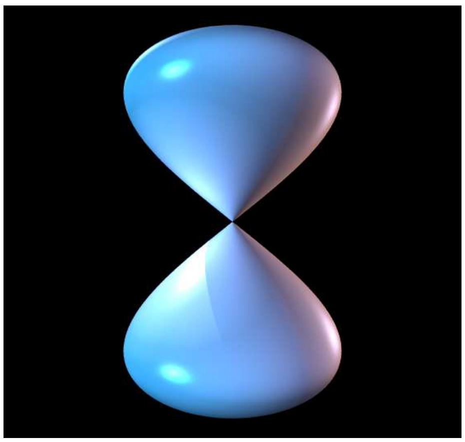
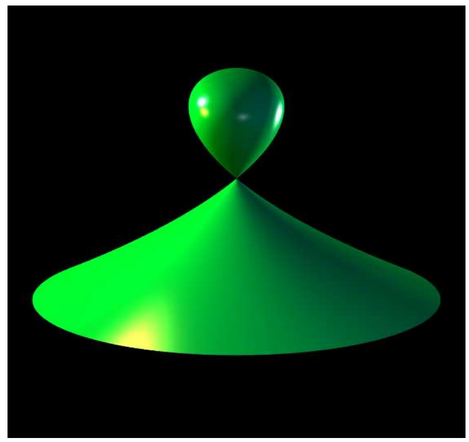
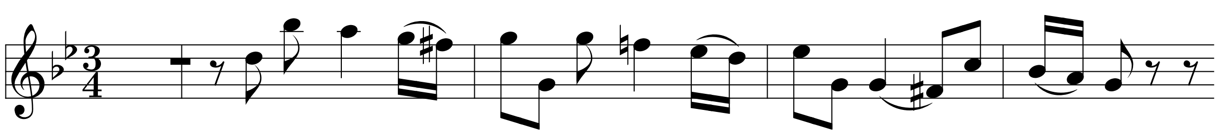
# Chapter 1

*We begin life with a seemingly blank slate - and, though the writing that gradually appears on that slate is not our own, our judgment of the things written thereon determines what we are and what we will become.*

(From the introduction to *The Codex of the Adept Riveda*)







# Chapter 1

# The simple singularity of type $A_1$

In the present Chapter we define the concept of a *hypersurface singularity* and fix various notations which are used throughout the text. We give moreover - by means of a special example - an introduction to some of the themes that are treated in this book.

## 1.1 Complex hyperplanes

A *hyperplane* $H$ in the complex n–dimensional affine space $\mathbb{A}^n$ - which can be identified with the complex vector space

$$\mathbb{C}^n = \{ x = {}^t(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{C} \}$$

of *column vectors* after the choice of a coordinate system - is described by the vanishing of a nontrivial *complex affine function*

$$\ell(x) = a_1 x_1 + \cdots + a_n x_n + \alpha , \quad a = (a_1, \ldots, a_n) \in (\mathbb{C}^n)^* \setminus \{ 0 \} , \quad \alpha \in \mathbb{C} ,$$

where we identify the space of *row vectors* with the *dual vector space* $(\mathbb{C}^n)^*$ of $\mathbb{C}^n$ (for which we sometimes write by abuse of language $\mathbb{C}^n$ as well). If we introduce new affine coordinates on $\mathbb{A}^n$ by

$$y = \Phi(x) := Cx + \gamma ,$$

where $C \in \mathrm{GL}(n, \mathbb{C})$ is an invertible complex $n \times n$ matrix and $\gamma$ is a column vector in $\mathbb{C}^n$, we immediately check that $\ell(x)$ is equal to the function $\tilde{\ell}(\Phi(x))$, where the affine function

$$\tilde{\ell}(y) = b_1 y_1 + \cdots + b_n y_n + \beta$$

is given by

$$b = (b_1, \ldots, b_n) = a \cdot C^{-1} , \quad \beta = \alpha - b \cdot \gamma .$$

Choosing $C$ and $\gamma$ appropriately, we can easily achieve that

$$b = (0, \ldots, 0, 1) \text{ and } \beta = 0$$

such that $H$ is always of the form $\{ y = {}^t(y_1, \ldots, y_n) \in \mathbb{C}^n : y_n = 0 \}$ with respect to some affine coordinate system.

## 1.2 Holomorphic functions

*Local complex analytic geometry* deals with (local) properties of zero sets of *holomorphic functions* instead of affine functions. More precisely, it deals with properties which are invariant under local biholomorphic coordinate transformations.

We collect here a few basic facts about holomorphic functions (for more details see Appendix A to this Chapter). To start with, we formulate

**Lemma 1.1 (Abel's Lemma)** *If a formal power series*

$$p(x) = \sum_{|\nu|=0}^{\infty} a_\nu x^\nu$$

*(where* $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ , $\mathbb{N} = \{ \nu \in \mathbb{Z} : \nu \geq 0 \}$ , $a_\nu = a_{\nu_1 \ldots \nu_n} \in \mathbb{C}$ , $|\nu| = \nu_1 + \cdots + \nu_n$ , $x^\nu = x_1^{\nu_1} \cdot \ldots \cdot x_n^{\nu_n}$ *) satisfies the condition*

$$|\, a_\nu (x^{(0)})^\nu \,| \leq M < \infty$$

*at a fixed point*

$$x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \in (\mathbb{C}^*)^n , \quad \mathbb{C}^* = \mathbb{C} \setminus \{ 0 \} ,$$

*in particular if it converges (absolutely) - with respect to any order of summation - at a point* $x^{(0)}$ *as above, then it converges absolutely and uniformly on each compact polydisk*

$$\overline{P}_r = \{ x \in \mathbb{C}^n : |\, x_j \,| \leq r_j , \; j = 1, \ldots, n \} , \quad 0 < r_j < |\, x_j^{(0)} \,| .$$

*Proof*. By assumption, $|\, a_\nu \rho^\nu \,| \leq M$ for all $\nu \in \mathbb{N}^n$ with a certain positive constant $M$ , where $\rho_j := |\, x_j^{(0)} \,| > 0$ . Take now $0 < r_j < \rho_j$ and $h = (h_1, \ldots, h_n)$ with $|\, h_j \,| \leq r_j$ . Then,

$$|\, a_\nu h^\nu \,| \leq |\, a_\nu \,|\, r^\nu = |\, a_\nu \, \rho^\nu \,| \cdot \left| \frac{r^\nu}{\rho^\nu} \right| \leq M \, \vartheta^\nu ,$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ , $\vartheta_j = r_j / \rho_j < 1$ . Now, the *multiple* geometric series

$$\sum_{|\nu|=0}^{\infty} \vartheta^\nu$$

is absolutely convergent with limit

$$\frac{1}{(1 - \vartheta_1) \cdot \ldots \cdot (1 - \vartheta_n)}$$

such that the series $\sum a_\nu h^\nu$ is *normally* (in particular absolutely and uniformly) convergent in the polydisk with polyradius $r = (r_1, \ldots, r_n)$ . □

A complex–valued function $f$ defined on an open set $U \subset \mathbb{A}^n$ is called *holomorphic*, if for each point $x \in U$ there exists a nonempty open polydisk

$$P_r = \overset{o}{\overline{P}}_r = \{ h \in \mathbb{C}^n : |\, h_j \,| < r_j , \; j = 1, \ldots, n \}$$

and a power series $p(h)$ converging absolutely on $P_r$ such that

$$x + P_r := \{ y \in \mathbb{A}^n : y = x + h , \; h \in P_r \} \subset U$$

and

$$f(x + h) = p(h) \text{ for all } h \in P_r .$$

By Abel's *Lemma*, $p(h)$ converges *absolutely* and *uniformly* on every compact polydisk contained in $P_r$ . It follows as in multidimensional calculus that all complex partial derivatives of first order

$$\frac{\partial f}{\partial x_j}(x) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x + he_j) - f(x)}{h} , \quad x \in U , \quad j = 1, \ldots, n$$

exist (where $e_j = (\delta_{1j}, \ldots, \delta_{nj})$, $\delta_{ij}$ denoting the *Kronecker symbol*) and that the functions $\partial f / \partial x_j$ are holomorphic on $U$. Hence, $f$ has complex derivatives of all orders, and a straightforward generalization of the *Cauchy Integral Formula* in one variable implies that the *Taylor series expansion*

$$T_{f,x}(h) = \sum_{|\nu|=0}^{\infty} \frac{1}{\nu!} \frac{\partial^{|\nu|} f}{\partial x^{\nu}} (x) \, h^{\nu} ,$$

where

$$\nu! = \nu_1! \cdot \ldots \cdot \nu_n! , \quad \frac{\partial^{|\nu|} f}{\partial x^{\nu}} = \frac{\partial^{\nu_1 + \cdots + \nu_n} f}{\partial x_1^{\nu_1} \cdot \ldots \cdot x_n^{\nu_n}} ,$$

converges to $f(x + h)$ for all $x \in U$ and all $h$ contained in the union of the polydisks $P_r$ satisfying $x + P_r \subset U$. (For more details consult Appendix A to this Chapter).

The set of all holomorphic functions on a (nonempty) open set $U$ in $\mathbb{A}^n$ forms a (commutative and associative) $\mathbb{C}$–algebra under the natural addition and multiplication of functions. We denote it by the symbol

$$\mathcal{O}_{\mathbb{C}^n}(U) \text{ or } \mathcal{O}(U) .$$

Since the coordinate functions $x_j$ are holomorphic, we have an inclusion of the polynomial ring in $n$ variables into any ring $\mathcal{O}(U)$ by restriction:

$$\mathbb{C}[x_1, \ldots, x_n] \hookrightarrow \mathcal{O}(U) , \quad P \longmapsto P_{|U} .$$

It is clear from the definition that holomorphy of a function is preserved under affine coordinate transformations.

## 1.3 Complex hypersurfaces

A *(complex analytic) hypersurface* $X$ in an open set $U \subset \mathbb{A}^n$ is locally the set of zeros of a nontrivial holomorphic function: to each point $x^{(0)} \in U$ there exists a connected open neighborhood $V$ of $x^{(0)}$ in $U$ and a holomorphic function $f_V \in \mathcal{O}(V)$, $f_V$ not identically zero, such that

$$X \cap V = \{ x \in V : f_V(x) = 0 \} .$$

Such a hypersurface is necessarily closed in $U$; moreover, it does not contain interior points: otherwise there would exist a connected open set $V \subset U$ and a defining function $f_V \in \mathcal{O}(V)$ for $X \cap V$ which would vanish on a nonempty open subset of $V$ and hence, by the *Identity Theorem* for holomorphic functions (see Appendix A), on all of $V$, contradicting our assumption.

In particular, if $U$ is connected and if $f \in \mathcal{O}(U)$ is not the zero element, then the *zero set* (or *null set*)

$$N(f) := \{ x \in U : f(x) = 0 \}$$

is a hypersurface in $U$. It is empty, if and only if $f$ is a *unit* in $\mathcal{O}(U)$, i.e. if there exists a holomorphic function $g$ on $U$ with $fg \equiv 1$. We always denote the set of units in $\mathcal{O}(U)$ by $\mathcal{O}^*(U)$.

In contrast to the real situation, where such zero sets may consist of isolated points (take e.g. the function $f(x) = x_1^2 + \cdots + x_n^2$ whose zero set

$$N_{\mathbb{R}}(f) = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : f(x) = 0 \}$$

contains the origin in $\mathbb{R}^n$ only), hypersurfaces in complex analytic geometry are complex $(n - 1)$–dimensional objects. This is a consequence of the WEIERSTRASS *Preparation Theorem* which constitutes the fundamental result in the function theory of several complex variables. The proof will be given in Chapter 3.

**Theorem 1.2 (Weierstraß Preparation Theorem)** *Let $f$ be a holomorphic function defined in a neighborhood of the origin $0 \in \mathbb{C}^n$ such that*

$$f(0) = 0, \quad \textit{but } f(0, \ldots, 0, x_n) \not\equiv 0 .$$

*Then there exists a polydisk $P_1$ with $0 \in P_1 \subset \mathbb{C}^{n-1}$, a disk $P_2$ about the origin in $\mathbb{C}$, holomorphic functions $a_1, \ldots, a_b \in \mathcal{O}(P_1)$ vanishing at $0$ and a unit $e \in \mathcal{O}^*(P_1 \times P_2)$ such that*

$$f(x', x_n) = e(x', x_n)\,(x_n^b + a_1(x')\,x_n^{b-1} + \cdots + a_b(x'))$$

*for all $x' = (x_1, \ldots, x_{n-1}) \in P_1$ and all $x_n \in P_2$.*

In fact, under the assumptions of Theorem 2, the projection

$$N(f) = \{(x', x_n) \in P_1 \times P_2 : f(x', x_n) = 0\} \longrightarrow P_1$$

is surjective and has finite fibers, if $P_1$ is chosen small enough, and locally around a point of a hypersurface $X \subset U$ we can find an affine coordinate system and a defining function for $X$ which satisfies the conditions of Theorem 2.

One might visualize the (much more "complex") situation by a "real" picture when restricting the projection to a real line in $P_1$.

