K}\langle x_1, \dots, x_n\rangle$ and $\mathbb{K}\{x_1, \dots, x_n\}$ are noetherian. The rings $\mathcal{C}^\infty_{n,0}$, however, are *not* noetherian (c.f. the Corollary 18).

Theorem 2.15 *A commutative ring A is noetherian if and only if each* increasing *chain of ideals*

$$\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset A$$

is stable : *there exists $k \in \mathbb{N}$ with $\mathfrak{a}_j = \mathfrak{a}_k$ for all $j \ge k$.*

Proof. i) If A is not noetherian, i.e. if there exists an ideal $\mathfrak{a}$ in A which is not finitely generated, there is an infinite sequence $f_j \in \mathfrak{a}$ such that the sequence of ideals $\mathfrak{a}_j := (f_0, \dots, f_j)\,A$ increases properly and, therefore, can not become stable.

ii) Let, conversely, A be noetherian and $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$ an increasing chain of ideals. Then,

$$\mathfrak{a} := \bigcup_{j\in\mathbb{N}} \mathfrak{a}_j$$

is an ideal, too. If one chooses finitely many generators $f_\rho \in \mathfrak{a}$ then there is a number $k \in \mathbb{N}$ with $f_\rho \in \mathfrak{a}_k$ for all ρ. Hence, $\mathfrak{a} = \mathfrak{a}_k$, and consequently $\mathfrak{a}_j = \mathfrak{a}_k$ for all $j \geq k$. □

Definition. An A–module M is said to be *noetherian* if every submodule $N \subset M$ is finitely generated on A.

If the base ring itself is noetherian, we have a simple criterion for a module to be noetherian.

Theorem 2.16 (Hilbert) *Let A be a noetherian ring and M an A–module. Then M is noetherian if and only if M is finitely generated.*

Proof. If the module M is noetherian then it is finitely generated by definition. Let, conversely, M be finitely generated, say by the elements $m_1, \ldots, m_t$, and let N be a submodule of M. We shall show by induction on t that N is finitely generated. For $t = 1$, the map $A \to M$ sending 1 to m_1 is surjective. The preimage of N is an ideal in A which is finitely generated by assumption on A. The images of its generators generate N. If $t > 1$ the image $\widetilde{N}$ of N in M/Am_1 is finitely generated since the module M/Am_1 is generated by the residue classes of the elements $m_2, \ldots, m_t$. Let $n_1, \ldots, n_s$ be elements of N whose images generate $\widetilde{N}$. Since $Am_1 \subset M$ is generated by one element, its submodule $N \cap Am_1$ is finitely generated, say by $p_1, \ldots, p_r$. Then it is easy to see that the elements $n_1, \ldots, n_s$ and $p_1, \ldots, p_r$ together generate N. □

In application of our considerations made before we now regard in our geometric algebras R_n the ideal

$$\mathfrak{m}_n^\infty = \bigcap_{r \geq 1} \mathfrak{m}_n^r .$$

Using a representation $f = \sum x_j g_j \in \mathfrak{m}_A^\infty$ and applying the LEIBNIZ rule one can immediately perceive that all g_j must be contained in $\mathfrak{m}_n^\infty$, too. Necessarily,

$$\mathfrak{m}_n^\infty = \mathfrak{m}_n \cdot \mathfrak{m}_n^\infty .$$

If $\mathfrak{m}_n^\infty$ is finitely generated it follows that $\mathfrak{m}_n^\infty = 0$. - We summarize:

Theorem 2.17 *For the rings $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ and $\mathbb{K}\{x_1, \ldots, x_n\}$ we have*

$$\mathfrak{m}_n^\infty := \bigcap_{r \geq 1} \mathfrak{m}_n^r = 0 .$$

Remark. Of course, this result follows easily without recourse to the Noether property of the rings we are considering. Just expand any germ f into a power series $\sum_\nu c_\nu x^\nu$ and deduce from $f \in \mathfrak{m}_n^\infty$ that $c_\nu = 0$ for all $\nu \in \mathbb{N}^n$. In fact, we use the Theorem implicitly in the proof of Theorem 3.17.

Concerning $\mathcal{C}^\infty$–functions, the circumstances are completely different. If one associates to a germ $f \in \mathcal{C}^\infty_{n,0}$ its TAYLOR *series*

$$j_0^\infty f \in \mathbb{K}\{x_1, \ldots, x_n\}$$

one obtains a $\mathbb{K}$–algebra homomorphism

$$\mathcal{C}^\infty_{n,0} \longrightarrow \mathbb{K}\{x_1, \ldots, x_n\}$$

which, in fact, is *surjective* (Theorem of E. BOREL). The kernel of this homomorphism is equal to $\mathfrak{m}_n^\infty$, the ideal of *flat* functions which is, as is well known, not trivial. - Therefore, we may conclude the

Corollary 2.18 *The local ring $\mathcal{C}^\infty_{n,0}$ is* not *noetherian.*

2.6 Krull's Intersection Theorem

For finitely generated modules on noetherian rings A, the preceding Theorem generalizes to KRULL's *Intersection Theorem*.

Theorem 2.19 *Let A be a noetherian local ring, M a finitely generated A–module, and $N \subset M$ a submodule. Then,*

$$N = \bigcap_{j=1}^{\infty} (N + \mathfrak{m}^j M) .$$

We will postpone the *proof* to Chapter 6 where it will be deduced from the ARTIN–REES *Lemma*. For a direct proof c.f. GRAUERT - REMMERT [1 - 02], Anhang pp. 211, 212.

Corollary 2.20 *Let N, N' be submodules of a finitely generated A–module M, A a noetherian local ring, and let*

$$N' \subset N + \mathfrak{m}^j M , \quad j \geq 1 .$$

Then, $N' \subset N$.

For: $N' \subset \bigcap_{j=1}^{\infty} (N + \mathfrak{m}^j M) = N$. □

We demonstrate on two examples how useful this Corollary is.

Theorem 2.21 *Let $\mathfrak{a} \subset \mathcal{O}_{n,0}$ be an ideal, $f_1, \ldots, f_k \in \mathfrak{a}$ fixed elements, and suppose that for all $f \in \mathfrak{a}$ we have a relation*

$$f = g_1 f_1 + \cdots + g_k f_k \ \text{ in } \ \widetilde{\mathcal{O}}_{n,0} , \ \text{ i.e. } \ g_1, \ldots, g_k \in \widetilde{\mathcal{O}}_{n,0} .$$

Then, the elements $f_1, \ldots, f_k$ form a system of generators for $\mathfrak{a}$ in $\mathcal{O}_{n,0}$.

Proof. The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ lies densely in $\mathcal{O}_{n,0}$ with respect to the $\mathfrak{m}$–adic topology, $\mathfrak{m} = \mathfrak{m}(\mathcal{O}_{n,0})$, and correspondingly in $\widetilde{\mathcal{O}}_{n,0}$ for $\widetilde{\mathfrak{m}} = \mathfrak{m}(\widetilde{\mathcal{O}}_{n,0})$; i.e.: for every $h \in \mathcal{O}_{n,0}$ (resp. $\widetilde{\mathcal{O}}_{n,0}$) and each number $j \in \mathbb{N}$ there exists a polynomial $P = P_{h,j}$ satisfying

$$h \equiv P \mod \mathfrak{m}^j \ \text{ resp. } \mod \widetilde{\mathfrak{m}}^j .$$

(C.f. Lemma 14). Therefore, there exist to each $j \in \mathbb{N}$ polynomials $g_i^{(j)}$, $i = 1, \ldots, k$, with

$$(*) \qquad f - \left(g_1^{(j)} f_1 + \cdots + g_k^{(j)} f_k \right) \in \widetilde{\mathfrak{m}}^j \cap \mathcal{O}_{n,0} .$$

But, as one can easily deduce from Theorem 4, $\widetilde{\mathfrak{m}}^j \cap \mathcal{O}_{n,0} = \mathfrak{m}^j$. Set now $\mathfrak{b} := (f_1, \ldots, f_k)\, \mathcal{O}_{n,0} \subset \mathfrak{a}$. Due to $(*)$, it follows that

$$\mathfrak{a} \subset \mathfrak{b} + \mathfrak{m}^j , \quad j \geq 1 ,$$

i.e. $\mathfrak{a} \subset \bigcap_{j=1}^{\infty} (\mathfrak{b} + \mathfrak{m}^j) = \mathfrak{b}$ and hence $\mathfrak{a} = \mathfrak{b} = (f_1, \ldots, f_k)\, \mathcal{O}_{n,0}$. □

Remark. We have used the full power of KRULL's Intersection Theorem in the last proof, in particular the fact that the ring $\mathcal{O}_{n,0}$ is noetherian. If we denote by B the analytic algebra $\mathcal{O}_{n,0}/\mathfrak{b}$ and by π the canonical epimorphism $\mathcal{O}_{n,0} \to B$ then the proof above implies that the ideal $\pi(\mathfrak{a})$ is contained in all powers $\mathfrak{m}_B^j$. Therefore, we need the Intersection Theorem only in the form $\bigcap \mathfrak{m}_B^j = 0$ for all analytic algebras B.

