37 elementary axiomatic characterizations of the real number field*

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Introduction

In my last Analysis I course during winter semester 1997/98 I tried to group the fundamental statements on sequences and series of real numbers and on continuous, differentiable and integrable functions on bounded closed intervals in such a way that one can derive their equivalence to the completeness of the real number field without too much work. The prize one has (or better: the students have) to pay for this is a very early initiation to quite general *topological* concepts (which have to be introduced anyway during the second or third semester). So, one doesn't loose time in the long run, but students might gain deeper insight into the basics of mathematics.

I also addressed in bypassing the question if one can weaken the quite often assumed property of the field to be "archimedean". If possible without extra labor, I eliminated that property or replaced it by the assumption that the ground field under consideration possesses *nontrivial sequences converging to zero* (see the condition $(*)$ in the supplement to the first circle and the beginning of the second circle). Since this class of ordered fields is properly larger than that of archimedean ordered fields, but yet proofs can be performed as in the case of classical real analysis via convergent sequences I consider it as a "didactically permissible" superclass.

Most of the material presented here should be well known although some of it (e. g. in connection with differentiable functions and the main theorem of differential and integral calculus) I couldn't find explicitly in this form in other publications. However, I made no effort to study the extensive literature devoted to this circle of ideas thoroughly (partly because it is not easily accessible). A manuscript of similar spirit which I became aware of only after the end of my course is [7].

In any case it is easy to carry out many, if not all of these considerations oneself. The present note may therefore be considered merely as a "Leitfaden" for interested students and colleagues. It goes without saying that they form only the sheer skeleton of my course (a complete elaboration of which is in preparation; see [6]). For the same reason I didn't find it necessary to give here any kind of motivating comments. Let me add the remark that the number appearing in the title of the present note has no mystical significance. In fact, it can be altered at will by either deleting those which are not really needed for classical analysis or by adding some of the many other rather fancy characterizations of the real numbers.

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I would be glad about constructive criticism and hints on further sources with similar intention. My thanks are due to my former collaborator Dr. Andreas Leipelt and my assistant Dr. Jörg Schürmann for their willingness to discuss this not quite world shaking issue with me. My special thanks go to my colleague Prof. Dr. Alexander Prestel of Konstanz who, in letters and oral communication, helped me considerably to improve my knowledge on nonarchimedean fields, especially by his reference to the η_1–fields, and directed a question on the independence of certain systems of axioms to the right path.

1 The first circle

In the following, $\mathbb{K}$ denotes always an ordered field. In order to save labor one should introduce the notions of *convergence of a sequence* and of a *Cauchy sequence* etc. from the beginning in the more general context of $\mathbb{K}$–*metric spaces* and prove instantly the trivial consequences (e. g.: Convergent sequences are necessarily bounded Cauchy sequences). After that, one can show the equivalence of the following statements (1) till (5) e. g. by a *round trip* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1). Here, *Cauchy completeness* means that every Cauchy sequence possesses a (automatically uniquely determined) limit. The *principle of monotone convergence* states that every increasing sequence in $\mathbb{K}$ which is bounded from above has a limit. With respect to the principles of nested intervals, one has to consider closed intervals $I_j = [\,a_j,\, b_j\,] \subset \mathbb{K}$ with $I_j \supset I_{j+1}$, $j \in \mathbb{N}$. The *strong* resp. *weak principle of nested intervals* demands that their intersection is not void, in the second case under the extra assumption that the sequence $b_j - a_j$ of the lengths of the intervals I_j tends to zero. The *theorem of Bolzano and Weierstraß* asserts that from every bounded sequence in $\mathbb{K}$ one can extract a *convergent subsequence*. If one wants to avoid the early appearance of the axiom of Archimedes and the notion of a Cauchy sequence one can start the journey as well from e. g. the second place.

(1) $\mathbb{K}$ is archimedean & Cauchy complete

(2) Principle of monotone convergence

(3) $\mathbb{K}$ is archimedean & Strong principle of nested intervals

(4)	$\mathbb{K}$ is archimedean & Weak principle of nested intervals

(5)	Theorem of Bolzano and Weierstraß

The *proofs* present no particular difficulties and could therefore be left to the reader. For the sake of completeness they shall be added.