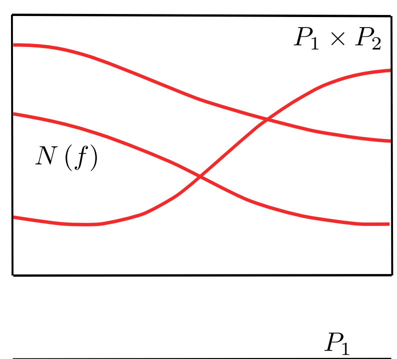



Figure 1.1

Let us remark here, that our notion of hypersurfaces is not interesting in the case of *differentiable* functions. The reason for this claim is the following Theorem of Whitney.

**Theorem 1.3 (Whitney)** *Let $A$ be a closed subset of the open subset $U \subset \mathbb{R}^n$. Then, there exists a function $f \in \mathcal{C}^\infty(U)$, such that $f \geq 0$ and $f(x) = 0$ if and only if $x \in A$.*

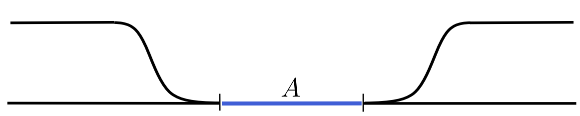



Figure 1.2

The *proof* will be given in Appendix B to this Chapter.

## 1.4 Smooth and singular points of hypersurfaces

A point $x^{(0)}$ of a hypersurface $X$ will be called a *smooth point* or a *regular point* of $X$, if $X$ can be described (locally around $x^{(0)}$) by the vanishing of a holomorphic coordinate function; otherwise it is called a *singular point* or a *singularity* of $X$. We lay particular stress on the remark that this set–theoretical notion of a smooth point is a *preliminary* one. We discuss several other concepts in Chapter 3.7. The refined function–theoretical definition shall be given later in Chapter 7.

To make this definition more precise, let us collect some facts concerning holomorphic and biholomorphic maps. A map $\varphi$ from an open set $U$ in $\mathbb{C}^n$ to $\mathbb{C}^m$ is given by $m$ functions $\varphi_1, \ldots, \varphi_m : U \to \mathbb{C}$. $\varphi$ is a *holomorphic* map, if all the functions $\varphi_k$, $k = 1, \ldots, m$, are holomorphic. If $m = n$, if $V = \varphi(U)$ is open in $\mathbb{C}^n$ and if there exists a holomorphic map $\psi = (\psi_1, \ldots, \psi_n) : V \to U$ with $\psi \circ \varphi = \mathrm{id}_U$, then the *chain rule* implies

$$\sum_{\ell=1}^{n} \frac{\partial \psi_k}{\partial y_\ell}(\varphi(x)) \frac{\partial \varphi_\ell}{\partial x_j}(x) = \delta_{jk}, \quad j, k = 1, \ldots, n,$$

such that necessarily the *complex Jacobi matrix*

$$\frac{\partial \varphi}{\partial x}(x) = \left( \frac{\partial \varphi_\ell}{\partial x_j}(x) \right)_{\substack{\ell=1,\ldots,n \\ j=1,\ldots,n}}$$

must have rank $n$ at each point $x \in U$. The converse follows, at least locally, from the *Inverse Function Theorem*.

***Theorem 1.4** *If $\varphi$ is a holomorphic map from an open set $W$ in $\mathbb{C}^n$ into $\mathbb{C}^n$, and if at $x^{(0)} \in W$ the* Jacobi determinant

$$\det \frac{\partial \varphi}{\partial x}(x^{(0)})$$

*does not vanish, then $\varphi$ is a biholomorphic map (locally around $x^{(0)}$). I.e. there exist open neighborhoods $U$ of $x^{(0)}$ (contained in $W$) and $V$ of $\varphi(x^{(0)})$ such that $\varphi : U \to V$ is bijective and the inverse $\psi = \varphi^{-1}$ is also holomorphic.*

There is now a simple criterion for a point of a hypersurface to be smooth in the sense introduced above:

**Theorem 1.5** *Let the origin $0$ be a point of the hypersurface $X = N(f)$, $f \in \mathcal{O}(U)$, $0 \in U \subset \mathbb{C}^n$. Then $0$ is a smooth point of $X$ if, and only if, the function $f$ can be written in a neighborhood of $0$ in the form*

$$f = g^b$$

*where the linear term of the Taylor series expansion of $g$ at $0$ is nontrivial.*

*Proof*. a) Let $0 \in X$ be a smooth point. By definition, there exist local holomorphic coordinates $y = \varphi(x)$ near the origin such that in some neighborhood $V$ of $\varphi(0) = 0$:

$$(*) \qquad \varphi(X) \cap V = \{ y \in V : f \circ \varphi^{-1}(y) = 0 \} = \{ y \in V : y_n = 0 \}.$$

Therefore, $f \circ \varphi^{-1}(0, \ldots, 0, y_n) = 0$ if and only if $y_n = 0$. By the Weierstraß Preparation Theorem, we have

$$f \circ \varphi^{-1}(y) = e(y) \cdot \omega(y)$$

with a unit $e$ and a Weierstraß polynomial

$$\omega(y) = y_n^b + a_1(y')\, y_n^{b-1} + \cdots + a_b(y'), \quad y' = (y_1, \ldots, y_{n-1}).$$

But then $(*)$ can only hold, if $\omega(y) = y_n^b$; hence

$$f(x) = (e \circ \varphi)(x)\, \varphi_n^b(x)$$

in some neighborhood of $0$. If we choose this neighborhood to be a sufficiently small polydisk $P$, then $(e \circ \varphi)(x) \neq 0$ for all $x \in P$. Consequently, it exists a branch of the logarithm

$$\gamma(x) = \log((e \circ \varphi)(x))$$

on $P$. We finally get

$$f(x) = g(x)^b, \quad x \in P,$$

with $g = h \cdot \varphi_n$, $h(x) = \exp\left(\frac{1}{b}\gamma(x)\right) \neq 0$ for all $x \in P$, and

$$dg(0) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(0)\, dx_j = h(0)\, d\varphi_n(0) + \varphi_n(0)\, dh(0) \neq 0,$$

since $\varphi_n(0) = 0$ and $d\varphi_n(0) \neq 0$, $(\varphi_1, \ldots, \varphi_n)$ being a locally invertible holomorphic map.
b) Since $N(g^b) = N(g)$, we may assume that

$$f(x) = g(x) = \ell(x) + r(x),$$

$\ell$ the linear part of $T_{g,0}$. Then $d\ell(0) = dg(0) \neq 0$, $dr(0) = 0$. Putting $y_j = \varphi_j(x) = x_j$, $j = 1, \ldots, n-1$, $y_n = \varphi_n(x) = g(x)$ - if, without loss of generality, $(\partial g / \partial x_n)(0) \neq 0$ - we get a locally invertible holomorphic map $\varphi = (\varphi_1, \ldots, \varphi_n)$ since

$$\det \frac{\partial \varphi}{\partial x}(0) = \frac{\partial \varphi_n}{\partial x_n}(0) \neq 0$$

and $\varphi(X) = \{ y : g \circ \varphi^{-1}(y) = 0 \} = \{ y : y_n = 0 \}$. □

## 1.5 Isolated critical points of holomorphic functions

We must carefully distinguish between regular points of a *zero set* $N(f) = \{ f = 0 \}$ and regular points of the *function* $f$ itself. Recall that a point $x^{(0)}$ is called *regular* for $f \in \mathcal{O}(U)$, if

$$df(x^{(0)}) \neq 0.$$

Otherwise, it is called a *critical point* of $f$. From the proof of Theorem 3, part b), one concludes the following version of the *Implicit Function Theorem*:

**Theorem 1.6** *If $x^{(0)}$ is a regular point for $f \in \mathcal{O}(U)$, then there exists a biholomorphic map $\psi : W \to V$, $W$ an open neighborhood of $0 \in \mathbb{C}^n$ with linear coordinates $y_1, \ldots, y_n$, $x^{(0)} \in V \subset U$, such that*

$$f \circ \psi(y) - f \circ \psi(0) = y_n \text{ for all } y \in W.$$

*In particular, if $x^{(0)} \in N(f)$ is a regular point for $f$, then it is a smooth point of $N(f)$.*

The converse of the last statement is not true in general. It holds, if and only if the number $b$ in Theorem 3 is equal to 1, since, for $b \geq 2$,

$$df = bg^{b-1}dg = 0$$

at points, where $f$ vanishes. - This observation leads us to a simple criterion for detecting *isolated* hypersurface singularities:

**Theorem 1.7** *Let $x^{(0)} \in N(f)$ be an isolated critical point for the function $f \in \mathcal{O}(U)$, $U \subset \mathbb{C}^n$, $n \geq 2$. Then $x^{(0)}$ is an isolated singularity of $N(f)$.*

Here are some *Examples* of *curves* with *singularities*.

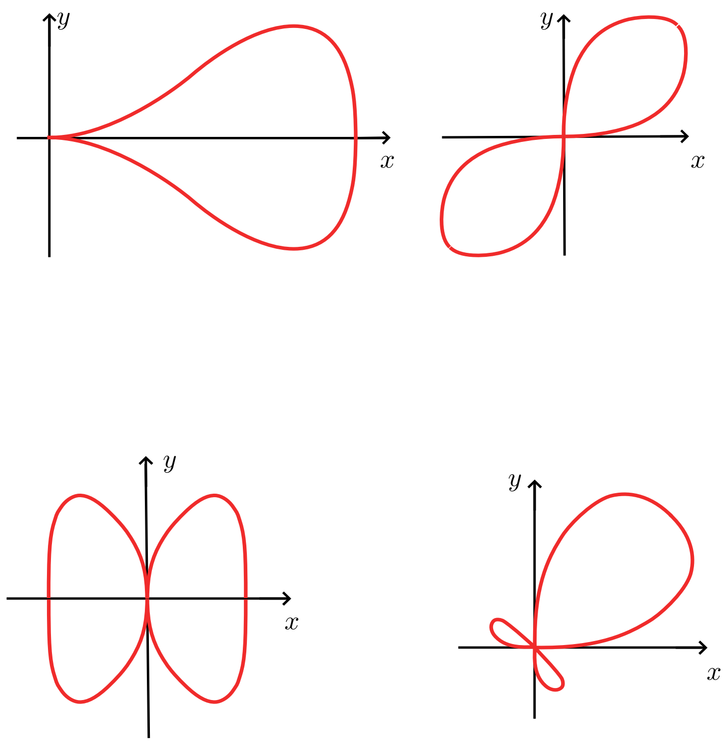



Figure 1.4

These are more precisely curves in the plane with exactly one singularity at the origin. They belong to the following equations (in parenthesis the name of the corresponding singularity):

$$
\begin{array}{rcll}
x^2 &=& x^4 + y^4 & \text{(tacnode)} \\
xy &=& x^6 + y^6 & \text{(node = ordinary double point)} \\
x^3 &=& y^2 + x^4 + y^4 & \text{(cusp)} \\
x^2 y + x y^2 &=& x^4 + y^4 & \text{(ordinary triple point)}\,.
\end{array}
$$

*Exercise* (Hartshorne [4 - 03], p. 35): Which equation corresponds to which picture? Hint: it suffices to study the symmetry properties of the curves (provided one can be sure that each equation belongs to one of the curves). Or, one has to study the singularity type at the origin for each given equation. So, let us do some local computations. From $x^2 = x^4 + y^4$ we get $x^2\,(1 - x^2) = y^4$. If we put $\xi = x\sqrt{1 - x^2}$, $\mu = y$, we obtain (locally around $0$) $\xi^2 = \mu^4$, i.e $0 = \xi^2 - \mu^4 = (\xi - \mu^2)\,(\xi + \mu^2)$. Hence, we have locally (up to analytic diffeomorphism) two parabolae which touch each other at the origin (as in Figure 5b below). For the third equation one deduces in the same manner that after analytic coordinate change at the origin we find the equation of the *cusp*: $x^3 = y^2$. The reader may amuse himself to consider also the other cases.

*Remark*. He or she may also have noticed that the singularities above lie on *irreducible* curves. This, of course, is not satisfied in general as the following examples show ( $y^2 = x^2$ resp. $y^2 = x^4$ ):

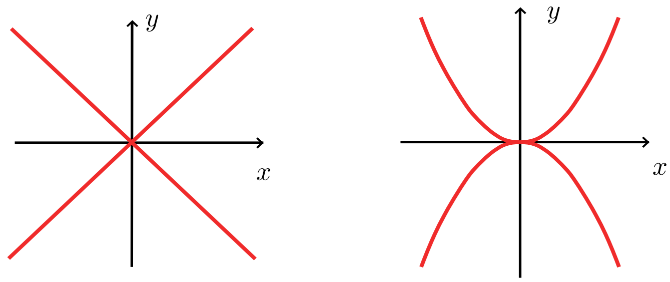



Figure 1.5

The *node* or *ordinary double point* as in Figure 5a above is the most simple curve singularity conceivable. It appears on many of the famous singular plane curves. For the pleasure of the reader, we include pictures of some *Examples*.