2.7 Local homomorphisms

Quite important for the theory to be developed later are (special) *ring homomorphisms*. If M, N are differentiable (real analytic, complex analytic) manifolds, and if $f : N \to M$ denotes a differentiable (resp. analytic) map then, to each $x^{(0)} = f(y_0)$, $y_0 \in N$ and the corresponding *germs of sets* $(M, x^{(0)})$ and (N, y_0), there exists an associated *map germ* $f_{y_0} : (N, y_0) \longrightarrow (M, x^{(0)})$ and a *ring homomorphism*

$$\varphi = f_{y_0}^* : \mathcal{C}^\infty_{M,x^{(0)}} \longrightarrow \mathcal{C}^\infty_{N,y_0} \quad \text{etc.}$$

defined by substitution $\mathcal{C}^\infty_{M,x^{(0)}} \ni g \mapsto g \circ f$. In local charts this amounts to a map

$$\begin{cases} \mathcal{C}^\infty_{m,0} \longrightarrow \mathcal{C}^\infty_{n,0} \\ \quad g \quad \longmapsto g \circ f \end{cases}$$

with $f = (f_1, \ldots, f_m)$, $f_j \in \mathfrak{m}_n = \mathfrak{m}(\mathcal{C}^\infty_{n,0})$. In particular, if $g(0) = 0$ then $(g \circ f)(0) = g(f(0)) = g(0) = 0$.

Definition. A ring homomorphism $\varphi : A \to B$ of local rings is termed *local* if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

Theorem 2.22 *Each algebra homomorphism* $\varphi : A \to B$ *of local* $\mathbb{K}$*–algebras is local.*

Proof. Automatically, as $\mathbb{K}$–vector spaces, $A = \mathbb{K} \oplus \mathfrak{m}_A$ and $B = \mathbb{K} \oplus \mathfrak{m}_B$. Let $f \in \mathfrak{m}_A$ and $\varphi(f) = c + g$, $c \in \mathbb{K}$, $g \in \mathfrak{m}_B$. If $c \neq 0$, then $h = f - c$ would be a unit, hence $h\,h_0 = 1$, and, since φ is a $\mathbb{K}$–algebra homomorphism:

$$g\,\varphi(h_0) = (c + g - c)\,\varphi(h_0) = (\varphi(f) - \varphi(c))\,\varphi(h_0) = \varphi((f - c)\,h_0) = \varphi(h\,h_0) = \varphi(1) = 1\,,$$

whence $g \notin \mathfrak{m}_B$. Contradiction! □

A beautiful Corollary from Krulls *Intersection Theorem* is a uniqueness result for local homomorphisms which, in particular, applies to local *noetherian* algebras.

Theorem 2.23 *Let* $\varphi_1, \varphi_2 : A \to B$ *be (local) algebra homomorphisms of local* $\mathbb{K}$*–algebras with*

$$(*) \qquad \varphi_1(x_j) = \varphi_2(x_j)\,, \quad j = 1, \ldots, n\,,$$

$x_1, \ldots, x_n$ *a finite system of generators for* $\mathfrak{m} = \mathfrak{m}_A$, B *satisfying* $\bigcap_j \mathfrak{m}_B^j = 0$. *Then,* $\varphi_1 = \varphi_2$.

Proof. Using Lemma 14, we find to each element $f \in A$ and each integer $\ell \in \mathbb{N}$ a polynomial $P = P_{f,\ell}$ in $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$ satisfying

$$f - P \in \mathfrak{m}_A^\ell\,.$$

Moreover, $\varphi_1(\mathfrak{m}_A^\ell) \subset \mathfrak{m}_B^\ell$ and similarly $\varphi_2(\mathfrak{m}_A^\ell) \subset \mathfrak{m}_B^\ell$ for all $\ell \in \mathbb{N}$. Due to $(*)$, $\varphi_1(P) = \varphi_2(P)$; hence by subtraction

$$\varphi_2(f) - \varphi_1(f) = \varphi_2(f - P) - \varphi_1(f - P) \in \mathfrak{m}_B^\ell \text{ for all } \ell\,,$$

and thus

$$\varphi_2(f) - \varphi_1(f) \in \bigcap_{\ell=1}^{\infty} \mathfrak{m}_B^\ell = 0\,.$$

□

Corollary 2.24 *The* $\mathbb{K}$*–algebra homomorphisms*

$$\varphi : \mathcal{O}_{m,0} = \mathbb{K}\langle y_1, \ldots, y_m \rangle \longrightarrow \mathbb{K}\langle x_1, \ldots, x_n \rangle = \mathcal{O}_{n,0}$$

are exactly the substitution *homomorphisms*

$$g \longmapsto g(f_1(x), \ldots, f_m(x))\,, \quad f_1, \ldots, f_m \in \mathfrak{m}(\mathcal{O}_{n,0})\,.$$

Proof. Substitution homomorphisms are $\mathbb{K}$–algebra homomorphisms. Let, conversely, $\varphi : \mathcal{O}_{m,0} \to \mathcal{O}_{n,0}$ be given and $f_j := \varphi(y_j) \in \mathfrak{m}(\mathcal{O}_{n,0})$, $\widetilde{\varphi}$ the substitution homomorphism $g \mapsto g(f_1, \ldots, f_m)$. Then, $\widetilde{\varphi}(y_j) = \varphi(y_j)$, $j = 1, \ldots, m$, and hence, because of Theorem 23, $\widetilde{\varphi} = \varphi$. □

Remark. It is obvious that the proof of the Corollary also works in the formal case. In the $\mathcal{C}^\infty$–category, it is *false*; counterexamples are, however, not easy to construct. C.f. K. Reichard, *Nichtdifferenzierbare Morphismen differenzierbarer Räume*. Manuscripta Math. 15, 243–250 (1975).

2.8 Rückert's Nullstellensatz

Given finitely many holomorphic functions $f_1, \ldots, f_m \in \mathcal{O}(U)$ on some open subset U of a complex affine space $\mathbb{A}^n$, we denote by $N(f_1, \ldots, f_m)$ the *simultaneous zero set* of the functions $f_1, \ldots, f_m$:

$$N(f_1, \ldots, f_m) = \{ x \in U : f_1(x) = \cdots = f_m(x) = 0 \} .$$

Definition. An *analytic subset* A of an open set U is by definition locally (with respect to any point $x^{(0)} \in U$) the simultaneous zero set of finitely many holomorphic functions. This is the same as to say that A is *closed* in U and that for all $x^{(0)} \in A$ there exists a neighborhood $V \subset U$ of $x^{(0)}$ and functions $g_1, \ldots, g_r \in \mathcal{O}(V)$ such that

$$A \cap V = N(g_1, \ldots, g_r) .$$

In particular, *algebraic sets* are analytic subsets of number space $\mathbb{C}^n$ itself.

If $f_1, \ldots, f_m$ are just *germs* of holomorphic functions at the origin, i.e. elements of $\mathcal{O}_{\mathbb{C}^n,0} = R_n$, they define at least a *germ* of an analytic set at 0 which we denote by the same symbol $N(f_1, \ldots, f_m)$ as above if there is no risk of misunderstanding. (Germs of *sets* at a point of a topological space are defined precisely in the same manner as we introduced germs of functions and maps). If $g \in R_n$ is an element such that a power g^t is contained in the ideal generated by $f_1, \ldots, f_m$, then

$$N(f_1, \ldots, f_m) \subset N(g^t) = N(g) .$$

This remark implies the following statements:

a) $N(f_1, \ldots, f_m)$ depends *only* on the *ideal* $\mathfrak{a} \subset R_n$ generated by the elements $f_1, \ldots, f_m$ such that we write $N(\mathfrak{a})$ instead in the following;

b) If $\mathfrak{b} \subset \mathfrak{a}$, then $N(\mathfrak{a}) \subset N(\mathfrak{b})$;

c) $N(\mathfrak{a}) = N(\operatorname{rad} \mathfrak{a})$, where $\operatorname{rad} \mathfrak{a}$ denotes the ideal

$$\operatorname{rad} \mathfrak{a} = \{ g \in R_n : \exists\, t \text{ with } g^t \in \mathfrak{a} \}$$

which contains $\mathfrak{a}$.