(1) $\Longrightarrow$ (2) It is sufficient to show: *Every increasing sequence in an archimedean field which has an upper bound is a Cauchy sequence*. For, if this is not true, one can construct by induction a strongly increasing subsequence $(x_j)_{j\in\mathbb{N}}$ which is bounded from above such that $x_{j+1} - x_j \geq \varepsilon_0$ with a fixed $\varepsilon_0 > 0$. Hence, $x_j \geq x_0 + j\varepsilon_0$ for all $j \in \mathbb{N}$; however, due to the axiom of Archimedes, the righthand side would surpass every given bound for sufficiently large j. Contradiction!

(2) $\Longrightarrow$ (3) If $\mathbb{K}$ wouldn't be archimedean the sequence $(j)_{j\in\mathbb{N}}$ would have to be bounded, hence it is convergent and in particular a Cauchy sequence. But then, e. g. $(j+1)-j=1$ has for large enough j to be smaller than any given $\varepsilon > 0$ which is impossible (choose $\varepsilon = 1$). Contradiction! The strong principle of nested intervals follows immediately from applying the principle of monotone convergence to the sequences of left (resp. right) endpoints of the intervals I_j.

(3) $\Longrightarrow$ (4) Trivial.

(4) $\Longrightarrow$ (5) Well known. (Use the method of dividing the intervals into halves).

(5) $\Longrightarrow$ (1) That the theorem of Bolzano and Weierstraß implies the axiom of Archimedes follows exactly as in the proof of (2) $\Longrightarrow$ (3); one has only to replace the sequence $(j)_{j\in\mathbb{N}}$ by a suitable subsequence. The remaining part is a consequence of the following general fact which is easy to prove: *A Cauchy sequence in a $\mathbb{K}$–metric space which possesses a convergent subsequence is convergent.* $\square$

Remark. In the step from (1) to (2) we used the ordinary formulation of the axiom of Archimedes: *For each pair $a, b \in \mathbb{K}$ of positive elements there exists a positive integer j such that $a < jb$.* Before one can carry out the proof of the next step one has to demonstrate that this condition is equivalent to the unboundedness of the sequence $(j)_{j\in\mathbb{N}}$ in $\mathbb{K}$. For economical reasons, one should introduce the notion of *convergence to infinity* for sequences in ordered fields, show that a sequence (a_j) is convergent to ∞ if and only if there exists $N \in \mathbb{N}$

with $a_j > 0$ for all $j \geq N$ and

$$\lim_{j\to\infty} \frac{1}{a_{j+N}} = 0 \,,$$

and prove the following Proposition.

Proposition *For any ordered field $\mathbb{K}$ the following assertions are equivalent:*

i) *The sequence $(j)_{j\in\mathbb{N}}$ is unbounded in $\mathbb{K}$, i. e. $\lim\limits_{j\to\infty} j = \infty$.*

ii) *The sequence $\left(\frac{1}{j}\right)_{j\in\mathbb{N}^*}$ converges in $\mathbb{K}$ to zero.*

iii) *For every pair $a,\, b \in \mathbb{K}$ of positive elements it exits a positive integer j such that $a < jb$.*

iv) *For each $a \in \mathbb{K}$ with $|\,a\,| < 1$ one has $\lim\limits_{j\to\infty} a^j = 0$.*

v) *For given fixed integer $g \geq 2$ each positive element $a \in \mathbb{K}$ has an g–*adic expansion

$$a = \sum_{k=\ell}^{\infty} a_k\, g^{-k} \,, \quad \ell \in \mathbb{Z} \,, \quad a_k \in \{\,0,\, 1, \ldots, g-1\,\} \subset \mathbb{N} \,.$$

vi) *Each element $a \in \mathbb{K}$ is the limit of a convergent sequence of rational numbers.*

vii) *Every interval $[\,a,\, b\,] \subset \mathbb{K}$ with $a < b$ contains elements of $\mathbb{Q}$.*

Addendum *In* vi) *the sequence of rational numbers can be chosen as an increasing one.*

Proof. i) $\Longrightarrow$ ii) This has been explained before in greater generality.

ii) $\Longrightarrow$ iii) b/a is positive. Hence there is an j with $1/j < b/a$ and consequently $a < jb$.