1. The *folium cartesium* is given by the equation $x^3 + y^3 = 3a\,xy$, $a > 0$ a fixed parameter:

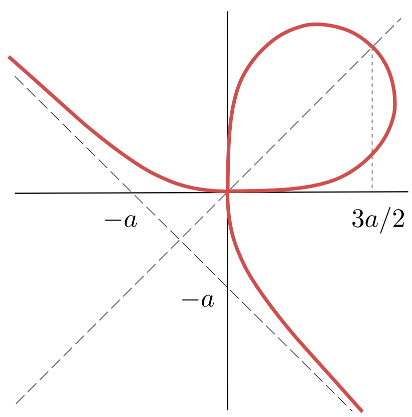



Figure 1.6

2. Another one is a *singular elliptic curve* $y^2 = x^2(x + 1)$:

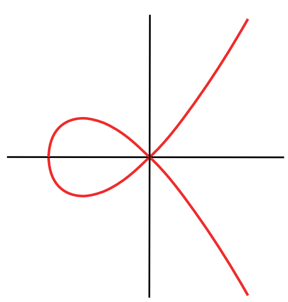


Figure 1.7

3. The last one is the so called "Ampersand" curve $(y^2 - x^2)(x - 1)(2x - 3) = 4\,(x^2 + y^2 - 2x)^2$. Here, *ampersand* stands for *and per se and* where *per se and* means the roman "et–symbol" &.

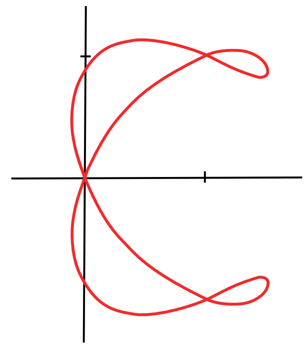


Figure 1.8

We conclude this Section with pictures of some *surface singularities*. The first one is the *conical* or *ordinary double point*, also called the $A_1$–singularity, the second is called a *pinch point* or the WHITNEY–*umbrella*, and the last one is the *limao* in the gallery of surface pictures by HERWIG HAUSER (see Notes and References). They have the equations $x^2 + y^2 = z^2$ , $x^2 = zy^2$ , $x^2 = y^3z^3$ (resp.). Please remark that the $A_1$–singularity is the unique singularity in both surfaces at the beginning of the Chapter that are called the *octdong* and the *dingdong* in the gallery cited above.

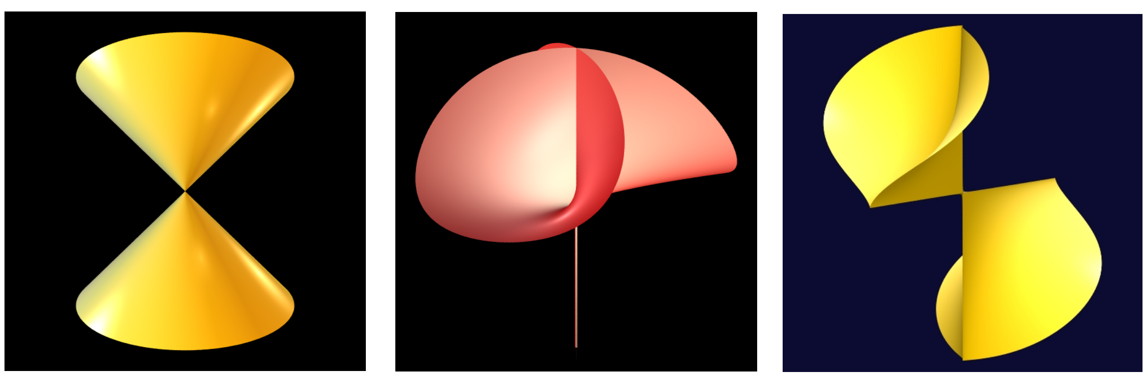


Figure 1.9

## 1.6 Quadratic forms

In this Section we are concerned with the zero set of a *quadratic form* $q$ , i.e. a function $q$ which can be written as

$$q\,(x) \;=\; \sum_{j,k=1}^{n} a_{jk}x_jx_k \;, \quad a_{jk} \in \mathbb{C} \;, \quad x \;=\; (x_1, \ldots, x_n) \in \mathbb{C}^n \;.$$

We notice that $N\,(q)$ is invariant under the natural action of $\mathbb{C}$ on the complex vector space $\mathbb{C}^n$ :

$$\left\{ \begin{array}{ccc} N\,(q) \times \mathbb{C} & \longrightarrow & N\,(q) \\ (x \;\;,\;\; c) & \longmapsto & cx \end{array} \right. .$$

Recall that sets $C$ satisfying this invariance property are usually called (complex) *cones* in $\mathbb{C}^n$ . More on cones can be found in Chapter 4.

Since, without loss of generality, we may assume that $a_{jk} = a_{kj}$ for all $j, k$, and since symmetric matrices can be diagonalized over the complex numbers, we can always find a linear coordinate transformation given by an invertible matrix system $C$ such that

$$\widetilde{q}\,(y) := q\,(Cy) = \sum_{j=1}^{r} y_j^2 \,,$$

where $r$ is the rank of the matrix $(a_{jk})$, $0 \le r \le n$.

In order to understand the geometry of $N\,(q)$, it is sufficient to study the special case $r = n$, since in general

$$\Big\{\, x \in \mathbb{C}^n : \sum_{j=1}^{r} x_j^2 = 0 \,\Big\} = \Big\{\, x \in \mathbb{C}^r : \sum_{j=1}^{r} x_j^2 = 0 \,\Big\} \times \mathbb{C}^{n-r} \,.$$

It turns out that the cases $n = 1$, $n = 2$ and $n \ge 3$ behave completely differently. For $n = 1$, $N\,(q)$ is a point and hence (set–theoretically) a smooth variety. Since the origin is the only critical point of $q$, $N\,(q)$ has an isolated hypersurface singularity at the origin for $n \ge 2$ due to Theorem 5. By the factorization

$$q\,(x_1,\, x_2) = x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)\,, \quad i = \sqrt{-1}$$

we see that (in case $n = 2$) $N\,(q)$ is the union of two lines intersecting transversely at the origin: the *node* or the *ordinary double point*.

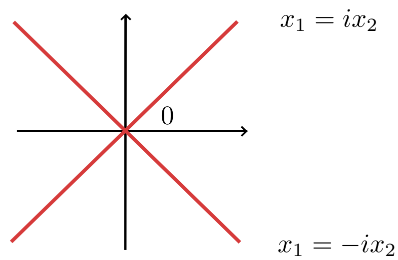



Figure 1.10

In higher dimensions there is no proper decomposition of $N\,(q)$ into two or more analytic components near $0 \in \mathbb{C}^n$; in other words: it is impossible to find a nontrivial factorization of $q\,(x)$ in a neighborhood of the origin. To prove this recall that the ring $\mathbb{C}\,\langle\, x \,\rangle = \mathbb{C}\,\langle\, x_1, \ldots, x_n \,\rangle$ of convergent power series is *factorial*, and that a Weierstraß polynomial

$$\omega = x_n^b + a_1 x_n^{b-1} + \cdots + a_b\,, \quad a_j \in \mathbb{C}\,\langle\, x_1, \ldots, x_{n-1} \,\rangle\,, \quad j = 1, \ldots, b\,,$$

is a prime element in $\mathbb{C}\,\langle\, x \,\rangle$, if and only if it is prime in the polynomial ring $\mathbb{C}\,\langle\, x_1, \ldots, x_{n-1} \,\rangle\,[\, x_n \,]$ which is factorial by the Gauss *Lemma* (for proofs of these statements see Chapter 3). Hence, if the polynomial $q\,(x) = x_1^2 + \cdots + x_n^2 = q'(x') + x_n^2$ were not a prime element in $\mathbb{C}\,\langle\, x \,\rangle$, there would exist a decomposition

$$q\,(x) = (x_n + g_1)\,(x_n + g_2)$$

with elements $g_1,\ g_2 \in \mathbb{C}\,\langle\, x' \,\rangle$ satisfying $g_2 = -g_1$ due to the uniqueness of the Taylor series expansion. Therefore, $q'(x') = x_1^2 + \cdots + x_{n-1}^2$ would be a square in $\mathbb{C}\,\langle\, x_1, \ldots, x_{n-1} \,\rangle$ which is not possible (for $n = 3$ thanks to the known unique factorization, for $n \ge 4$ by reduction to the case $n = 3$).

The reader who is familiar with *Eisenstein's irreducibility criterion* will notice that our argument is equivalent to a proof of this result in a special case.

## 1.7 Germs of holomorphic functions and automorphisms

One can rephrase Theorem 4 in the following manner: If $0$ is a regular point of the function $f \in \mathcal{O}(U)$, then $f$ is near the origin biholomorphically equivalent to the function

$$f(0) + \sum \frac{\partial f}{\partial x_j}(0)\, x_j \; .$$

We want to generalize this statement. In order to do so, we introduce some notions and concepts which are better adapted to our *local* considerations. Recall that two $\mathbb{C}$–valued functions $f$ and $g$, given in some neighborhoods $U$ resp. $V$ of a point $x^{(0)}$, are called equivalent (with respect $x^{(0)}$), if there is an open set $W$ with $x^{(0)} \in W \subset U \cap V$ such that $f_{|W} = g_{|W}$. The class of a function $f$ with respect to this equivalence relation is called the *germ* $f_{x^{(0)}}$ of $f$ at $x^{(0)}$. It is an easy exercise to show that the set of all germs at a given point forms, in a natural way, a $\mathbb{C}$–algebra. Restricting to germs of *holomorphic* functions, we find a subalgebra which usually is denoted by

$$\mathcal{O}_{\mathbb{C}^n, x^{(0)}} \;\text{ or briefly }\; \mathcal{O}_{n, x^{(0)}} \;\text{ or }\; \mathcal{O}_{x^{(0)}} \; .$$

We always represent elements $f_{x^{(0)}} \in \mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ by functions $f \in \mathcal{O}(U)$, $x^{(0)} \in U$, where $U$ can be replaced by smaller open neighborhoods of $x^{(0)}$, if necessary. By assigning to each function $f \in \mathcal{O}(U)$ its Taylor series expansion at the point $x^{(0)}$:

$$f \longmapsto \sum_{|\nu|=0}^{\infty} \frac{\partial^{|\nu|} f}{\partial x^{\nu}}(x^{(0)})\, t^{\nu} \, , \quad t_j := x_j - x_j^{(0)} \, ,$$

we obtain a $\mathbb{C}$–algebra isomorphism

$$\mathcal{O}_{\mathbb{C}^n, x^{(0)}} \xrightarrow{\sim} \mathbb{C}\langle t \rangle = \mathbb{C}\langle t_1, \ldots, t_n \rangle \, ,$$

the last symbol denoting the $\mathbb{C}$–algebra of *convergent power series* centered at the origin in $\mathbb{C}^n$. If $\varphi = (\varphi_1, \ldots, \varphi_m) : U \to \mathbb{C}^m$ is a holomorphic map, $y^{(0)} = \varphi(x^{(0)})$, then every germ $g_{y^{(0)}} \in \mathcal{O}_{\mathbb{C}^m, y^{(0)}}$ has a representative which can be composed from the right by a suitable restriction $\varphi_{|V}$. By this procedure, we get a $\mathbb{C}$–algebra homomorphism

$$\widehat{\varphi}_{x^{(0)}} : \mathcal{O}_{\mathbb{C}^m, y^{(0)}} \longrightarrow \mathcal{O}_{\mathbb{C}^n, x^{(0)}}$$

which only depends on the germs of $\varphi_1, \ldots, \varphi_m$ at $x^{(0)}$. In the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n, x^{(0)}} & \xrightarrow{\sim} & \mathbb{C}\langle t_1, \ldots, t_n \rangle \\
\Big\downarrow \widehat{\varphi}_{x^{(0)}} & & \Big\downarrow \widehat{\varphi}_{x^{(0)}} \\
\mathcal{O}_{\mathbb{C}^m, y^{(0)}} & \xrightarrow{\sim} & \mathbb{C}\langle s_1, \ldots, s_m \rangle
\end{array}$$

the induced vertical homomorphism on the right is determined by the *substitutions*

$$s_k = \varphi_k(x^{(0)} + t) - y_k^{(0)} \, , \quad k = 1, \ldots, m \, .$$

Obviously, the set

$$\mathfrak{m}_{\mathbb{C}^n, x^{(0)}} = \mathfrak{m}_{x^{(0)}} = \{\, f_{x^{(0)}} \in \mathcal{O}_{\mathbb{C}^n, x^{(0)}} : f(x^{(0)}) = 0 \,\}$$

is an ideal in $\mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ which is maximal, since $\mathcal{O}_{\mathbb{C}^n, x^{(0)}} / \mathfrak{m}_{\mathbb{C}^n, x^{(0)}} \cong \mathbb{C}$ is a field. (See Chapter 2 for this and further claims). Moreover, this ideal is the unique maximal ideal, the elements of $\mathcal{O}_{\mathbb{C}^n, x^{(0)}} \setminus \mathfrak{m}_{\mathbb{C}^n, x^{(0)}}$ being precisely the units in $\mathcal{O}_{\mathbb{C}^n, x^{(0)}}$. In other words: $\mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ is a *local ring* with maximal ideal

$\mathfrak{m}_{\mathbb{C}^n, x^{(0)}}$. Any morphism $\widehat{\varphi}_{x^{(0)}} : \mathcal{O}_{\mathbb{C}^m, y^{(0)}} \to \mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ induced by a map germ $\varphi = (\varphi_1, \ldots, \varphi_m)$ is automatically *local* in the sense that it maps $\mathfrak{m}_{\mathbb{C}^m, y^{(0)}}$ into $\mathfrak{m}_{\mathbb{C}^n, x^{(0)}}$. On the other hand, if $\varphi$ is a local homomorphism from $\mathbb{C}\langle s_1, \ldots, s_m \rangle$ to $\mathbb{C}\langle t_1, \ldots, t_n \rangle$ then $\varphi$ is induced by representatives $\varphi_1, \ldots, \varphi_m$ of the germs $\varphi(s_1), \ldots, \varphi(s_m)$.