It is also easily checked that

d) $N(\mathfrak{a} + \mathfrak{b}) = N(\mathfrak{a}) \cap N(\mathfrak{b})$;

e) $N(\mathfrak{a} \cap \mathfrak{b}) = N(\mathfrak{a} \cdot \mathfrak{b}) = N(\mathfrak{a}) \cup N(\mathfrak{b})$.

Only part e) needs some hints. It is clear that $N(\mathfrak{a}) \cup N(\mathfrak{b}) \subset N(\mathfrak{a} \cap \mathfrak{b}) \subset N(\mathfrak{a} \cdot \mathfrak{b})$, where the last inclusion is a consequence of the inclusion of ideals $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$. If a point x does not belong to $N(\mathfrak{a}) \cup N(\mathfrak{b})$, then there exist germs $f \in \mathfrak{a}$, $g \in \mathfrak{b}$ which do not vanish at x. Thus, $(f \cdot g)(x) \neq 0$. □

If, on the other hand, A is a germ of any subset of $\mathbb{C}^n$ at 0, we define

$$\mathfrak{i}(A) = \{ f \in R_n : \text{a representative of } f \text{ vanishes on a representative of } A \} .$$

Plainly, $\mathfrak{i}(A)$ is an *ideal* in R_n having the following properties:

f) $A \subset B \Longrightarrow \mathfrak{i}(B) \subset \mathfrak{i}(A)$;

g) $\mathfrak{i}(A \cup B) = \mathfrak{i}(A) \cap \mathfrak{i}(B)$;

h) $\mathfrak{a} \subset \mathfrak{i}(N(\mathfrak{a}))$.

By the preceding properties the inclusions

$$\mathfrak{a} \subset \operatorname{rad}\mathfrak{a} \subset \mathfrak{i}(N(\operatorname{rad}\mathfrak{a})) = \mathfrak{i}(N(\mathfrak{a}))$$

are obvious. In fact, we have

***Theorem 2.25 (Rückert's Nullstellensatz)** *For any ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^n,0}$ we have equality*

$$\mathfrak{i}(N(\mathfrak{a})) = \operatorname{rad}\mathfrak{a}\,.$$

Remark. This Theorem is sometimes called the *Nullstellensatz* of Hilbert - Rückert. The reason for this misuse of historical truth is the fact that Hilbert proved (earlier) the corresponding result for ideals $\mathfrak{a}$ in the *polynomial* ring $\mathbb{C}[x_1, \ldots, x_n]$.

Important Note. Our main concern in this manuscript is to study certain analytic sets $X = N(g_1, \ldots, g_r) \subset U$ *locally* at a fixed point $x \in X$, i.e. to study the germ $(X, x) := X_x = (N(g_1, \ldots, g_r))_x$ which is obviously the same as $N(\mathfrak{a}_x)$ with the ideal $\mathfrak{a}_x$ generated by the germs $g_{1,x}, \ldots, g_{r,x}$. However, if $f_1, \ldots, f_m$ is another set of analytic functions in a neighbourhood of x whose germs at x generate the ideal $\mathfrak{a}_x$, then also $(X, x) = (N(f_1, \ldots, f_m))_x = N(f_{1,x}, \ldots, f_{m,x})$. Hence, there exists a neighborhood $V \subset U$ of x such that

$$X \cap V = \{ x \in V : f_1(x) = \cdots = f_m(x) = 0 \}\,.$$

In other words: *With regard to the analytic sets, one may* replace *the functions* $g_1, \ldots, g_r$ *locally around* x *by the functions* $f_1, \ldots, f_m$. But much more is true: *Even the two* ideal sheaves *generated by these systems of functions are* identical *locally around the point* x. This is a special instance of what we call the *permanence principle* for finitely generated sheaves of modules (see the Supplement).

2.9 Germs of maps

Any (substitution) homomorphism $\psi : R_m \to R_n$ is uniquely determined by m germs $f_1, \ldots, f_m \in \mathfrak{m}_n$ which define a map germ

$$f = (f_1, \ldots, f_m) : (\mathbb{K}^n)_0 \longrightarrow (\mathbb{K}^m)_0\,.$$

If we have a commutative diagram

$$\begin{tikzcd} R_m \arrow[r, "\psi"] \arrow[d] & R_n \arrow[d] \\ R_m/\mathfrak{b} = B \arrow[r, "\varphi"'] & A = R_n/\mathfrak{a} \end{tikzcd}$$

then we have the following result.

Lemma 2.26 *The map germ $f : (\mathbb{K}^n)_0 \to (\mathbb{K}^m)_0$ associated to ψ induces a map from $N(\mathfrak{a})$ to $N(\mathfrak{b})$ which only depends on φ. If φ is an isomorphism, then the induced map is a homeomorphism with respect to the relative topology on $N(\mathfrak{a})$ and $N(\mathfrak{b})$.*

Proof. Take a point x in (a representative of) $N(\mathfrak{a})$ (for the sake of brevity we do not distinguish between the representative and its germ). By definition, $g(x) = 0$ for all $g \in \mathfrak{a}$. Commutativity of the diagram above is the same as to say that $\psi(\mathfrak{b}) \subset \mathfrak{a}$, or, in other terms, if $h \in \mathfrak{b}$, then

$$g := h \circ f \in \mathfrak{a} .$$

Hence, $h(f(x)) = g(x) = 0$, i.e. $f(x) \in N(\mathfrak{b})$. If $\widetilde{\psi}$ is another extension inducing a map germ $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_m)$, then $\widetilde{\psi}(h) - \psi(h) \in \mathfrak{a}$ for all $h \in R_m$. In particular, $f_j - \widetilde{f}_j \in \mathfrak{a}$ which implies $f(x) = \widetilde{f}(x)$ for $x \in N(\mathfrak{a})$.

If φ is an isomorphism, the inverse homomorphism induces a map germ $F : (\mathbb{K}^m)_0 \to (\mathbb{K}^n)_0$ such that, according to what we have just seen,

$$(F \circ f)_{|N(\mathfrak{a})} = \mathrm{id}_{N(\mathfrak{a})} \quad \text{and} \quad (f \circ F)_{|N(\mathfrak{b})} = \mathrm{id}_{N(\mathfrak{b})} . \qquad \square$$

2.10 Several notions of singular points

We want to address now once more the problematic nature of the notion of a *singularity*. In fact, we consider in this manuscript at least two different kinds of *singular* objects.

1) At the one hand we regard (differentiable, analytic) germs of *maps*

$$f_0 : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^m, 0)$$

and call these *singular* if

$$r := \operatorname{rank} Df_0(0) < \min(n, m) .$$

In the *regular*, i.e. the nonsingular, case *all* fibers $f^{-1}(y)$, $y \in V$, are for a suitable representative

$$f := (f_1, \ldots, f_m) : U = U(0) \longrightarrow V = V(0) , \quad U \subset \mathbb{K}^n , \; V \subset \mathbb{K}^m$$

submanifolds of $U \subset \mathbb{K}^n$ of the (with regard to the known situation for linear maps in *Linear Algebra*) *expected* (fixed) dimension, i.e.

a) of dimension 0 if $r = n \le m$,

b) of dimension $n - m$ if $r = m \le n$, resp..

More precisely, one finds after differentiable resp. analytic coordinate change (local) *normal forms* in these cases, as in Linear Algebra, namely

a) $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$ if $r = n \le m$,

b) $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ if $r = m \le n$.

Type a) is called an *immersion*, type b) a *submersion*.

In particular we can state: If $f = (f_1, \ldots, f_m)$ is the germ of a *regular* map then the algebra

$$R_n / (f_1, \ldots, f_m) R_n \cong R_\ell , \quad \ell = \max(n - m, 0) .$$

Here, the symbol R_n etc. stands, according to the situation, for $\mathcal{O}_{n,0}$ or $\mathcal{C}^\infty_{n,0}$.

Remark. If the *map* f is *singular* in 0 then the *fiber* $f^{-1}(0)$ may, nevertheless, be *smooth*. This is, due to the so called *Rank Theorem*, the case e.g. if the rank of f is in a fixed neighborhood of the point 0 constant and smaller than $\min(m, n)$ (see Chapter 3).

In general, a *singularity* in this context means, more exactly, an *equivalence class* of germs of maps with respect to a suitable equivalence relation. Such are e.g. the notions of R–equivalence (see below for functions) and RL–equivalence (as above), hence equality of germs of maps up to composition with germs of automorphisms from right or from right and left.

On the other hand, m germs of functions $f = (f_1, \ldots, f_m) \in \mathfrak{m}_n R_n^{\oplus m}$ also determine:

2) an *ideal* $\mathcal{I}_f = \displaystyle\sum_{j=1}^{m} R_n f_j \subset \mathfrak{m}_n \subset R_n$;

3) a local $\mathbb{K}$–*algebra* $R_n / \mathcal{I}_f$;

4) a *zero locus germ* $N = N(\mathcal{I}_f) \subset (\mathbb{K}^n, 0)$.