Therefore, there exists an inverse to $\widehat{\varphi}_{x^{(0)}}$, if and only if $\varphi$ is invertible locally near $\varphi(x^{(0)})$ as a holomorphic map germ. In particular, in the special case $x^{(0)} = y^{(0)} \in \mathbb{C}^n$, the invertible holomorphic map germs fixing $x^{(0)}$ form a group (under the natural composition of maps) which is isomorphic to the group $\mathrm{Aut}\, \mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ of local $\mathbb{C}$–algebra isomorphisms of $\mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ by sending $\varphi_{x^{(0)}}$ to $\widehat{(\varphi^{-1})}_{x^{(0)}}$.

We are now in the position to introduce the important notion of *(right) equivalence* for germs of holomorphic functions: Two germs $f_{x^{(0)}}, g_{x^{(0)}} \in \mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ are called *right equivalent*, if there exists an element $\widehat{\varphi}_{x^{(0)}} \in \mathrm{Aut}\, \mathcal{O}_{\mathbb{C}^n, x^{(0)}}$ with $\widehat{\varphi}_{x^{(0)}}(g_{x^{(0)}}) = f_{x^{(0)}}$. This is the same as to say that $f_{x^{(0)}}$ and $g_{x^{(0)}}$ have representatives $f \in \mathcal{O}_{\mathbb{C}^n}(U)$ and $g \in \mathcal{O}_{\mathbb{C}^n}(V)$, resp., on suitable open sets $U$, $V$, and $\widehat{\varphi}_{x^{(0)}}$ is induced by a biholomorphic map $\varphi : U \to V$ such that $g \circ \varphi = f$. In particular, we have in this situation

$$\varphi(N(f)) = N(g)$$

in a neighborhood of $x^{(0)}$ so that - up to holomorphic coordinate transformations - the hypersurfaces $N(f)$ and $N(g)$ may be considered to be identical near $x^{(0)}$.

## 1.8 Morse Lemma

Henceforth, we encounter the need for studying the *group action* (for more details on group actions in general see Chapter 6):

$$\begin{cases} \mathcal{O}_{\mathbb{C}^n, x^{(0)}} \times \mathrm{Aut}\, \mathcal{O}_{\mathbb{C}^n, x^{(0)}} \longrightarrow \mathcal{O}_{\mathbb{C}^n, x^{(0)}} \\ (f_{x^{(0)}}, \widehat{\varphi}_{x^{(0)}}) \longmapsto \widehat{\varphi}_{x^{(0)}}(f_{x^{(0)}}) , \end{cases}$$

whose *orbits* are precisely the right equivalence classes of germs of holomorphic functions. Since equivalent functions have the same value at $x^{(0)}$, it suffices to consider the induced action on the maximal ideal $\mathfrak{m}_{\mathbb{C}^n, x^{(0)}}$. Plainly, there is the trivial orbit consisting of the zero germ only. By the Implicit Function Theorem, the *regular* germs

$$f_{x^{(0)}} \in \mathfrak{m}_{\mathbb{C}^n, x^{(0)}}, \quad df(x^{(0)}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^{(0)})\, dx_j \neq 0$$

are seen to form another orbit. The simplest *critical* germs $f_{x^{(0)}}$ are those for which the *Hesse form*

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(x^{(0)})\, \xi_j \xi_k$$

is nondegenerate. Under this assumption, $x^{(0)}$ is also called a *nondegenerate critical point* of $f$; the germ $f_{x^{(0)}}$ is sometimes called a MORSE germ. In accordance with the previous Section, we find coordinates near such a point such that $x^{(0)} = 0$ and $f$ is of the form

$$f(x) = \sum_{j=1}^{n} x_j^2 + g(x), \quad g_0 \in \mathfrak{m}_{\mathbb{C}^n, 0}^3 .$$

Here, $\mathfrak{m}_{\mathbb{C}^n, 0}^k$ denotes the ideal of germs of holomorphic functions $g$ which vanish to at least $(k-1)$–th order at the origin:

$$g_0 \in \mathfrak{m}_{\mathbb{C}^n, 0}^k \iff \frac{\partial^{|\nu|} g}{\partial x^{\nu}}(0) = 0 \text{ for all } \nu \text{ with } |\nu| \leq k-1 .$$

Of course, if $\mathfrak{a}\,\mathfrak{b}$ stands as usual for the ideal generated by all products $f \cdot g$, $f \in \mathfrak{a}$, $g \in \mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{b}$ given ideals in a ring $R$, then $\mathfrak{m}^k$ coincides with the $k$–fold product of $\mathfrak{m} = \mathfrak{m}_{\mathbb{C}^n,0}$.

The Morse *Lemma* asserts that all nondegenerate critical germs $f_{x^{(0)}} \in \mathfrak{m}_{\mathbb{C}^n,x^{(0)}}$ are equivalent in the complex–analytic category.

**Theorem 1.8** *Each nondegenerate critical germ $f_0 \in \mathfrak{m}_{\mathbb{C}^n,0}$ is right equivalent to the germ of the function*

$$\sum_{j=1}^{n} x_j^2 \, .$$

Although we can get quite easily the Morse *Lemma* in a more conceptual framework as a simple Corollary from the theory of finitely determined germs (see Chapter 2) we give here a completely elementary *proof*. Notice that the proof goes through mutatis mutandis (see formula $(+++)$ below) in the $\mathcal{C}^\infty$–category.

By Hadamard's *Lemma* (see below), we write

$$f\,(x) \,=\, x_1 g_1(x) + \cdots + x_n g_n(x) \, .$$

By differentiation,

$$\frac{\partial f}{\partial x_j}\,(x) \,=\, g_j(x) + \sum_{k=1}^{n} x_k \,\frac{\partial g_k}{\partial x_j} \;,$$

such that, due to the assumptions,

$$g_j(0) \,=\, 0 \,, \quad j \,=\, 1, \ldots, n \, .$$

Invoking Hadamard's Lemma again, we can write

$$(+) \qquad f\,(x) \,=\, \sum_{j,k=1}^{n} x_j\, x_k\, g_{jk}(x) \,,$$

where, obviously, we may assume $g_{kj}(x) = g_{jk}(x)$. Consequently,

$$(++) \qquad g_{jk}(0) \,=\, \frac{1}{2}\,\frac{\partial^2 f}{\partial x_j \partial x_k}\,(0) \, .$$

Hence, using the "Main Axes Theorem" of Linear Algebra and a corresponding linear coordinate change, we even may assume that

$$2\, g_{jk}(0) \,=\, \delta_{jk} \, .$$

(Notice that in the (real) differentiable case we are only allowed to conclude $2 g_{jk}(0) = \pm\delta_{jk}$. This leads to the *real differentiable* standard form

$$(+++) \qquad f\,(x) \,=\, -\,(x_1^2 + \cdots + x_r^2) + x_{r+1}^2 + \cdots + x_n^2$$

for a Morse germ in which the Morse *index* $r$ is well–defined). - We claim:

**Lemma 1.9** *Suppose that*

$$f\,(x) \,=\, \sum_{j,l=1}^{n} x_j x_k g_{jk}(x)$$

*with $g_{kj}(x) = g_{jk}(x)$ and $\partial^2 f / \partial x_j \partial x_k\,(0) = \delta_{jk}$. Then, for each $r$ with $0 \le r \le n$, there exist functions $h_1^{(r)}, \ldots, h_r^{(r)}$ and $h_{jk}^{(r)}$, $j = k = r+1, \ldots, n$ with $h_{kj} = h_{jk}$ in a neighborhood of the origin such that*

i) $h_\rho^{(r)}(0) = 0$ *and* $\dfrac{\partial h_\rho^{(r)}}{\partial x_\sigma}\,(0) = \delta_{\rho\sigma}$, $\quad \rho, \sigma = 1, \ldots, r$,

ii) $f(x) = \sum_{\rho=1}^{r} h_\rho^{(r)}(x)^2 + \sum_{j,k=r+1}^{n} h_{jk}^{(r)}(x)\, h_j^{(r)}(x)\, h_k^{(r)}(x)$ .

*Proof* (Morse Lemma). Using the preceding Lemma for $r = n$ yields

$$f(x) = \sum_{j=1}^{n} h_j(x)^2 \,, \quad h_j := h_j^{(r)} \,,$$

where $h_j(0) = 0$ and $\partial h_j / \partial x_k\,(0) = \delta_{jk}$. Setting $\xi_j = h_j(x)$ leads to a coordinate change $\xi = h(x)$ after which

$$\varphi(\xi) := f(h^{-1}(\xi)) = f(x) = \sum_{j=1}^{n} \xi_j^2 \,. \qquad \square$$

*Proof* (Lemma). The Lemma will be proved by induction on $r$. For $r = 0$ just put $h_j(x) = x_j$ and $h_{jk}(x) = g_{jk}(x)$. Assume that the induction hypothesis is true for $r - 1$ such that in particular

$$f(x) = \sum_{\rho=1}^{r-1} h_\rho^{(r-1)}(x)^2 + \sum_{j,k=r}^{n} h_{jk}^{(r-1)}(x)\, h_j^{(r-1)}(x)\, h_k^{(r-1)}(x) \,.$$

Necessarily, we have

$$h_{rr}^{(r-1)}(0) = \frac{1}{2}\,\frac{\partial^2 f}{\partial x_r^2}\,(0) = g_{rr}(0) = 1 \,.$$

Now, put $h_\rho^{(r)}(x) = h_\rho^{(r-1)}(x)$, $\rho = 1, \ldots, r-1$, and

$$h_r^{(r)}(x) = \left(h_{rr}^{(r-1)}(x)\right)^{1/2} \left( h_r^{(r-1)}(x) + \frac{1}{h_{rr}^{(r-1)}(x)} \sum_{j=r+1}^{n} h_{rj}^{(r-1)}(x)\, h_j^{(r-1)}(x) \right)$$

which is well–defined in a neighborhood of the origin. A straightforward calculation shows that

$$h_r^{(r)}(0) = 0 \text{ and } \frac{\partial h_r^{(r)}}{\partial x_j}\,(0) = \delta_{rj} \,.$$

Moreover, with suitable new functions,

$$f(x) = \sum_{\rho=1}^{r} h_\rho^{(r)}(x)^2 + \sum_{j,k=r+1}^{n} h_{jk}^{(r)}(x)\, h_j^{(r)}(x)\, h_k^{(r)}(x) \,. \qquad \square$$

HADAMARD's Lemma which we used above says:

**Lemma 1.10** *Every holomorphic function (germ) $f$ vanishing at the origin can be written in the form*

$$f(x) = \sum_{j=1}^{n} x_j f_j$$

*with holomorphic functions $f_j$. In other words: the maximal ideal of the ring of convergent power series is generated by the coordinate functions $x_1, \ldots, x_n$.*

In the case of *formal* power series this is trivial, and in the convergent case one deduces easily from a formal representation $f = x_1 f_1 + \cdots + x_n f_n$ the convergence of the series $f_1, \ldots, f_n$ near $0$. $\square$

*Remark*. The Lemma is also true in the $\mathcal{C}^\infty$–situation. But here, one has to use a different argument. Choose a sufficiently small ball $B$ with center $0$ and take for $x \in B$ the connecting line segment $\alpha(t) = tx$, $t \in I = [0, 1]$. If $f \in \mathcal{C}^\infty(B)$, $f(0) = 0$, then

$$f(x) = \int_\alpha df = \int_I df \circ \alpha = \sum_{j=1}^{n} \left( \int_0^1 \frac{\partial f}{\partial x_j}\,(tx)\, dt \right) x_j \,.$$

The functions in brackets are in $\mathcal{C}^\infty(B)$. $\square$

## 1.9 The surface singularity of type $A_1$

According to the Morse Lemma, we may and do call any isolated hypersurface singularity defined by a *nondegenerate critical germ* the *singularity of type* $A_1$ or simply the $A_1$–singularity. We intend to discuss here some of its special features in the surface case, i.e. in the case, where the ambient space is three–dimensional.