A *singularity* is in all these cases again an equivalence class of such (singular) germs with respect to a certain *equivalence relation*. In the last cases these are more or less "geometrically" self–evident:

Definition.

i) The ideals $\mathcal{I}_f, \mathcal{I}_g \subset \mathfrak{m}_n$ are called *embedded isomorphic* if it exists a germ $\varphi \in \operatorname{Aut} R_n$ with $\varphi(\mathcal{I}_f) = \mathcal{I}_g$.

ii) The rings $R_n / \mathcal{I}_f$ and $R_\ell / \mathcal{I}_g$ are called *abstract isomorphic* if there exist substitution homomorphisms
$$\varphi : R_n \longrightarrow R_\ell \, , \quad \psi : R_\ell \longrightarrow R_n$$
which induce local $\mathbb{K}$–algebra homomorphisms
$$\overline{\varphi} : R_n / \mathcal{I}_f \longrightarrow R_\ell / \mathcal{I}_g \, , \quad \overline{\psi} : R_\ell / \mathcal{I}_g \longrightarrow R_n / \mathcal{I}_f \, .$$
that are inverse to each other.

iii) The germs of sets $N, \ M \subset (\mathbb{K}^n, 0)$ are called *geometrically embedded isomorphic* if it exists a germ of an automorphism $\Phi \in \operatorname{Aut}(\mathbb{K}^n, 0)$ with $\Phi(N) = M$.

Remarks. 1. If $\mathcal{I}_f , \ \mathcal{I}_g \subset \mathfrak{m}_n$ are embedded isomorphic then the automorphism $\varphi : R_n \to R_n$ with $\varphi(\mathcal{I}_f) = \mathcal{I}_g$ induces an *abstract* isomorphism
$$\overline{\varphi} : R_n / \mathcal{I}_f \xrightarrow{\sim} R_n / \mathcal{I}_g \, .$$

2. If $R_n / \mathcal{I}_f$ and $R_n / \mathcal{I}_g$ (notice that $n = \ell$!) are abstract isomorphic and $\mathcal{I}_f$ and $\mathcal{I}_g \subset \mathfrak{m}_n^2$ then $\mathcal{I}_f$ and $\mathcal{I}_g$ are embedded isomorphic. For, let $\varphi : R_n \to R_n$ be a lifting of the isomorphism $\overline{\varphi} : R_n / \mathcal{I}_f \longrightarrow R_n / \mathcal{I}_g$ then the induced map

$$\begin{tikzcd} \mathfrak{m}_n / \mathcal{I}_f \arrow[r] \arrow[d] & \mathfrak{m}_n / \mathcal{I}_g \arrow[d] \\ \mathfrak{m}_n / \mathfrak{m}_n^2 \arrow[r] & \mathfrak{m}_n / \mathfrak{m}_n^2 \end{tikzcd}$$

in the lower row is surjective, hence an isomorphism. Then, due to the *Inverse Mapping Theorem*, φ is an isomorphism, too. Since $\overline{\varphi^{-1}} = \overline{\varphi}^{-1}$, one concludes with $\varphi(\mathcal{I}_f) \subset \mathcal{I}_g$, $\varphi^{-1}(\mathcal{I}_g) \subset \mathcal{I}_f$ immediately that $\varphi(\mathcal{I}_f) = \mathcal{I}_g$.

3. If $\mathcal{I}_f, \mathcal{I}_g \subset \mathfrak{m}_n$ are embedded isomorphic then the zero sets $N = N(\mathcal{I}_f)$ and $M = N(\mathcal{I}_g)$ are geometrically embedded isomorphic since every automorphism of R_n induces an automorphism of the germ $(\mathbb{K}^n, 0)$.

4. The converse to 3. is, of course, not valid: for $f = f(x) = x^2$ and $g = g(x) = x^3$ in R_1 the zero sets $N(\mathcal{I}_f) = N(\mathcal{I}_g) = \{ 0 \}$ but $\mathcal{I}_f$ and $\mathcal{I}_g$ are not embedded isomorphic since $\varphi(\mathfrak{m}_n^\ell) = \mathfrak{m}_n^\ell$ for

all $\ell \in \mathbb{N}$, $\varphi \in \operatorname{Aut} R_n$. In the complex analytic case it follows however by the HILBERT - RÜCKERT *Nullstellensatz* that at least the *radicals* of the ideals

$$\operatorname{rad} \mathcal{I}_f \ \text{ and } \ \operatorname{rad} \mathcal{I}_g$$

are embedded isomorphic. In particular, if $N(\mathfrak{a})$ is a point, then $N(\mathfrak{a}) = N(\mathfrak{m})$ and therefore $\mathfrak{m}_n^\ell \subset \mathfrak{a}$, i.e. $\mathfrak{a}$ is of finite codimension (Theorem 13). This is the background for the truth of Corollary 34.

We introduce still another equivalence relation.

Definition. Two m–tuples $f, g \in \mathfrak{m}_n R_n^{\oplus m}$ are called V–*equivalent* if there exists an invertible $m \times m$–matrix M with entries in R_n such that

$$g = M f ,$$

where f and g have to be regarded as column vectors.

Theorem 2.27 *Let $f, g \in \mathfrak{m}_n R_n^{\oplus m}$ be given with the corresponding ideals $\mathcal{I}_f$, $\mathcal{I}_g \subset \mathfrak{m}_n$. Then the following are equivalent*:

i) *f and g are V–equivalent as map germs* $(\mathbb{K}^n, 0) \to (\mathbb{K}^m, 0)$;

ii) *$\mathcal{I}_f$ and $\mathcal{I}_g$ are embedded isomorphic as ideals in R_n*.

Remark. Hence, f and g determine the same *variety* up to isomorphism. This is the reason why the corresponding equivalence relation is termed "V–equivalence" in [1 - 16].

Proof. i) $\Longrightarrow$ ii) is easy to see.

ii) $\Longrightarrow$ i). After the application of an automorphism from the right we may assume that $\mathcal{I}_f = \mathcal{I}_g$. Then we find $m \times m$–matrices A and B with entries in R_n such that (f and g considered as column vectors)

$$g = A f , \quad f = B g .$$

We have to modify A in such a way that $A_0 := A(0)$ is invertible. First, look at an arbitrary matrix M of the form

$$M = C(E_m - BA) + A , \quad C \in M(m \times m, \mathbb{K}) ;$$

then we have

$$M f = A f = g \ \text{ and } \ M_0 = C(E_m - B_0 A_0) + A_0 .$$

Hence, it suffices to establish the following Lemma from Linear Algebra.

Lemma 2.28 *Let $A, B \in M(m \times m, \mathbb{K})$ be arbitrary matrices. Then, there exists an $m \times m$–matrix C such that*

$$M := C(E_m - BA) + A$$

is invertible.

Proof. Decompose $V = \mathbb{K}^m$ into

$$V = V_0 \oplus \ker A = V_1 \oplus \operatorname{im} A .$$

Then, there exists an endomorphism C of V with $C_{|V_0} = 0$ such that $C : \ker A \to V_1$ is an isomorphism. Choose such a C in the formula above; if then $M v = 0$ we have

$$Av = -C(E_m - BA) v \in \operatorname{im} A \cap \operatorname{im} C = \{0\} ,$$

hence $A v = C v = 0$, i.e. $v \in \ker A \cap \ker C = \{0\}$. □

In the literature one uses instead of "V–equivalence" also the notion of *contact–equivalence*. We want to expound the reason for this notion at this place a little further. Let two pairs of (germs of) submanifolds

$$(X_1, Y_1) \ \text{ and } \ (X_2, Y_2) \ \text{ in } \ (\mathbb{K}^N, 0)$$

be given. The pairs are called *contact–equivalent* if there exists an automorphism φ of $(\mathbb{K}^N, 0)$ satisfying $\varphi(X_1) = X_2$, $\varphi(Y_1) = Y_2$. Let now $h_1 : (\mathbb{K}^{n_1}, 0) \to (\mathbb{K}^N, 0)$ be a parametrization of X_1, in particular $n_1 := \dim X_1$, and $g_1 : (\mathbb{K}^N, 0) \to (\mathbb{K}^{m_1}, 0)$ a submersion with $g_1^{-1}(0) = Y_1$, in particular $m_1 = \mathrm{codim}_{\mathbb{K}^N} Y_1$. We further put $f_1 = g_1 \circ h_1$ and construct f_2 correspondingly for the pair (X_2, Y_2). Clearly, the map germs f_j are not uniquely determined by the pairs (X_j, Y_j), but the integers n_j and m_j are. - Then, the following holds true:

Theorem 2.29 *The pairs (X_1, Y_1) and (X_2, Y_2) are contact–equivalent if and only if $n_1 = n_2 =: n$, $m_1 = m_2 =: m$ and the map germs*

$$f_1, f_2 : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^m, 0)$$

are V–equivalent.