We first want to show that a generic linear projection $\varrho$ of the set $X = \{ x \in \mathbb{C}^3 : x_1^2 + x_2^2 + x_3^2 = 0 \} \subset \mathbb{C}^3$ to a plane through $0$ realizes $X$ as a twofold analytic covering over $\mathbb{C}^2$ which is branched along two lines intersecting transversely at the origin (as do the projections parallel to the coordinate axes). Let $x_j = \ell_j + c_j z$, $\ell_j = a_j x + b_j y$, $j = 1, 2, 3$, be a generic linear coordinate transformation; then the defining equation for $X$ is

$$f(x, y, z) = \left( \sum c_j^2 \right) z^2 + 2 \left( \sum c_j \ell_j \right) z + \left( \sum \ell_j^2 \right) .$$

Generically, we have $c = \sum c_j^2 \neq 0$ such that under this assumption $f(x, y, z)$ may be considered as a quadratic equation in $z$ with *discriminant*

$$\Delta = \Delta\,(x, y) = \left( \sum c_j^2 \right) \left( \sum \ell_j^2 \right) - \left( \sum c_j \ell_j \right)^2$$

which, as a function of $x$ and $y$, can be written in the form

$$\Delta\,(x, y) = Ax^2 + 2Bxy + Cy^2$$

with polynomials $A$, $B$, $C$ in the nine variables $a_1$, $a_2$, $a_3$,;$b_1$, $b_2$, $b_3$, $c_1$, $c_2$, $c_3$. So, the restriction of the projection $(x, y, z) \mapsto (x, y)$ to $X = \{ (x, y, z) : f\,(x, y, z) = 0 \}$ is a twofold covering of the $(x, y)$–plane with branch locus consisting of two *distinct* lines, if and only if

$$\begin{cases} c = \sum c_j^2 \neq 0 \\ AC - B^2 \neq 0 \,. \end{cases}$$

Our statement then follows from the fact that the complement of an *algebraic set* $A$ in $\mathbb{C}^n$ - that is the set $A$ of common zeros of finitely many *polynomials* in $\mathbb{C}\,[\, x_1, \ldots, x_n \,]$ - is, for $A \neq \mathbb{C}^n$, necessarily (open and) dense in $\mathbb{C}^n$, when applied to $\mathrm{GL}\,(3, \mathbb{C})$ interpreted as the open set

$$\left\{ (a_1, \ldots, c_3) \in \mathbb{C}^9 : \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \neq 0 \right\} \subset \mathbb{C}^9$$

and to the intersection of $\mathrm{GL}\,(3, \mathbb{C})$ with the set $\{ c \cdot (AC - B^2) \neq 0 \}$.

In Chapter 6, we will develop a general theory of multiplicities for all (normal) surface singularities by showing that almost all linear projections $\varrho$ of such a singularity to a complex plane have the same number $m$ of sheets (i.e. the cardinality of the fibers $\varrho^{-1}(u)$, $u$ outside the branch locus). For a singularity, we necessarily must have $m \geq 2$; singularities with $m = 2$ are called *double points*. (Observe that the ordinary double point in $\mathbb{C}^2$ satisfies the condition that almost all of its projections to a complex *line* are twofold). Hence, the $A_1$–singularity is a double point. It will be shown in Chapter 6 that it is the unique double point whose (generic) branch locus has an ordinary double point as singularity.

To a certain extent, the method of analytic coverings reduces the study of $d$–dimensional singularities to the study of the $(d - 1)$–dimensional branch locus $\Sigma$ in $\mathbb{C}^d$. However, it is not merely the *abstract* singularity $\Sigma$ which comes into play; what really matters is the *pair* $(\Sigma, \mathbb{C}^d)$, since e.g. the topological (and even holomorphic) types of unbranched coverings of $\mathbb{C}^d \setminus \Sigma$ are classified by the fundamental group $\pi_1(\mathbb{C}^d \setminus \Sigma)$. For the special case, where $\Sigma$ is the union of the coordinate axes in $\mathbb{C}^2$, we shall determine all possible surface singularities branched over $\Sigma$ in Chapter 6.

Another way to visualize the (topological) complexity of an isolated hypersurface singularity $x^{(0)} \in X$ in $\mathbb{C}^n$ lies in forming the so–called *link* of $X$: We take a small ball

$$B_\varepsilon(x^{(0)}) = \{ x \in \mathbb{C}^n : |\, x - x^{(0)} \,| < \varepsilon \} \, , \quad \varepsilon > 0 \, ,$$

and intersect $X$ with the sphere $S_\varepsilon(x^{(0)}) = \partial B_\varepsilon(x^{(0)})$ to get

$$X_\varepsilon = X \cap S_\varepsilon(x^{(0)}) \, .$$

We will prove in Chapter 15 that, for small enough $\varepsilon > 0$, the link $X_\varepsilon$ is a real $(2n - 3)$–dimensional differentiable manifold and that, topologically, $X$ is the real cone over $X_\varepsilon$:

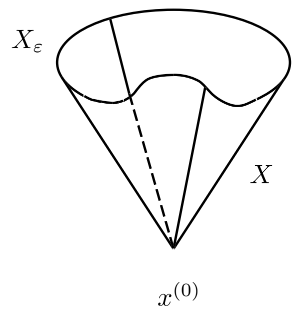



Figure 1.11

Of course, if we form $X_\varepsilon$ for a hyperplane $X \subset \mathbb{C}^n$ or, more generally, for a smooth point $x^{(0)}$ of a hypersurface $X$, then $X_\varepsilon$ will be homeomorphic to the $(2n - 3)$–dimensional standard sphere. What, however, is the link of the $A_1$–singularity?

Without violating the topological structure we can perform an analytic coordinate transformation. Therefore, we may assume that $X$ is given by the equation

$$z^2 - 2xy = 0 \, .$$

We define a holomorphic map

$$\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$$

by

$$\pi\,(u,\, v) = (u^2,\, v^2,\, \sqrt{2}\, uv) \, .$$

It is an easy exercise to prove that $\pi$ maps $\mathbb{C}^2$ surjectively onto $X$ and that, for all $(u,\, v) \in \mathbb{C}^2$, the fiber $\pi^{-1} \circ \pi\,(u,\, v)$ consists of the points $(u,\, v)$ and $(-u,\, -v)$. Obviously,

$$|\, x \,|^2 + |\, y \,|^2 + |\, z \,|^2 = (|\, u \,|^2 + |\, v \,|^2)^2$$

for $(x,\, y,\, z) = \pi\,(u,\, v)$. This implies that the link

$$X_\varepsilon = X \cap \partial B_\varepsilon \, , \quad B_\varepsilon = B_\varepsilon(0)$$

is homeomorphic to the quotient of the 3–sphere in $\mathbb{C}^2(u,\, v)$ (with center $0$ and radius $\sqrt{\varepsilon}$) by the action of the antipodal map $(u,\, v) \mapsto (-u,\, -v)$, i.e. $X_\varepsilon$ is topologically the real projective space $\mathbb{P}_3(\mathbb{R})$ which is not homeomorphic to the 3–sphere.

We should remark here that for $n \geq 4$ it can happen that $X_\varepsilon$ is homeomorphic to the sphere without $x^{(0)}$ being a smooth point of $X$. However, in Chapter 15, we prove that this cannot occur for surface singularities.

There is yet another description of the $A_1$–singularity $X$ following from the existence of the map $\pi : \mathbb{C}^2 \to X$. As we remarked earlier, two points of $\mathbb{C}^2$ are mapped to the same point in $X$, if and only if they are conjugate under the action of the holomorphic map $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ which is given by

$\tau(u, v) = (-u, -v)$. Therefore, as a set, $X$ is equal to the quotient $\mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts by the generator $\tau$. But, since the action of $\mathbb{Z}_2$ is given by biholomorphic automorphisms, the topological quotient $\mathbb{C}^2/\mathbb{Z}_2$ can be equipped with the structure of a complex analytic space (for more details, see Chapter 8) on which the holomorphic functions are precisely the holomorphic functions on $\mathbb{C}^2$ that are invariant under the action of $\mathbb{Z}_2$. We will show that $X$ is indeed isomorphic to this *holomorphic* quotient $\mathbb{C}^2/\mathbb{Z}_2$.

The general theory of quotients implies in particular that the integral domain $A = \mathcal{O}_{\mathbb{C}^3,0}/(x_1^2 + x_2^2 + x_3^2)\mathcal{O}_{\mathbb{C}^3,0}$ is *normal*, i.e. algebraically closed in its quotient field. This, however, is true for any factor ring $\mathcal{O}_{\mathbb{C}^3,0}/f\,\mathcal{O}_{\mathbb{C}^3,0}$ with an arbitrary function $f \in \mathcal{O}(U)$, $0 \in U \subset \mathbb{C}^3$, having an isolated critical point at the origin, as we shall see in Chapter 13.

As a last description of the $A_1$–singularity, we identify it with the special fiber of a holomorphic map between complex analytic manifolds which arises canonically in the theory of complex Lie groups.

The special linear group $\mathrm{SL}(2, \mathbb{C})$ of $2 \times 2$–matrices with determinant $1$ has the structure of a smooth hypersurface in $\mathbb{C}^4$:

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha\delta - \beta\gamma = 1 \right\} .$$

To each matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ we associate its *trace* $\mathrm{tr}(M) = \alpha + \delta \in \mathbb{C}$; the holomorphic map $\mathbb{C}^4 \to \mathbb{C}$, given by $(\alpha, \beta, \gamma, \delta) \mapsto \alpha + \delta$, induces a holomorphic map

$$\mathrm{tr} : \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathbb{C} .$$

The special fiber over the point $t_0 = 2$ consists of all matrices $M$ with eigenvalue 1 (of multiplicity 2); in other words: $\mathrm{tr}^{-1}(2)$ is the *unipotent variety* of $\mathrm{SL}(2, \mathbb{C})$ (a matrix $M$ is called unipotent, if $M - E$, $E$ the unit $2 \times 2$–matrix, is nilpotent). The unipotent variety of $\mathrm{SL}(2, \mathbb{C})$ can henceforth be described analytically by the equation

$$\alpha(2 - \alpha) - \beta\gamma = 1$$

in $\mathbb{C}^3$ which is critical only at $\alpha = 1$, $\beta = \gamma = 0$. Introducing the new variables $\xi = \alpha - 1$, $\eta = \beta$, $\zeta = \gamma$, the singularity is given by

$$\zeta^2 - \xi\eta = 0 .$$

Hence we see that the unipotent variety of $\mathrm{SL}(2, \mathbb{C})$ turns out to be singular precisely at the unit matrix $E$, the singularity being isomorphic to the $A_1$–singularity. We will sketch a more conceptual approach to this phenomenon in Appendix B to Chapter 16.

## 1.10 Barth's sextic

Up to now, we have mainly seen some "global" curves and surfaces with exactly *one* singularity. But there is at least one exception: The surface on the frontispiece has obviously *many* singularities, all of type $A_1$. This leads to a completely different question: *Given a certain isolated singularity. How many of them can have a global object of this dimension?* Of course, one has to make precise what we mean by a *global* object. Since, for instance, a *hypersurface* in $\mathbb{C}^n$ may have singularities at *infinity* it might be reasonable to take them also into account. Therefore, one is more or less forced to consider *projective varieties*, i.e. curves, surfaces etc. in $\mathbb{P}^n$ (see Chapter 4) and to impose eventually stronger restrictions.

For varieties in $\mathbb{P}^n$, a useful invariant is their *degree* (see again Chapter 4), which in case of hypersurfaces is nothing else but the degree of a defining homogeneous polynomial. Barth's polynomial of degree 6 (hence the word *sextic*) is the following:

$$4(\rho^2 x^2 - y^2)(\rho^2 y^2 - z^2)(\rho^2 z^2 - x^2) - (1 + 2\rho)(x^2 + y^2 + z^2 - 1)^2 = 0 , \quad \rho = \frac{1 + \sqrt{5}}{2}$$

the *golden ratio*. It has exactly 65 nodes and no other singularities. In fact, Barth's sextic reached a *world record*.

***Theorem 1.11** *No hypersurface in $\mathbb{P}^3$ of degree 6 can have more than 65 nodes.*

## 1.11 Determinantal varieties

Linear algebra provides us with some other very special albeit extremely instructive *Examples*. Instead of $\mathbb{C}^n$ we regard the space $\mathbb{C}^{n\times n} = M(n\times n, \mathbb{C})$ of all $n\times n$–matrices. The *regular*, i.e. *invertible* matrices form the open dense subset

$$\mathrm{Reg}\,(n\times n, \mathbb{C}) = \{ A\in M(n\times n, \mathbb{C}) : \det A \neq 0 \} .$$

The complement

$$H = \{ A\in M(n\times n, \mathbb{C}) : \det A = 0 \} = \{ A\in M(n\times n, \mathbb{C}) : \mathrm{rank}\, A \leq n-1 \}$$

of *singular* matrices forms therefore a complex–analytic (in fact, by the well–known expression of the determinant, a complex–algebraic) subset of $\mathbb{C}^{n\times n}$. We write for the general matrix

$$A = (x_{jk})_{j,k=1,\ldots,n} , \quad x_{jk}\in\mathbb{C} .$$

Then, by Laplace's rule,

$$\frac{\partial}{\partial x_{\ell m}} \det A = \pm \det A_{\ell m} ,$$

where $A_{\ell m}$ denotes the $(n-1)\times(n-1)$–minor of the matrix $A$ which one forms out of $A$ by deleting the $\ell$–th row and the $m$–th column. Therefore, $A$ is a singular point of the hypersurface $H$ of singular matrices of size $n\times n$, if and only if

$$\det A_{jk} = 0 , \quad j, k = 1,\ldots,n ,$$

i.e. if and only if $\mathrm{rank}\, A \leq n-2$.

In other terms, if we set

$$M^{(r)}(n\times n, \mathbb{C}) := \{ A\in M(n\times n, \mathbb{C}) : \mathrm{rank}\, A \leq r \} ,$$

we have complex–algebraic subsets of $\mathbb{C}^{n^2}$ (since $M^{(r)}(n\times n, \mathbb{C})$ is described by the vanishing of all the $(r+1)\times(r+1)$–minors of $A$), such that

$$H = M^{(n-1)}(n\times n, \mathbb{C}) = \mathrm{sing}\, M(n\times n, \mathbb{C}) = \mathrm{sing}\, M^{(n)}(n\times n, \mathbb{C})$$

and

$$\mathrm{sing}\, H = \mathrm{sing}\, M^{(n-1)}(n\times n, \mathbb{C}) = M^{(n-2)}(n\times n, \mathbb{C}) .$$

We shall prove in Chapter 4 that, more generally, the algebraic sets

$$M^{(r)}(m\times n, \mathbb{C}) = \{ A\in M(m\times n, \mathbb{C}) : \mathrm{rank}\, A \leq r \}$$

in $\mathbb{C}^{m\times n} = M(m\times n, \mathbb{C})$ satisfy

$$\mathrm{sing}\, M^{(r)}(m\times n, \mathbb{C}) = M^{(r+1)}(m\times n, \mathbb{C}) ,$$

and that the submanifold

$$M^{(r)}(m\times n, \mathbb{C}) \setminus M^{(r+1)}(m\times n, \mathbb{C})$$

of $M^{(n)}(m\times n, \mathbb{C}) = \mathbb{C}^{m\times n}$ has codimension $(m-r)(n-r)$.