For the *Proof* see J. A. Montaldi: *On contact between submanifolds*. Michigan Math. J. 33 (1986), pp. 195–199.

2.11 Right equivalent germs

We denote in the following by R_n either of the local rings $\mathcal{C}^\infty_{n,0}$ or $\mathcal{O}_{n,0}$ with the maximal ideal $\mathfrak{m} = \mathfrak{m}_n$. Correspondingly, $\mathrm{Aut}\, R_n$ stands for the group of *germs* of *invertible* differentiable resp. analytic maps $\varphi : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ (with respect to the composition of (germs of) maps as group multiplication). An element $\varphi \in \mathrm{Aut}\, R_n$ determines in a unique fashion n germs of functions $h_1, \ldots, h_n \in \mathfrak{m}_n$ satisfying $Dh(0) = \det \partial(h_1, \ldots, h_n)/\partial(x_1, \ldots, x_n)(0) \neq 0$ where

$$\varphi(f) = f \circ \varphi = f(h_1, \ldots, h_n).$$

Thus, the group $\mathrm{Aut}\, R_n$ operates also on R_n by composition from the right:

$$\begin{cases} R_n \times \mathrm{Aut}\, R_n \longrightarrow R_n \\ (f \ , \ \varphi) \longmapsto f \circ \varphi . \end{cases}$$

We are interested, as we already stated in Chapter 1.7 for the complex analytic category, in the *orbits* in R_n under this operation.

Definition. Two germs $f, g \in R_n$ are called *right equivalent* or R–*equivalent* for short if they lie in the same orbit with respect to this $\mathrm{Aut}\, R_n$–action, i.e. if there exists an automorphism germ φ with $f \circ \varphi = g$. We write under these circumstances $f \sim_r g$.

Remarks. 1. If $f \circ \varphi = g$, $f(x) = 0$ and $x = \varphi(y)$ then $g(y) = (f \circ \varphi)(y) = f(x) = 0$ and vice versa. Therefore,

$$\varphi(\{ y : g(y) = 0 \}) = \{ x : f(x) = 0 \}$$

near $x = 0$, $y = 0$; in other words: the germs of the zero sets of R–equivalent function germs f and g coincide near 0 after coordinate change.

2. R_n is *not* a finite dimensional manifold and $\mathrm{Aut}\, R_n$ is *not* a Lie group. Nevertheless, it is possible by reducing the considerations to finite dimensional subspaces that classical results on operations of Lie groups on manifolds are applicable.

2.12 Jets of functions and finitely determined germs

We have previously seen that certain germs of *holomorphic* functions are equivalent to a finite part of their Taylor series expansion. We would like to study this phenomenon here in more detail, also for *differentiable* functions.

Definition. A germ $f \in R_n$ is called *k–(right) determined* if it is right equivalent to each germ g which has the same k–jet as f. In symbols: $j^k g = j^k f \Longrightarrow g \sim_r f$. Here, of course,

$$j^k f = j_0^k f := \sum_{|\nu| \le k} \frac{D^\nu f(0)}{\nu!} x^\nu .$$

In particular, f is then equivalent to its k–jet $j^k f$; moreover – in the holomorphic case – the *complex analytic* hypersurface $N(f)$ is (after *holomorphic* base change) equal to the *algebraic* set $N(j^k f)$. We say that f is *finitely determined* if it is k–determined for some $k \in \mathbb{N}$.

Remark. Since $j^k(j^k f) = j^k f$, k–determinacy of f is rather a property of the *polynomial* $j^k f$ than of the *germ* f itself.

Examples. The Implicit Function Theorem in the form of Theorem 1.4 asserts that every regular germ $f \in \mathfrak{m}_n$ is 1–determined. The Morse Lemma is equivalent to the statement that every nondegenerate critical germ $f \in \mathfrak{m}_n^2$ is in fact 2–determined. Both results follow from a general criterion which we formulate in the next Section.

Warning. The example

$$f(x, y) = \left(x + \frac{y}{1 - y} \right)^2$$

shows that the jets $j^k f$ may be finitely determined for $k \ge k_0$ without f being finitely determined.

2.13 The Mather - Tougeron criterion

We shall see that the property of k–determinacy is closely related to the following inclusion which, after all, only concerns finite dimensional vector spaces:

$$(\,*\,)_k \qquad \mathfrak{m}_n^k \subset \mathfrak{m}_n J_f + \mathfrak{m}_n^{k+1} .$$

Here, J_f denotes the Jacobi *ideal* of f which, by definition, is generated by the (germs of) partial derivatives of f:

$$J_f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) R_n = \left\{ \sum_{j=1}^n g_j \frac{\partial f}{\partial x_j} : g_j \in R_n , \quad j = 1, \ldots, n \right\} .$$

To be precise, the following Theorem of Mather holds true which we shall deduce in the course of the present and the next Section.

Theorem 2.30 (Mather)

a) *If $(\,*\,)_k$ holds then f is k–determined.*

b) *If conversely f is k–determined then $(\,*\,)_{k+1}$ holds.*

Let us first test the potential of part a) by elaborating some examples; after that we shall give different versions for the condition $(\,*\,)_k$ before we prove a). Part b) will be proven in the Appendix.

As a *Corollary*, we get another proof for Theorem 1.4 : If 0 is a regular point of $f \in R_n$ then at least one of the germs $\partial f / \partial x_j$ is a unit in R_n. Hence, $J_f = R_n$ and $\mathfrak{m}_n \cdot R_n = \mathfrak{m}_n \cdot J_f$. Therefore, f is 1–determined.

To prove the Morse *Lemma* by means of the criterion above, take a nondegenerate critical point $x^{(0)} = 0$ of f and choose local coordinates such that

$$j^2 f = \sum_{j=1}^n \pm x_j^2$$

(see Chapter 1.6). Then $\partial f / \partial x_j = \pm 2\, x_j + g_j$ with $g_j \in \mathfrak{m}_n^2$, and therefore

$$\mathfrak{m}_n^2 \subset \mathfrak{m}_n J_f + \mathfrak{m}_n^3 .$$

Hence, f is 2–determined.

We give two more

Examples. 1. $f(x_1, x_2) = x_1^2 + x_2^{k+1}$, $k \geq 1$. Here, J_f is generated by x_1 and x_2^k, and we have $\mathfrak{m}_2^{k+1} \subset \mathfrak{m}_2 J_f$; thus f is $(k+1)$–determined. This is more generally true for the A_k–*singularities*

$$f(x_1, \ldots, x_n) = x_1^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^{k+1} , \quad k \geq 1 .$$

2. In the case of the D_k–*singularity* with equation $f(x_1, x_2) = x_1^2 x_2 + x_2^{k-1}$, $k \geq 4$, we find

$$J_f = (x_1 x_2, x_1^2 + (k-1)\, x_2^{k-2})\, R_2 .$$

Because of

$$(k-1)\, x_2^{k-1} = x_2\, (x_1^2 + (k-1)\, x_2^{k-2}) - x_1\, (x_1 x_2)$$

and

$$x_1^{k-1} = x_1^{k-3}(x_1^2 + (k-1)\, x_2^{k-2}) - (k-1)\, x_1^{k-4}\, x_2^{k-3}\, (x_1 x_2) ,$$

we see that

$$\mathfrak{m}_2^{k-1} \subset \mathfrak{m}_2 \cdot J_f .$$

3. The germ $x^3 + xy^4 + y^6$ is 6–determined.

More examples will be discussed later. - We are next going to formulate another version of the Theorem of Mather which partly goes back to Tougeron.

Theorem 2.31 *Given* $f \in R_n$ *then the following statements are equivalent*:

i) f *is* finitely *determined, i.e.* ℓ*–determined for a certain number* $\ell \in \mathbb{N}$;

ii) *there exists a number* k *with* $\mathfrak{m}_n^k \subset \mathfrak{m}_n J_f$;

iii) *the ideal* $\mathfrak{m}_n J_f$ *is of finite codimension in* R_n ;

iv) *the ideal* J_f *is of finite codimension in* R_n .

f *is in the case* ii) k*–determined and in the case* iii) ℓ*–determined if* ℓ *denotes the* $\mathbb{K}$*–dimension of* $R_n / \mathfrak{m}_n J_f$.