# 1.A Appendix A: Fundamental properties of holomorphic functions

## 1.A.1 Real and complex differentiability

Recall that a real or complex valued function $f$ in the *real* variables $x_1, \ldots, x_n$ is called *(totally) real differentiable* at a point $x^{(0)}$ if there is a neighborhood $U$ of $x^{(0)}$ and there are functions $\Delta_1, \ldots, \Delta_n$ on $U$ which are continuous at $x^{(0)}$ such that

$$f(x) = f(x^{(0)}) + \sum_{j=1}^{n} \Delta_j(x)(x_j - x_{0j}), \quad x \in U.$$

In particular, $f$ is necessarily continuous at $x^{(0)}$. The functions $\Delta_j(x)$ are in general not uniquely determined, but there values

$$\frac{\partial f}{\partial x_j}(x^{(0)}) := \Delta_j(x^{(0)})$$

are.

A complex valued function $f$ in $n$ *complex* variables $x_1, \ldots, x_n$ is called *(totally) real differentiable* if it is differentiable in the sense defined above with respect to the $2n$ real variables $\xi_j = \mathrm{Re}\, x_j$, $\eta_j = \mathrm{Im}\, x_j$, $j = 1, \ldots, n$. Due to Euler formulas

$$\xi_j = \frac{x_j + \overline{x_j}}{2}, \quad \eta_j = \frac{x_j - \overline{x_j}}{2i},$$

this is equivalent to an expansion of the form

$$f(x) = f(x^{(0)}) + \sum_{j=1}^{n} A_j(x)(x_j - x_{0j}) + \sum_{j=1}^{n} B_j(x)(\overline{x_j} - \overline{x_{0j}})$$

with functions $A_j, B_j : U \to \mathbb{C}$ which are continuous at $x^{(0)}$. The Wirtinger *derivatives* are defined by

$$\frac{\partial f}{\partial x_j}(x^{(0)}) := A_j(x^{(0)}), \quad \frac{\partial f}{\partial \overline{x_j}}(x^{(0)}) = B_j(x^{(0)})$$

and can easily be expressed by linear combinations of the partial derivatives $\dfrac{\partial f}{\partial \xi_j}$ and $\dfrac{\partial f}{\partial \eta_j}$. Such a function $f$ is called (totally) *complex* differentiable at $x^{(0)}$ if one can choose $B_j \equiv 0$ for all $j = 1, \ldots, n$. $f$ is then necessarily (totally) real differentiable at $x^{(0)}$ and satisfies the Cauchy–Riemann equations

$$\frac{\partial f}{\partial \overline{x_j}}(x^{(0)}) = 0, \quad j = 1, \ldots, n,$$

which are equivalent to

$$\frac{\partial g}{\partial \xi_j}(x^{(0)}) = \frac{\partial h}{\partial \eta_j}(x^{(0)}), \quad \frac{\partial g}{\partial \eta_j}(x^{(0)}) = -\frac{\partial h}{\partial \xi_j}(x^{(0)}), \quad g = \mathrm{Re}\, f, \quad h = \mathrm{Im}\, f.$$

## 1.A.2 The Cauchy integral formulas

Recall from classical Complex Analysis that a function in one complex variable is holomorphic (in an open set $U \subset \mathbb{C}$), if and only if it is totally real differentiable and satisfies the Cauchy–Riemann equations at each point of $U$. Therefore, the implications ii) $\Longrightarrow$ iii) $\Longrightarrow$ iv) in the following Theorem are immediately clear.

**Theorem 1.12** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ a function. Then, the following are equivalent* :

i) $f$ *is holomorphic on* $U$;

ii) $f$ *is totally complex differentiable on* $U$;

iii) $f$ *is totally real differentiable on* $U$ *and satisfies the Cauchy–Riemann equations*;

iv) $f$ *is continuous on* $U$ *and holomorphic in each complex variable separately*;

v) *for each point* $x^{(0)} \in U$ *and each polydisk* $P$ *with* $x^{(0)} + \overline{P} \subset U$ *one has the Cauchy integral formula*

$$f\,(x^{(0)} + h) = \frac{1}{(2\pi i)^n} \int_T \frac{f\,(x^{(0)} + \zeta)}{(\zeta_1 - h_1) \cdot \ldots \cdot (\zeta_n - h_n)}\, d\zeta\,, \quad h \in P\,,$$

*where* $T = \{\,|\,x_1\,| = r_1\,\} \times \cdots \times \{\,|\,x_n\,| = r_n\,\} \subset \partial P$.

**Corollary 1.13** *In the situation above,* $f$ *is arbitrarily often (totally) complex differentiable, and* $f$ *satisfies the Cauchy integral formulas*

$$D^\nu f\,(x^{(0)} + h) = \frac{\nu!}{(2\pi i)^n} \int_T \frac{f\,(x^{(0)} + \zeta)}{(\zeta_1 - h_1)^{\nu_1+1} \cdot \ldots \cdot (\zeta_n - h_n)^{\nu_n+1}}\, d\zeta\,.$$

*In particular,*

$$D^\nu f\,(x^{(0)}) = \frac{\nu!}{(2\pi i)^n} \int_T \frac{f\,(x^{(0)} + \zeta)}{\zeta_1^{\nu_1+1} \cdot \ldots \cdot \zeta_n^{\nu_n+1}}\, d\zeta\,.$$

*Proof* of Theorem 11. Only a few implications are left. i) $\Longrightarrow$ ii). By a formal power series expansion, one can find formal power series $\Delta_1, \ldots, \Delta_n$ around $0$ such that formally

$$(*) \qquad f\,(x^{(0)} + h) = f\,(x^{(0)}) + h_1\,\Delta_1(h) + \cdots + h_n\,\Delta_n(h)$$

and $h_j\,\Delta_j(h)$ is a subseries of the power series expansion of $f$ at $x^{(0)}$. By assumption, all $\Delta_j$ converge at some point $h \in (\mathbb{C}^*)^n$. Hence, all $\Delta_j$ are convergent in a neighborhood of $0$, and therefore, $(*)$ holds as an equation for functions near $h = 0$. Since the functions $\Delta_1, \ldots, \Delta_n$ are at least continuous at the origin, $f$ is totally complex differentiable at any point $x^{(0)} \in U$.

iv) $\Longrightarrow$ v). Since $f$ is continuous, the Cauchy integral on the right hand side exists and is equal to an iterated Cauchy integral

$$\frac{1}{2\pi i} \int_{\partial D_1} \frac{d\zeta_1}{\zeta_1 - h_1} \left( \cdots \left( \frac{1}{2\pi i} \int_{\partial D_n} \frac{f\,(x^{(0)} + \zeta)}{\zeta_n - h_n}\, d\zeta_n \right) \cdots \right).$$

Since $f$ is partially holomorphic one can now apply the classical Cauchy integral formula.

v) $\Longrightarrow$ i). This works as in the classical case by replacing the geometric series by the "multiple" geometric series. □

*Proof* of Corollary 12. Just differentiate the Cauchy integral formula. □

### 1.A.3 The Identity Theorem and other applications

The *Identity Theorem* for holomorphic functions in several complex variables reads as follows.

**Theorem 1.14** *Let* $G$ *be a domain in* $\mathbb{C}^n$ *and let* $f, g$ *be holomorphic functions on* $G$. *Then, the following are equivalent*:

i) $f = g$,

ii) $f_{|U} = g_{|U}$ *for some nonempty open subset* $U \subset G$,

iii) *all partial derivatives of* $f$ *and* $g$ *agree at one point* $x^{(0)} \in G$.

*Remark*. It is obviously not sufficient (as in dimension 1) that $f$ agrees with $g$ on a *non discrete* subset $D \subset G$: Just take $f(x_1, x_2) := x_1 x_2$ and $g = 0$.

*Proof* of Theorem 13. Plainly, i) $\Longrightarrow$ ii) $\Longrightarrow$ iii). For the remaining implication iii) $\Longrightarrow$ i) define $h := f - g$ and

$$A := \{ x \in G : D^\nu h(x) = 0 \text{ for all } \nu \in \mathbb{N}^n \}.$$

By assumption, $A$ is not empty; moreover $A$ is open since $h = 0$ locally near an arbitrary point $x \in A$ due to the *Identity Theorem* for power series. Since all partial derivatives of $h$ are continuous functions, $A$ is also closed in $G$. Hence, $A = G$ and $h = 0$ on $G$. $\square$

**Corollary 1.15** *If $G$ is a domain in $\mathbb{C}^n$, then the ring of holomorphic functions on $G$ is integer, i.e. does not contain zerodivisors.*

*Proof*. As in the classical case of one variable. $\square$

As a more important application, we formulate and proof

**Theorem 1.16 (Hartogs' Extension Theorem)** *Let $0 < \rho_j < r_j$, $j = 1, \ldots, n$, $n \geq 2$, be real numbers, and define*

$$\begin{aligned} P &:= P_r,\ r = (r_1, \ldots, r_n),\ P' = P_{\rho'},\ \rho' = (\rho_1, \ldots, \rho_{n-1}, r_n), \\ Q &:= \{ x \in P : |x_j| \geq \rho_j,\ j = 1, \ldots, n-1,\ |x_n| \leq \rho_n \}, \\ H &:= P \setminus Q. \end{aligned}$$

*Then, every holomorphic function $f$ on $H$ can (uniquely) be extended to a holomorphic function on $P$. In other words: the canonical restriction homomorphism*

$$\mathcal{O}(P) \longrightarrow \mathcal{O}(H)$$

*is bijective.*

*Proof*. Choose $\vartheta_n$ with $\rho_n < \vartheta_n < r_n$ and $x \in P'' := \{ x \in P : |x_n| < \vartheta_n \}$.

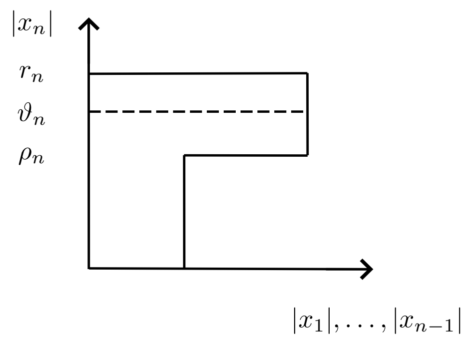



Figure 1.12

Then, the function

$$F(x) := \frac{1}{2\pi i} \int_{|\xi_n| = \vartheta_n} \frac{f(x_1, \ldots, x_{n-1}, \xi_n)}{\xi_n - x_n}\, d\xi_n$$

is continuous on $P''$ and satisfies the Cauchy–Riemann equations, i.e. $F \in \mathcal{O}(P'')$. Moreover, $F$ coincides with $f$ on the polydisk $P' \cap P''$ due to Cauchy's integral formula in one variable. By the Identity Theorem, $F = f$ on the domain $H \cap P''$. Hence,

$$x \longmapsto \begin{cases} f(x), & x \in H \\ F(x), & x \in P'' \end{cases}$$

defines a holomorphic extension of $f$. The injectivity of the homomorphism $\mathcal{O}(P) \to \mathcal{O}(H)$ follows from the Identity Theorem. □

*Remark*. In dimension $1$, each domain $G \subset \mathbb{C}$ is a *domain of holomorphy*, i.e. there exist holomorphic functions $f$ on $G$ which cannot be extended to any larger domain. In dimension $n \geq 2$, this is no longer true as the example of the Hartogs' domain $H$ shows. Examples of domains of holomorphy are balls and polydisks.

By the same technique or as a direct consequence of Hartogs' Theorem one easily deduces the following

**Corollary 1.17** *Let $0 \leq \rho < r$ be given and $K := P_r \setminus \overline{P_\rho}$. Then, for $n \geq 2$, the restriction homomorphism $\mathcal{O}(P_r) \to \mathcal{O}(K)$ is an isomorphism. In particular, there are no isolated "singularities" for holomorphic functions in $n \geq 2$ variables (and thus no isolated zeroes either).*

## 1.A.4 The Riemann Extension Theorems

Let $G \subset \mathbb{C}^n$ be a domain and $A \subset G$ a closed subset of $G$. $A$ is called an *analytic set* in $G$ if to each $x^{(0)} \in A$ there exists an open neighborhood $x^{(0)} \in U \subset G$ and finitely many holomorphic functions $f_1, \ldots, f_r \in \mathcal{O}(U)$ such that

$$A \cap U = \{ x \in U : f_1(x) = \cdots = f_r(x) = 0 \} .$$

We say that $A$ is of *codimension* $\geq k$ at $x^{(0)} \in A$ if one can find an affine subspace $L$ of dimension $k$ through $x^{(0)}$ such that $A \cap L = \{ x^{(0)} \}$ in a neighborhood of $x^{(0)}$.

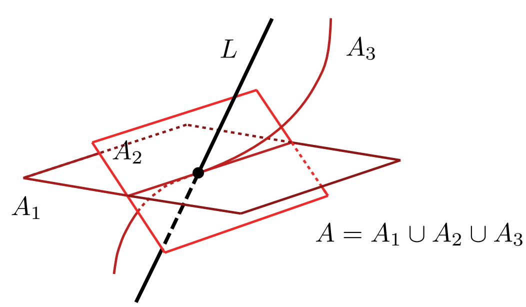



Figure 1.13

The *codimension* $\operatorname{codim}_{x^{(0)}} A$ of $A$ in $G$ at $x^{(0)}$ is the *maximal* $k$ with that property, and the codimension of $A$ in $G$ is defined by

$$\operatorname{codim} A := \min_{x^{(0)} \in A} \operatorname{codim}_{x^{(0)}} A .$$

It is an easy exercise to show that $\operatorname{codim}_{x^{(0)}} A = 0$ at some point $x^{(0)} \in A$ implies $A = G$.

We say that *Riemann's Extension Theorem* holds for $A$ in $G$, if the homomorphism $\mathcal{O}(G) \to \mathcal{O}(G \setminus A)$ is surjective. Clearly, this cannot be satisfied without further restrictions on $A$ and/or on $f$. If, for instance, $f \in \mathcal{O}(G)$ is nontrivial with non empty zero set $A = \{ x \in G : g(x) = 0 \}$, then $f := 1/g$ has no holomorphic continuation to $G$. - We now state

**Theorem 1.18 (Riemann's Extension Theorems)**

1. *Let $A \subset G$ be an analytic subset of codimension $\geq 1$. Then, $G \setminus A$ is a domain, and each holomorphic function $f$ on $G \setminus A$ which is locally bounded at the points of $A$ has a holomorphic extension to $G$.*

2. *If* $\operatorname{codim} A \geq 2$ *, no restrictions on* $f$ *are needed, i.e. the canonical homomorphism*

$$\mathcal{O}(G) \longrightarrow \mathcal{O}(G \setminus A)$$

*is an isomorphism.*

*Proof*. It is sufficient to show that $G \setminus A$ is dense in $G$ and that we have extendability locally at any point $x^{(0)} \in A$ since by the first fact local holomorphic extensions are uniquely determined and hence patch together to a global function on $G$. Connectedness of $G \setminus A$ can from this be deduced as follows: If this were not true one could find nontrivial locally constant functions $f_1,\, f_2 \in \mathcal{O}(G \setminus A)$ with $f_1 f_2 = 0$. But then $F_1 F_2 = 0$ for extensions $F_1,\, F_2$ of $f_1,\, f_2$, and by a well known consequence of the Identity Theorem, one of the $F_j$ must vanish identically (see Corollary xx).

So, we may assume that $x^{(0)} = 0 \in A = \{ x \in U : f_1(x) = \cdots = f_r(x) = 0 \}$, and that $A \cap \{ x_1 = \cdots = x_{n-1} = 0 \} = \{ 0 \}$. This implies that at least one of the functions $f_1, \ldots, f_r$, say $g := f_1$, is $x_n$–regular at the origin. Hence, due to the Weierstraß Preparation Theorem, there exists a Weierstraß polynomial $\omega$ in $x_n$ at $0$ such that (locally around $0$)

$$A \subset B = \{ x \in P : \omega(x) = 0 \}$$

for a certain polydisk $P$. It is immediately clear that $P \setminus B$ and consequently $P \setminus A$ as well is dense in $P$, thus taking care of our first claim. So, it remains to show that bounded holomorphic functions on $P \setminus B$ extend holomorphically to $P'$ for suitably chosen polydisks $P' \subset\subset P$ with center the origin. Without loss of generality we may assume that

$$P = P_r\,,\ r = (r_1, \ldots, r_n)\,, \quad P' = P_{r'}\,,\ r' = (r_1, \ldots, r_{n-1}, \rho_n)\,,\ 0 < \rho_n < r_n\,,$$

and that all solutions of $\omega(x', x_n) = 0$, $x' = (x_1, \ldots, x_{n-1})$ with $|x_j| < r_j$, $j = 1, \ldots, n-1$, satisfy $|x_n| < \rho_n$.

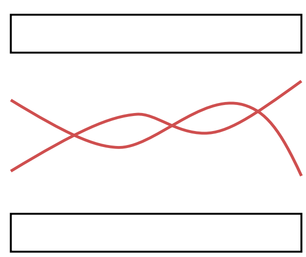


Figure 1.14

For fixed $x'$, $f(x', x_n)$ is holomorphic in $x_n$ outside of the finite set $\omega(x', x_n) = 0$ and bounded. Therefore, the classical Riemann Removable Singularity Theorem implies that this function can holomorphically be extended to all $x_n$ with $|x_n| < r_n$ such that the function

$$F(x) := \frac{1}{2\pi i} \int_{|\xi_n| = \rho_n} \frac{f(x_1, \ldots, x_{n-1}, \xi_n)}{x_n - \xi_n}\, d\xi_n\,, \quad x \in P'$$

is equal to $f$ on $P' \setminus B$. Again, $F$ is continuous on $P'$ and satisfies the Cauchy–Riemann equations, such that $F$ is a holomorphic extension of $f$ on $P'$.

The case $\operatorname{codim} A \geq 2$ can simply be reduced to the case handled before by invoking the *Maximum Principle* for holomorphic functions (following from the Cauchy integral formula as in the classical case $n = 1$) which implies that holomorphic functions near points $x^{(0)} \in A$ have to be bounded. Or else, one can use the general theory of branched coverings to find a similar picture as in Figure 10 where the base has dimension $n - 2$ instead of $n - 1$ but the fibers of the projection $A \to \mathbb{C}^{n-2}$ are still discrete. Then, the result is an immediate consequence of (the Corollary to) Hartogs' Extension Theorem. $\square$

# 1.B Appendix B: The Theorems of Whitney and Sard

## 1.B.1 The Theorem of Whitney

We proof the Theorem of Whitney (Theorem 3) in a more general version. Notice that we assume all topological spaces (especially differentiable and analytic manifolds) to satisfy the second axiom of countability and hence to be *paracompact*.

**Theorem 1.19 (Whitney)** *Any closed subset $A$ of the manifold $M$ equals the zero set of a nonnegative function $f \in \mathcal{C}^\infty(M)$.*

*Proof*. By the assumptions, $M$ carries a countable atlas and an underlying differentiable *partition of unity*. Therefore, we can restrict ourselves to the case $M = \mathbb{R}^n$; furthermore, let, without loss of generality, $A \neq \mathbb{R}^n$, i.e. $U = \mathbb{R}^n \setminus A$ is supposed to be open and not void. Then, we have

$$U = \bigcup_{j=0}^{\infty} \overline{B}_j$$

with open balls $B_j$, and there are functions $f_j \geq 0$ on $\mathbb{R}^n$ satisfying

$$f_j(x) > 0 \text{ if and only if } x \in B_j .$$

By multiplication with a suitable constant we can achieve that for given $\varepsilon_j > 0$ the estimates

$$\sup_{\mathbb{R}^n} | D^\nu f_j | \leq \varepsilon_j$$

are satisfied for all $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ with $|\nu| \leq j$. Choose the numbers $\varepsilon_j$ such that the series $\sum_{j=0}^\infty \varepsilon_j$ is convergent. Let now $\nu \in \mathbb{N}^n$ be fixed; then the series $\sum_{j\geq|\nu|} \varepsilon_j$ is a majorant to $\sum_{j\geq|\nu|} D^\nu f_j$. Hence, the series

$$\sum_{j=0}^{\infty} D^\nu f_j$$

are absolutely and uniformly convergent, and consequently $f = \sum_{j=0}^\infty f_j$ is a $\mathcal{C}^\infty$–function on $\mathbb{R}^n$. For $x \in A$ are all $f_j(x) = 0$, thus $f(x) = 0$; for $x \notin A$, i.e. $x \in U$, we have $x \in B_j$ for at least one $j$ and therefore

$$f(x) \geq f_j(x) > 0 . \qquad \square$$

Similarly, one can show the following Theorem.

**Theorem 1.20** *Let $A \subset M$ a closed subset. Then, there is a $\mathcal{C}^\infty$–function $f$ on $M$ such that $A$ is the critical locus of $f$.*

*Proof*. Exercise. $\square$

## 1.B.2 The Theorem of Sard

The more surprising is the following theorem.

**Theorem 1.21 (Sard)** *Let $f : M \to N$ be a differentiable map. Then, the set of critical values is meager in $N$, i.e. of Lebesgue measure zero in any coordinate chart. In particular, the set of regular values is dense in $N$, if* $\dim N > 0$.

First, some remarks on the formulation of the Theorem and some preparations for its *proof* are in order. A set $C \subset \mathbb{R}^n$ is called *of* LEBESGUE*–measure zero* (or *meager*) if for any $\varepsilon > 0$ there exists a countable covering by squares

$$\bigcup_{j\in\mathbb{N}} W_j \supset C$$

with

$$\sum_{j\in\mathbb{N}} \mathrm{Vol}_n W_j \, < \varepsilon \, .$$

This is equivalent to the statement that the characteristic function $\chi_C$ is LEBESGUE*–integrable* with

$$\int_{\mathbb{R}^n} \chi_C \, dx^n \, = \, 0 \, .$$

It is evident that the following hold true:

1. If $C$ is of measure zero and $D \subset C$, then $D$ is of measure zero, too.
2. If $C_j$, $j \in \mathbb{N}$, are meager sets, then also

$$C \, = \bigcup_{j\in\mathbb{N}} C_j \, .$$

Further, we remind the reader to the following well–known

**Theorem 1.22** *Let $U \subset \mathbb{R}^n$ be open, $C \subset U$ meager and $f : U \to \mathbb{R}^n$ a $\mathcal{C}^1$–map. Then, $f(C)$ is meager in $\mathbb{R}^n$.*

*Proof*. $U$ is a union of countably many compact cubes $W_j$. We may, therefore, suppose that $C \subset \overline{W}_0 \subset W \subset\subset U$ and that $f_{|\overline{W}}$ can be extended to a $\mathcal{C}^1$–map $\mathbb{R}^n \to \mathbb{R}^n$, i.e. we suppose without loss of generality $U = \mathbb{R}^n$. Putting $K := \sup_{\overline{W}} \| Df \|$, we get due to the Mean Value Theorem

$$(+) \qquad \| f(x_1) - f(x^{(0)}) \| \, \le \, K \cdot \| x_1 - x^{(0)} \| \, , \quad x^{(0)}, x_1 \in \overline{W} \, .$$

To every $\epsilon > 0$ and $L > 0$ there exists a (new) covering by cubes

$$C \subset \bigcup_{j=1}^{\infty} W_j \ \text{ with } \ \sum_{j=1}^{\infty} \mathrm{Vol}_n W_j \, \le \, \frac{\epsilon}{L}$$

and $W_j \subset W$ for all $j$. If $a_j$ denotes the length of the edges of the cube $W_j$ and $x_j$ its center, then $f(W_j)$ is, because of (+), contained in the ball with center $x_j$ and radius $K\, a_j \sqrt{n}/2$ and therefore also in a cube $\widetilde{W}_j$ of edgelength $K\, a_j \sqrt{n}$. From this, we deduce

$$f(C) \subset \bigcup_{j=1}^{\infty} \widetilde{W}_j \, , \quad \mathrm{Vol}_n \widetilde{W}_j \, = \, K^n \sqrt{n}^{\,n} \, \mathrm{Vol}_n W_j$$

and finally

$$\sum_{j=1}^{\infty} \mathrm{Vol}_n \widetilde{W}_j \, \le \, \varepsilon \ \text{ if } \ L := \, K^n \sqrt{n}^{\,n} \, . \qquad \square$$

**Corollary 1.23** *A subset $C$ of a manifold is meager if, and only if, its intersection with each chart of a fixed countable atlas is meager. In particular, the above Theorem remains true also for maps $f : M \to N$, $\dim M = \dim N$.*

With this result and the arguments of the proof of Theorem 19 we immediately see that it is sufficient to prove the following local version of the Theorem of Sard.

**Theorem 1.24** *The set of critical values of a differentiable map*

$$f : U \longrightarrow \mathbb{R}^p\,, \quad U \subset \mathbb{R}^n \text{ open}$$

*is meager.*

The *proof* will be performed in several steps. We set

$$C_0 := C = \{\, x \in U : \mathrm{rg}_x Df < p \,\}$$

and

$$C_j = \{\, x \in U : (D^\nu f_k)(x) = 0\,,\ 1 \le |\,\nu\,| \le j\,,\ k = 1, \ldots, p \,\}\,.$$

Then, $C_0 \supset C_1 \supset C_2 \supset \cdots$, and thus

$$C_0 = (C_0 \setminus C_1) \cup (C_1 \setminus C_2) \cup \cdots \cup (C_{j-1} \setminus C_j) \cup C_j$$

for each $j$.

Let us first prove that the set $f(C_0)$ is meager in $\mathbb{R}^p$. For this, the following suffices.