Proof. $\mathfrak{m}_n$ is a finitely generated ideal. Henceforth, because of the Nakayama Lemma, the assumption ii) is equivalent to $(\,*\,)_k$. The equivalence of i), ii) and iii) then follows from Mather's Theorem and Theorem 13. Since $\mathfrak{m}_n J_f \subset J_f$, iii) implies the statement iv). Let, on the other hand, iv) be satisfied. Then there exist elements $f_1, \ldots, f_r \in R_n$ such that each function germ $h \in R_n$ has a decomposition:

$$h = \sum_{\rho=1}^{r} c_\rho f_\rho + \sum_{j=1}^{n} h_j \, \frac{\partial f}{\partial x_j}$$

with $c_\rho \in \mathbb{K}$ and $h_j \in R_n$. If one decomposes the h_j into their constant part and their part in the maximal ideal, one deduces at once that the quotient $R_n / \mathfrak{m}_n J_f$ is at most of $\mathbb{K}$–dimension $r + n$. □

Remark. It is possible to formalize the last argument in the proof above a little further: From $R_n = \mathbb{K} \oplus \mathfrak{m}_n$ it follows that

$$\sum_{j=1}^{n} \mathbb{K}\, \frac{\partial f}{\partial x_j} + \mathfrak{m}_n J_f = J_f .$$

Hence, $R_n / \mathfrak{m}_n J_f$ is finite dimensional on $\mathbb{K}$ if and only if R_n / J_f is finite dimensional. - The same reasoning yields a supplement to Theorem 13.

Corollary 2.32 *If, under the assumptions of* Theorem 13, *the ideal* $\mathfrak{a}$ *is finitely generated then either of the conditions* i) *and* ii) *is equivalent to the following property*:

iii) *The ideal* $\mathfrak{m}_n \mathfrak{a}$ *is of finite codimension.*

The property iv) in Theorem 31 can be interpreted either algebraically or geometrically. Regard the substitution homomorphism $\varphi : R_n \to R_n$ given by the partial derivatives $\partial f / \partial x_1, \ldots, \partial f / \partial x_n$. The assumption that J_f is of finite codimension in R_n means exactly that the homomorphism φ is *quasi–finite* (cf. Chapter 3). The *Preparation Theorem* is now equivalent to the assertion that quasi–finite homomorphisms are automatically *finite* (see loc. cit.). - Therefore, we may conclude:

Theorem 2.33 (Tougeron) $f \in \mathfrak{m}_n$ *is finitely determined if and only if the map germ* $(Df)_0 : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ *of its differential is finite.*

In the *complex analytic* case this condition has the concrete meaning that the holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ given by the partial derivatives of f (locally around 0) is a *finite branched covering*. In this context, also the Theorem of Mather has a consequence which we at least want to formulate.

Corollary 2.34 *If* 0 *is an isolated critical point of the* complex analytic *function germ* $f \in \mathcal{O}_{n,0}$ *then* f *is finitely determined.*

Proof. By assumption, 0 is an isolated point in

$$\left\{ x \in \mathbb{C}^n : \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\} .$$

Then, by Rückert's Nullstellensatz and Theorem xx, $\mathcal{O}_{n,0} / J_f$ is a finite dimensional $\mathbb{C}$–vector space. □

Remark. The last statement and Corollary 32 are *false* in the *real–analytic* case.

Examples. 1. For $f(x, y) = x^2 + y^2$ we have over the reals $N(f) := \{ (x, y) \in \mathbb{R}^2 : f(x, y) = 0 \} = \{ 0 \}$. But $\mathbb{R}\langle x, y \rangle / f \mathbb{R}\langle x, y \rangle$ is not a finite dimensional $\mathbb{R}$–vector space since otherwise we would have a certain power $\mathfrak{m}_2^{\ell} \subset f R_2^{\mathbb{R}}$, and this then would also hold over the complex numbers $\mathbb{C}$ (f regarded as a polynomial in $\mathbb{C}[x, y]$). Hence, also in the complex case we would have

$$N(f) \subset N(\mathfrak{m}_2^{\ell}) = \{ 0 \} ,$$

which is nonsense because of $N(f) = \{ (x, y) \in \mathbb{C}^2 : y = \pm i\, x \}$!

2. Example 1 can be expanded to a counterexample to the Corollary over the real number field. This time, we put

$$f(x, y) = (x^2 + y^2)^2 .$$

Here, J_f is generated by $x(x^2 + y^2)$ and $y(x^2 + y^2)$, and $N(J_f)$ consists of the origin only. Thus, f has an isolated singular point at 0; f, however, is *not* finitely determined. Otherwise, $\mathfrak{m}_2 J_f$ would be of finite codimension, i.e. $\mathcal{O}_{2,0}^{\mathbb{R}} / \mathfrak{m}_2 J_f$ would be of finite dimension. But now $\mathfrak{m}_2 J_f \subset (x^2 + y^2)\, \mathcal{O}_{2,0}^{\mathbb{R}}$ such that there exists an epimorphism

$$\mathcal{O}_{2,0}^{\mathbb{R}} / \mathfrak{m}_2 J_f \longrightarrow \mathcal{O}_{2,0}^{\mathbb{R}} / (x^2 + y^2)\, \mathcal{O}_{2,0}^{\mathbb{R}}$$

of $\mathbb{R}$–algebras. Consequently, $\dim_{\mathbb{R}} \mathcal{O}_{2,0}^{\mathbb{R}} / (x^2 + y^2)\, \mathcal{O}_{2,0}^{\mathbb{R}} < \infty$ in contradiction to Example 1.

That both claims above are also false in the *differentiable* case is even much easier to see. Look at the germ at the origin of $f(x) = e^{-1/x^2}$, $x \neq 0$, continued by 0 at 0. Then, $N(f) = 0$, but $j_0^{\infty} f = 0$ and f is not finitely determined.

2.14 Proof of part a) in Mather's Theorem

Let us now start with the *proof* of direction a) in the Theorem of Mather. What do we really want to prove? Obviously, the condition of k–determinacy of f can be stated as follows (if $j^k g = j^k f$ then $\alpha = g - f \in \mathfrak{m}_n^{k+1}$):

(+) For all $\alpha \in \mathfrak{m}^{k+1}$, we have $f + \alpha \sim_r f$.

We attempt to prove this by a *homotopy* argument yielding the equivalence

$$f \sim_r f + t\,\alpha$$

for all $t \in [\,0,\,1\,]$. - To obtain this result, we need the following assertion which, for $t = 1$, immediately yields our claim (+).

(++) If $(\,*\,)_k$ is satisfied for f, then there exists to each element $\alpha \in \mathfrak{m}^{k+1}$ a continuous family g_t of automorphisms of R_n, $t \in [\,0,\,1\,]$, with $g_0 = \mathrm{id}$, such that

$$(f + t\,\alpha)\,(g_t) = f\;.$$

The last condition may also be formulated in an *infinitesimal* version which we actually prove first.

Lemma 2.35 *For f and α as above, there exist (differentiable, analytic, holomorphic) functions $w_1, \ldots, w_n$ in a neighborhood of $\{\,0\,\} \times [\,0,\,1\,] \subset \mathbb{K}^n \times \mathbb{K}$ such that*

$$(\times)\qquad \alpha\,(x) + \sum_{j=1}^{n} w_j(x,t)\,\frac{\partial(f + t\,\alpha)}{\partial x_j}\,(x) = 0\;,$$

where the functions w_j vanish for fixed $t \in [\,0,\,1\,]$ to at least second order at $x = 0$.

Proof. Let $\alpha_1, \ldots, \alpha_N$ denote the monomials of degree k in the generators $x_1, \ldots, x_n$ of the maximal ideal $\mathfrak{m}_n$. Because of $(*)_k$, we find germs $h_{\nu j} \in \mathfrak{m}_n$ satisfying

$$\alpha_\nu = \sum_{j=1}^{n} h_{\nu j}\,\frac{\partial f}{\partial x_j}\;,\quad \nu = 1, \ldots, N\;.$$

Hence, we have

$$\alpha_\nu = \sum_{j=1}^{n} h_{\nu j}\,\frac{\partial(f + t\,\alpha)}{\partial x_j} - t\sum_{j=1}^{n} h_{\nu j}\,\frac{\partial \alpha}{\partial x_j}\;.$$

Since $\alpha \in \mathfrak{m}_n^{k+1}$, the last sum also belongs to $\mathfrak{m}_n^{k+1}$, and consequently, we get for some $a_{\nu\mu} \in \mathfrak{m}_n$:

$$(+)\qquad \alpha_\nu = \sum_{j=1}^{n} h_{\nu j}\,\frac{\partial(f + t\,\alpha)}{\partial x_j} - t\sum_{\mu=1}^{N} a_{\nu\mu}\,\alpha_\mu\;,\quad \nu = 1, \ldots, N\,.$$

The matrix $A = (\delta_{\nu\mu} + t\,a_{\nu\mu})$ has determinant equal to 1 for $x = 0$ and all t. Therefore, it exists an $N \times N$–matrix B of functions of the given category in a neighborhood of $\{\,0\,\} \times [\,0,\,1\,]$ such that $B\,A$ is the unit matrix. Setting $B = (b_{\nu\mu})$, we derive from $(+)$ that

$$\alpha_\nu = \sum_{\mu=1}^{N} b_{\nu\mu} \sum_{j=1}^{n} h_{\mu j}\,\frac{\partial(f + t\,\alpha)}{\partial x_j}\;,\quad \nu = 1, \ldots, N\;,$$

and by writing

$$\alpha = \sum_{\nu=1}^{N} \alpha_\nu\,\alpha_\nu\;,\quad \alpha_\nu \in \mathfrak{m}_n\;,$$

we can define the sought–off functions w_j as

$$w_j(x,\,t) \;=\; -\sum_{\nu,\mu=1}^{N} \alpha_\nu(x)\,b_{\nu\mu}(x,\,t)\,h_{\mu j}(x)\;. \qquad \square$$

The proof of part a) in the Theorem of MATHER will be completed by means of the following theorem.

Theorem 2.36 *Suppose, that for the function germs* $f\,,\ \alpha \in R_n$ *the relation* $(\times)$ *holds. Then there exists a* differentiable *family* $g_t \;=\; g_t(x) \in \mathrm{Aut}\,R_n\,,\ t \in [\,0,\,1\,]\,,\ g_0(x) \;=\; x$ *such that*

$$(f \,+\, t\,\alpha)\,(g_t(x)) \;=\; f\,(x)\;.$$

In particular, $f \,+\, \alpha \sim_r f$ *.*

Proof. We look at the following system of ordinary differential equations, depending on the parameters $x_1, \ldots, x_n$:

$$\frac{dg_j(x,\,t)}{dt} \;=\; w_j(g_1(x,\,t), \ldots, g_n(x,\,t),\,t)\;, \quad j \;=\; 1, \ldots, n\;.$$

Obviously, $g_j(0,\,t) \;\equiv\; 0\,,\ j \;=\; 1, \ldots, n$, is a (unique) solution of this system with $g_j(0,\,0) \;=\; 0$ for all j. Therefore, for all x sufficiently close to 0, there exists a unique solution $(g_j(x,\,t))_{j=1,\ldots,n}$ with $g_j(x,\,0) \;=\; x_j$. Moreover, the total solution $(g_j(x,\,t))$ is differentiable (analytic, holomorphic) in x for all $t \in [\,0,\,1\,]$, and therefore, it defines a (germ of a) differentiable (analytic, holomorphic) map g_t of $\mathbb{K}^n$ into itself at the origin for all $t \in [\,0,\,1\,]$. In order to prove that the g_t are local differentiable (analytic, holomorphic) automorphisms, it suffices to compute the Jacobi determinants

$$J\,(t) \;:=\; \det\left.\frac{\partial g\,(x,\,t)}{\partial x}\right|_{x=0}\,, \quad t \in [\,0,\,1\,]\;.$$

Now, due to the vanishing properties of the functions w_j in Lemma 35,

$$\begin{aligned}\frac{d}{dt}\left(\left.\frac{\partial g_j(x,\,t)}{\partial x_k}\right|_{x=0}\right) \;&=\; \frac{\partial}{\partial x_k}\,w_j(g_1(x,\,t), \ldots, g_n(x,\,t),\,t)\Big|_{x=0}\\ &=\; \sum_{l=1}^{n} \frac{\partial w_j}{\partial x_l}(g_1(x,\,t), \ldots, g_n(x,\,t),\,t)\,\left.\frac{\partial g_l(x,\,t)}{\partial x_k}\right|_{x=0} \;=\; 0\;,\end{aligned}$$

and we finally arrive at

$$J\,(t) \;\equiv\; J\,(0) \;=\; 1\;.$$

The *proof* of Mather's Theorem will be finished by checking that

$$(f \,+\, t\,\varphi)(g_t(x)) \;=\; f\,(x)$$

for all x near the origin and all $t \in [\,0,\,1\,]$. We do this for fixed x by differentiating with respect to t (and by observing that the equation is correct for $t \;=\; 0$):

$$\begin{aligned}\frac{d}{dt}(f \,+\, t\,\varphi)\,(g_t(x)) \;&=\; \sum_{j=1}^{n} \frac{\partial(f \,+\, t\,\varphi)}{\partial x_j}(g_t(x))\,\frac{dg_j(x,\,t)}{dt} \,+\, \varphi\,(g_t(x))\\ &=\; \sum_{j=1}^{n} \frac{\partial(f \,+\, t\,\varphi)}{\partial x_j}(g_t(x))\,w_j(g_t(x),\,t) \,+\, \varphi\,(g_t(x))\\ &=\; -\,\varphi\,(g_t(x)) \,+\, \varphi\,(g_t(x)) \;=\; 0\;.\end{aligned}$$

2.15 Mather's Theorem and Nakayama's Lemma

The main purpose of the present Section is to demonstrate that the Nakayama Lemma may be used *directly* to prove the Mather–Tougeron criterion and even more general equivalence criteria. It is readily checked that - by the compactness of the interval $[0, 1] \subset \mathbb{K}$ - we need the functions $w_j(x, t)$ in Lemma 35 only locally with respect to t in order to insure the equivalence of $f + \alpha$ and f. Thus, we get the following result in which $\mathfrak{m}_n$ denotes, as always, the maximal ideal of R_n, and R_{n+1,t_0} the ring of differentiable (analytic, holomorphic) functions germs at $(0, t_0) \in \mathbb{K}^n \times [0, 1] \subset \mathbb{K}^{n+1}$. The ring R_n will be regarded as subring of R_{n+1,t_0}; in particular, $\mathfrak{m}_n \subset \mathfrak{m}_{n+1,t_0} = \mathfrak{m}(R_{n+1,t_0})$.

Theorem 2.37 *Let f and α be elements of $\mathfrak{m}_n$. Then $f + \alpha \sim_r f$ if, for all $t_0 \in [0, 1]$,*

$$\alpha \in \mathfrak{m}_n^2 \cdot \left(\frac{\partial(f + t\alpha)}{\partial x_1}, \dots, \frac{\partial(f + t\alpha)}{\partial x_n} \right) R_{n+1,t_0} .$$

Let us show that this criterion is fulfilled for a germ f satisfying the Mather criterion $(*)_k$ and all elements $\alpha \in \mathfrak{m}_n^{k+1}$. By assumption on f, we have

$$\mathfrak{m}_n^k \subset \mathfrak{m}_n J_f .$$

This implies

$$\mathfrak{m}_n^k R_{n+1,t_0} \subset \mathfrak{m}_n J_f \cdot R_{n+1,t_0} .$$

Since $\partial\alpha/\partial x_j \in \mathfrak{m}_n^k$ for all $\alpha \in \mathfrak{m}_n^{k+1}$, $j = 1, \dots, n$, we see that $t \cdot \partial\alpha/\partial x_j$ is an element of $\mathfrak{m}_n^k \cdot R_{n+1,t_0}$. From this, we deduce that

$$\mathfrak{m}_n^k R_{n+1,t_0} \subset \mathfrak{m}_n \left(\frac{\partial(f + t\alpha)}{\partial x_1}, \dots, \frac{\partial(f + t\alpha)}{\partial x_n} \right) R_{n+1,t_0} + \mathfrak{m}_{n+1} \left(\mathfrak{m}_n^k R_{n+1,t_0} \right) .$$

Since $\mathfrak{m}_n^k R_{n+1,t_0}$ is a finitely generated R_{n+1,t_0}–module, the Nakayama Lemma implies

$$\mathfrak{m}_n^k R_{n+1,t_0} \subset \mathfrak{m}_n \left(\frac{\partial(f + t\alpha)}{\partial x_1}, \dots, \frac{\partial(f + t\alpha)}{\partial x_n} \right) R_{n+1,t_0} . \qquad \square$$

2.A Appendix: Proof of part b) of Mather's Theorem

For the proof of the opposite direction in the Theorem of Mather we need some further notations. In order to avoid superfluous paperwork we think of $n \in \mathbb{N}$ being fixed and set

$$\mathcal{A}_k = \mathbb{K}[x_1, \dots, x_n] / (x_1, \dots, x_n)^{k+1} \mathbb{K}[x_1, \dots, x_n] \cong R_n / \mathfrak{m}_n^{k+1} .$$