**Theorem 1.25** *To each point $x^{(0)} \in U$ there exists a neighborhood $V = V(x^{(0)}) \subset U$ such that the sets*

$$f((C_0 \setminus C_1) \cap V)\,, \ldots, f((C_{j-1} \setminus C_j) \cap V)\,,\ f(C_j \cap V)$$

*are meager for $j \ge n/p - 1$.*

*Proof*. We proceed by induction on $n$, where for $n = 0$ nothing has to be shown. Therefore, we may assume in the following that the Theorem of Sard has been proven already for $n - 1 \ge 0$.

a) Regard $C_0 \setminus C_1$. Here, we may suppose moreover that $p \ge 2$, because otherwise $C_0 = C_1$. Let $x^{(0)} \in C_0 \setminus C_1$ and, without loss of generality,

$$x^{(0)} = 0\,, \quad \frac{\partial f_1}{\partial x_1}(0) \ne 0\,.$$

Denote by $\varphi : U \to \mathbb{R}^n$ the map $\varphi(x) = (f_1(x), x_2, \ldots, x_n)$. Due to the Invertible Mapping Theorem, $\varphi$ is invertible in a neighborhood $V$ of $0$, and in the diagram

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & V' \arrow[dl, "g\, =\, f \circ \varphi^{-1}"] \\ & \mathbb{R}^p & \end{tikzcd}$$

the map $g$ is of the form

$$g(x_1, \ldots, x_n) = (x_1, g_2(x), \ldots, g_p(x))\,.$$

Since $\varphi(C \cap V) = C'$ is the critical set of $g$, we also have $g(C') = f(C)$. We may therefore suppose that $f = g$ and, consequently, that $f$ maps the hyperplane $\{\, x_1 = t \,\}$ onto the hyperplane $\{\, y_1 = t \,\}$, $(y_1, \ldots, y_p) \in \mathbb{R}^p$. Let $g^t$ be the respective restriction, regarded as a map of an open part of $\mathbb{R}^{n-1}$ to $\mathbb{R}^{p-1}$. Because of

$$D\,g = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & \partial g^t_j/\partial x_k \end{array}\right)$$

a point in $(\{\, t \,\} \times \mathbb{R}^{n-1}) \cap V'$ is critical for $g$ if and only if it is critical for $g^t$. – In order to finish the proof in this case we need a Theorem of Fubini–*type*.

**Theorem 1.26** *Let $D \subset \mathbb{R}^p$ be the union of countably many compact sets, $p \geq 2$, and suppose that*

$$D_t = D \cap \mathbb{R}^p_t\,, \quad \mathbb{R}^p_t = \{\,(y_1, \ldots, y_p) \in \mathbb{R}^p : y_p = t\,\}$$

*is meager in $\mathbb{R}^p_t = \mathbb{R}^{p-1}$ for each $t \in \mathbb{R}$. Then, $D$ is meager in $\mathbb{R}^p$.*

This is an easy consequence of the Theorem of FUBINI since compact sets are LEBESGUE–*measurable*.
Sets which satisfy the more general assumptions made in the preceding Theorem are

a) *closed* subsets $A$ because of $A = \bigcup_{R>0} A \cap \overline{B}_R$,

b) *open sets* (since they are unions of countably many compact balls),

c) *continuous* images of such sets,

d) countable unions and finite intersections of such sets.

Since the sets $C_j$ are closed in $U$, the sets $f\,(C_j)$ and $f\,(C_{j-1} \setminus C_j)$ are of this type. – This argument finishes the proof of part a).

b) Let $x^{(0)} = 0 \in C_{j-1} \setminus C_j$, $j \geq 2$. Then, without loss of generality,

$$\frac{\partial^j f_1}{\partial x_1 \partial x_{s_1} \cdot \ldots \cdot \partial x_{s_{j-1}}}\,(0) \neq 0\,.$$

Set $w\,(x) = \partial^{j-1} f_1 \,/\, \partial x_{s_1} \cdot \ldots \cdot \partial x_{s_{j-1}}$. Due to $x^{(0)} \in C_{j-1}$, we infer $w\,(0) = 0$, but $\partial w / \partial x_1\,(0) \neq 0$. Regard now as in part a) the transformation $\varphi\,(x) = (w\,(x), x_2, \ldots, x_n)$ and conclude similarly by induction.

c) Let $W$ be a compact cube of side length $a$, $W \subset U$. From the TAYLOR formula we conclude

$$(++) \qquad \| f\,(x + h) - f\,(x) \| \leq K \, \| h \|^{j+1}$$

for $x \in C_j \cap W$, $x + h \in W$. Decompose $W$ in $r^n$ cubes of side length $a/r$. If $W_1$ denotes one of these cubes and if $x^{(0)} \in C_j \cap W_1$, then each point in $W_1$ can be written in the form $x^{(0)} + h$ with $\| h \| \leq \sqrt{n} \cdot a / r$. Thus, because of (++), $f\,(W_1)$ is lying in a cube of edge length

$$2K\,(\sqrt{n} \cdot a)^{j+1} / r^{j+1} = \frac{b}{r^{j+1}}$$

with a constant $b = b\,(W, f)$. These cubes have altogether a volume sum

$$r^n \cdot \frac{b^p}{r^{p(j+1)}} = b^p \, \frac{1}{r^{p(j+1)-n}}\,,$$

and this expression becomes arbitrarily small for sufficiently fine subdivisions of $W$ if $p\,(j+1) - n > 0$, i.e. $j > \dfrac{n}{p} - 1$. □

# Notes and References

(Local) complex analytic geometry will be treated to a higher extend in Chapters 2, 3 and 6 of this book. Our main source for this topic is the second volume of the Grauert–Remmert trilogy:

**[01 - 01]** H. Grauert, R. Remmert: Coherent Analytic Sheaves. Grundlehren der mathematischen Wissenschaften 265, Berlin–Heidelberg–New York–Tokyo: Springer–Verlag 1984.

For instance, the Weierstraß Preparation Theorem can be found there on p. 42. The factoriality of the ring $\mathbb{C}\,\langle\, x \,\rangle = \mathcal{O}_0^{(n)}$ which we used in Section 6 is proved on p. 44. We also make permanent use of the *noetherian* and *henselian* property of the local ring $\mathcal{O}_0^{(n)}$. Concerning such purely algebraic notions and results, we often refer to the first volume of the above mentioned trilogy:

**[01 - 02]** H. Grauert, R. Remmert: Analytische Stellenalgebren. Unter Mitarbeit von O. Riemenschneider. Die Grundlehren der mathematischen Wissenschaften 176, Berlin–Heidelberg–New York: Springer–Verlag 1971.

The English speaking reader or the impatient may find it more convenient to consult Chapter I, *Local Theory of Complex Spaces*, written by R. Remmert, in:

**[01 - 03]** H. Grauert, Th. Peternell, R. Remmert: Several Complex Variables VII. Sheaf–Theoretical Methods in Complex Analysis. Encyclopaedia of Mathematical Sciences, Volume 74, Berlin–Heidelberg–New York–London–Paris–Tokyo–Hong Kong–Barcelona–Budapest: Springer–Verlag 1994.

The *Chain Rule* and the *Inverse Mapping Theorem* are most easily derived from the corresponding results for *differentiable real–valued* maps using in addition the Cauchy–Riemann equations. A proof along these lines is found e.g. in the booklet

**[01 - 04]** H. Grauert, K. Fritzsche: Einführung in die Funktionentheorie mehrerer Veränderlichen. Springer Hochschultext, Berlin–Heidelberg–New York: Springer–Verlag 1974

resp. in its English edition which appeared under the title *Several Complex Variables* as Graduate Text in Mathematics 38 at Springer Heidelberg in 1976. The latter has been substantially rewritten and vastly expanded as

**[01 - 05]** K. Fritzsche, H. Grauert: From Holomorphic Functions to Complex Manifolds. Graduate Texts in Mathematics 213, New York etc.: Springer–Verlag 2002.

There exist now quite a lot of excellent and extensive introductions to several complex variables which treat the local and global theory at the same time. We list some of them in chronological order:

**[01 - 06]** R. Gunning, H. Rossi: Analytic Functions of Several Complex Variables. Prentice–Hall Series in Modern Analysis, Englewood Cliffs, N.Y.: Prentice–Hall, Inc. 1965.

**[01 - 07]** L. Hörmander: An Introduction to Complex Analysis in Several Variables. The University Series in Higher Mathematics, Princeton, N.J.: D.van Nostrand Company, Inc. 1966.

**[01 - 08]** H. Whitney: Complex Analytic Varieties. Addison–Wesley Series in Mathematics, Reading, Mass.: Addison–Wesley Company, Inc. 1972.

**[01 - 09]** L. Kaup, B. Kaup: Holomorphic Functions of Several Variables. de Gruyter Studies in Mathematics 3, Berlin–New York: Walter de Gruyter 1983.

We finally mention as a very useful general introduction and survey (without complete proofs):

**[01 - 10]** G. Fischer: Complex Analytic Geometry. Lecture Notes in Mathematics 538, Berlin–Heidelberg–New York: Springer–Verlag 1976,

and for the more algebraically oriented reader the books of

**[01 - 11]** S. S. Abhyankar: Local Analytic Geometry. Pure and Applied Mathematics, New York–London: Academic Press 1964

and

**[01 - 12]** Th. de Jong, G. Pfister: Local Analytic Geometry. Basic Theory and Applications. Advanced Lectures in Mathematics. Braunschweig/Wiesbaden: Friedr. Vieweg & Sohn.

The other books on singularities which we mentioned in the preface to this volume are the following:

**[01 - 13]** H. B. Laufer: Normal two–dimensional Singularities. Annals of Mathematics Studies 71, Princeton, New Jersey: Princeton University Press and University of Tokyo Press 1971.

**[01 - 14]** K. Lamotke: Regular Solids and Isolated Singularities. Advanced Lectures in Mathematics, Braunschweig–Wiesbaden: Friedr. Vieweg & Sohn 1986.

**[01 - 15]** D. Bättig, H. Knörrer: Singularitäten. Lectures in Mathematics. ETH Zürich. Basel–Boston–Berlin: Birkhäuser 1991.

**[01 - 16]** A. Dimca: Topics on Real and Complex Singularities. Advanced Lectures in Mathematics, Braunschweig–Wiesbaden: Friedr. Vieweg & Sohn 1987.

**[01 - 17]** T. Okuma: Plurigenera of Surface Singularities. Huntington, N. Y.: Nova Science Publishers 2000.

**[01 - 18]** W. Barth, C. Peters, A. Van de Ven: Compact Complex Surfaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin–Heidelberg–New York–Tokyo: Springer–Verlag 1984.

**[01 - 19]** E. Looijenga: Isolated Singular Points on Complete Intersections. London Math. Soc. Lecture Note Series 77, Cambridge: Cambridge University Press 1984.

**[01 - 20]** V. I. Arnol'd, S. M. Gusein–Zade, A. N. Varchenko: Singularities of Differentiable Maps, Volume I. Monographs in Mathematics, Boston–Basel–Stuttgart: Birkhäuser 1985.

**[01 - 21]** V. I. Arnol'd, S. M. Gusein–Zade, A. N. Varchenko: Singularities of Differentiable Maps, Volume II. Monographs in Mathematics, Boston–Basel–Stuttgart: Birkhäuser 1988.

The wonderful (real) pictures of surfaces with singularities are taken from Herwig Hauser's *Gallery* that can be found on his homepage under

www.homepage.univie.ac.at/herwig.hauser/gallery.html

The picture on the frontispiece shows Wolf Barth's famous *sextic* in a version that has been prepared by Andreas Leipelt. He also created a rotating icon that still appears on the homesite of my *Arbeitsbereich Analysis und Differentialgeometrie* at the Mathematics Department of the University Hamburg.

Barth published his construction in

**[01 - 22]** W. Barth: Two projective surfaces with many nodes, admitting the symmetries of the icosahedron. J. Algebraic Geom. 5, 173 - 186 (1996).

For more on this sextic, in particular the history of Theorem 11, and the life and other achievements of Wolf Barth, see:

**[01 - 23]** Th. Bauer et alii: Wolf Barth (1942-2016). Jahresber Dtsch Math-Ver 119, 273 - 292 (2017).

For the Appendix, our sources have been the following:

**[01 - 24]** Th. Bröcker: Differenzierbare Abbildungen. Der Regensburger Trichter, Band 3. 2. erweiterte und verbesserte Auflage. Regensburg 1973.

**[01 - 25]** Th. Bröcker, L. Lander: Differentiable Germs and Catastrophes. London Mathematical Society Lecture Notes 17. Cambridge: Cambridge University Press 1975.

**[01 - 26]** M. Golubitsky, V. Guillemin: Stable Mappings and Their Singularities. New York–Heidelberg–Berlin: Springer 1973.

A highly welcome addition to the literature on singularities is Shihoko Ishii's *Introduction to Singularities* that appeared in its original Japanese version in 1997 and in its first English printing in 2014. We consulted the second edition many times during the process of the revision of our text and recommend it warmly to everybody interested in the subject including higher dimensional singularities.

**[01 - 27]** Sh. Ishii: Introduction to Singularities. 2nd edition. Springer Japan K.K. 2018.

An extremely useful, but dense introduction - especially to the question, which topological information determines the analytic structure of a surface singularity (obviously the basis of a book project) - is given by

**[01 - 28]** A. Nemethi: Five Lectures on Normal Surface Singularities. Proceedings of the Summer School, Bolyai Society Mathematical Studies $\underline{8}$, Low Dimensional Topology, 269 - 351 (1999).