We interpret the $\mathbb{K}$–vector space $\mathcal{A}_k$ as the space of *k–jets* in R_n and at the same time as the vector space of polynomials of degree $\le k$. In particular,

$$\dim_{\mathbb{K}} \mathcal{A}_k = \binom{n + k}{k} .$$

There are canonical projections $R_n \to \mathcal{A}_k$ which we denote by $\pi_k : \pi_k(f) = j_0^k f$ modulo $(x_1, \dots, x_n)^{k+1} \mathbb{K}[x_1, \dots, x_n]$. Further, let $\mathcal{B}_k$ be the space of n–tuples in $\mathcal{A}_k$, a vector space of $\mathbb{K}$–dimension $n \binom{n+k}{k}$. For any elements $F = (f_1, \dots, f_n)$, $G = (g_1, \dots, g_n) \in \mathcal{B}_k$, one has a canonical composition:

$$F \circ G = (f_1 \circ G, \dots, f_n \cdot G) \bmod (x_1, \dots, x_n)^{k+1} \in \mathcal{B}_k .$$

Notice that the composition is well defined because of the *chain rule*: Derivatives of $F \circ G$ up to a fixed order k depend in an algebraic way only on such derivatives of F and G. By the same reason, this composition is associative with the neutral element $E = (x_1, \dots, x_n)$. We now regard in $\mathcal{B}_k$ the open dense subset

$$\mathcal{G}_k = \{ F = (f_1, \dots, f_n) : \det DF(0) \ne 0 \} \ni E .$$

To each such F there exists an inverse $\widetilde{G}$, and if one puts $G := \widetilde{G} \bmod (x_1, \dots, x_n)^{k+1}$, one has $F \circ G \equiv E \bmod (x_1, \dots, x_n)^{k+1}$. Hence, $\mathcal{G}_k$ is a group which moreover carries a structure of a manifold, and the compositions

$$\begin{cases} \mathcal{G}_k \times \mathcal{G}_k \longrightarrow \mathcal{G}_k \\ (F\ ,\ G) \longmapsto F \circ G \end{cases} \qquad \begin{cases} \mathcal{G}_k \longrightarrow \mathcal{G}_k \\ F \longmapsto F^{-1} \end{cases}$$

are differentiable (holomorphic, etc.) since the coefficients of $F \circ G$ resp. F^{-1} emerge from those of F and G algebraically (for the inverse this is a consequence of Cramer's rule). In other words: $\mathcal{G}_k$ is a *Lie group* that acts differentiably resp. analytically on the manifold $\mathcal{A}_k$ by insertion from the right. Moreover, it is plain that there is a commutative diagram:

$$\begin{tikzcd} R_n \times \mathrm{Aut}\ R_n \arrow[r] \arrow[d, "\pi_k \times \pi_k"'] & R_n \arrow[d, "\pi_k"] \\ \mathcal{A}_k \times \mathcal{G}_k \arrow[r] & \mathcal{A}_k\ , \end{tikzcd}$$

where we have denoted the projections for the automorphism groups again by π_k.

Let now $\widehat{f} \in \mathcal{A}_k$ be arbitrary and $\widehat{f}\,\mathcal{G}_k$ the orbit of $\widehat{f}$ under the action of $\mathcal{G}_k$ on $\mathcal{A}_k$. This is a (locally closed) submanifold of the vector space $\mathcal{A}_k$ whose tangent space at the place $\widehat{f}$ we want to determine. (In general, orbits are not locally closed submanifolds. However, here we are in a very good situation since we are dealing with an algebraic operation of an algebraic group on an affine space; for more details, cf. [1 - 16]). In particular, the orbit is a *homogeneous manifold* and therefore the tangent space of an orbit is the epimorphic image of the tangent space of the group $\mathcal{G}_k$ at its neutral element, i.e. of the *Lie algebra* of $\mathcal{G}_k$. To compute this, we have to describe all curves in $\mathrm{Aut}\ R_n$ resp. in $\mathcal{G}_k$ which start in $E = \mathrm{id}$. These are germs $\delta : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^n, 0)$ with $\delta(x, 0) = x$ and $\delta(0, t) = 0$, hence

$$\delta(x, t) = x + \varepsilon(x, t)\ , \quad \varepsilon(x, 0) = 0\ , \quad \varepsilon(0, t) = 0\ .$$

The tangent vectors of the orbit are then determined modulo $\mathfrak{m}_n^{k+1}$ by

$$\frac{\partial \widehat{f}}{\partial t}\,(x + \varepsilon(x, t))\Big|_{t=0} = \sum_{j=1}^{n} \frac{\partial \widehat{f}}{\partial x_j} \cdot \frac{\partial \varepsilon_j}{\partial t}\Big|_{t=0}\ .$$

It is obvious that $\partial\varepsilon_j / \partial t$ at $t = 0$ can be an arbitrary element of $\mathfrak{m}_n$, from which it follows that

$$(**) \qquad T_{\widehat{f}}\, \widehat{f}\,\mathcal{G}_k = \mathfrak{m}_n J_{\widehat{f}} \bmod \mathfrak{m}_n^{k+1} \subset \mathcal{A}_k\ .$$

If $\widehat{f} = j^k f$ is the k–jet of $f \in R_n$ then one can replace the right hand side of $(**)$ by $\mathfrak{m}_n J_f \bmod \mathfrak{m}_n^{k+1}$. From this consideration we shall conclude the following theorem which implies assertion b) in the Theorem of Mather.

Theorem 2.38 *Given $f \in R_n$ with a k–determined $(k+1)$–jet $j^{k+1} f$, then the following holds*:

$$(\,*\,)_{k+1} \qquad \mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n J_f\ .$$

Corollary 2.39 *If f is k–determined, then condition $(\,*\,)_{k+1}$ is necessarily satisfied.*

Proof (of Corollary). If f is k–determined, we have $f \sim_r j^{k+1} f$; hence $j^{k+1} f$ is k–determined, too. Thus $(\,*\,)_{k+1}$ holds due to the preceding theorem. □

We state another

Corollary 2.40 *With $j^{k+1}f$ also f is k–determined.*

Proof. Because of the theorem above and part a) of the Theorem of MATHER f is $(k+1)$–determined. If now $j^k g = j^k f = j^k(j^{k+1}f)$, we conclude that $g \sim_r j^{k+1}f \sim_r f$. □

Proof of Theorem 38. We look at

$$X = \{ g \in \mathcal{A}_{k+1} : \pi_k g = j^k f \} \ni j^{k+1} f =: \widehat{f} .$$

This is an affine subspace of $\mathcal{A}_{k+1}$, in particular a submanifold. Its tangent space is (at each place g) equal to the vector space of the *homogeneous* polynomials of degree $k+1$, hence

$$T_g X \cong \mathfrak{m}_n^{k+1} / \mathfrak{m}_n^{k+2} .$$

Now, $\widehat{f}$ is k–determined, i.e. $g \in X$ implies $g \sim_r \widehat{f}$ such that $X \subset \widehat{f} \cdot \mathcal{G}_{k+1}$. Therefore, we have $T_{\widehat{f}} X \subset T_{\widehat{f}} \widehat{f}\, \mathcal{G}_{k+1}$ and thus, because of (∗∗):

$$\mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n J_f + \mathfrak{m}_n^{k+2} .$$ □

Notes and References

The first part of the Chapter is an adaptation from the book [01 - 02]. As a main source for *Commutative Algebra* we recommend the wonderful book of

[02 - 01] D. Eisenbud: Commutative Algebra with a View Toward Algebraic Geometry. Corrected third printing. New York: Springer–Verlag 1999.

Other standard books are:

[02 - 02] M.F. Atiyah, I.G. Macdonald: Introduction to Commutative Algebra. Reading, M.A. : Addison–Wesley 1969.

[02 - 03] N. Bourbaki: Commutative Algebra. Chapters 1–7. New York: Springer–Verlag 1985.

[02 - 04] M. Nagata: Local Rings. New York: Wiley 1962.

[02 - 05] O. Zariski, P. Samuel: Commutative Algebra, Vol.I and II. Van Nostrand: Princeton 1958 and 1960. Reprint in: Graduate Texts in Mathematics 28 and 29, New York–Heidelberg–Berlin: Springer–Verlag 1975 and 1976.

A booklet which promises (correctly) to "cover the basic algebraic tools and results behind the scenes in the foundations of Real and Complex Analytic Geometry" is

[02 - 06] J. M. Ruiz: The Basic Theory of Power Series. Advanced Lectures in Mathematics. Braunschweig/Wiesbaden: Friedr. Vieweg & Sohn 1993.

The Sections on finite determined germs follow closely Chapter 6.4 of [01 - 20]. On p. 122 of this book, the reader will find a discussion of *Whitney's Example*

$$f(x, y, z) = x\,y\,(x+y)\,(x - z\,y)\,(x - e^z y)$$

for which the whole z–axis consists of critical points. f is at 0 not finitely determined.

The use of Nakayama's Lemma for proving the sufficiency of Mather's criterion is taken from [01 - 15].