iii) $\Longrightarrow$ iv) With $|\,a\,| < 1$ we have $|\,a^{-1}\,| > 1$ and $a^{-1} = 1 + x$ where $x \in \mathbb{K}$ is positive. Because of iii) we find to each $\varepsilon > 0$ an integer $N \in \mathbb{N}$ such that for all $j \geq N$:

$$jx > \varepsilon^{-1} - 1 \,.$$

The *Bernoulli* inequality implies $(1+x)^j \geq 1 + jx$ and hence

$$|\,a^j\,| = |\,a\,|^j = \frac{1}{(1+x)^j} \leq \frac{1}{1+jx} < \varepsilon \,.$$

iv) $\Longrightarrow$ v) Since $\lim\limits_{j\to\infty} g^j = \infty$, there exists to each given $a > 0$ a uniquely determined integer $\ell \in \mathbb{Z}$ satisfying $g^{-\ell} \leq a < g^{-\ell+1}$ and further a unique element $a_\ell \in \{\,1, \ldots, g-1\,\}$ such that $a_\ell g^{-\ell} \leq a < (a_\ell + 1)g^{-\ell} \leq g^{-\ell+1}$ and thus

$$0 \leq a - a_\ell g^{-\ell} < g^{-\ell} \,.$$

If $a^{(1)} := a - a_\ell g^{-\ell} = 0$, we are done. If this is not the case there exists a unique $k > \ell$ with $g^{-k} \leq a^{(1)} < g^{-k+1}$ and an element $a_k \in \{\,1, \ldots, g-1\}$ such that $a_k g^{-k} \leq a^{(1)} < (a_k+1)g^{-k} \leq g^{-k+1}$ whence

$$0 \leq a^{(2)} := a^{(1)} - a_k g^{-k} = a - \left(a_\ell g^{-\ell} + a_k g^{-k}\right) < g^{-k} \,.$$

Inductively proceeding we gain a g–adic sequence which, by the very construction and the assumption iv), does converge to a.

v) $\Longrightarrow$ vi) Choose for positive a the partial sums of a g–adic expansion. For negative a look at the expansion of $-a$.

vi) $\Longrightarrow$ vii) The center $m := (a + b)/2$ of the interval $[a, b]$ satisfies $a < m < b$. Since m can arbitrarily closely approximated by rational numbers there are elements $q \in \mathbb{Q}$ with $a < q < b$.

vii) $\Longrightarrow$ i) Let $K \in \mathbb{K}$ be an arbitrary (positive) bound. Then we find a positive rational number $q = n_0/m$, $n_0, m \in \mathbb{N}^*$ with $K < q < K + 1$ which implies $K < q \leq n$ for all $n \geq n_0$.

The addendum to vi) is clear for positive elements a due to the proof. If a is negative we find, since $\mathbb{K}$ must be archimedean by the preceding considerations, a positive integer n with $-a < n$. Hence $a + n$ is positive and thus an increasing limit of rational numbers. Then the claim is correct for a, too. $\square$

Remarks. 1. The g–adic expansion v) is especially valid for each *rational* number. It is well known that exactly the *finally periodic* expansions describe the rational numbers.

2. A field $\mathbb{K}$ which satisfies the equivalent conditions (1) up to (5) consists of *all* such series (and their negatives); see also the last section. From this it is clear (and could be elaborated on at this place) that each archimedean ordered field can be embedded in such a field $\mathbb{K}$ as an ordered subfield and that the field $\mathbb{K}$ is (up to order preserving isomorphism of fields) uniquely determined.

Supplement to the first circle

We document with the aid of an example that in characterizing the real numbers via the axiom of Cauchy completeness or the principles of nested intervals one can't dispense with the axiom of Archimedes. Further examples show that it is even not possible to relax it, say by assuming

$(*)$ $\mathbb{K}$ has nontrivial sequences converging to zero

or

$(**)$ $\mathbb{K}$ possesses analytically nilpotent elements, i. e. elements $a \neq 0$ with $\lim_{j\to\infty} a^j = 0$.

Notice that $(**)$, and consequently $(*)$, is in fact true in archimedean ordered fields because of item iv) in the proposition above.

The first examples we would like to mention are the so called η_1*–fields* (see e. g. [5]); these are ordered fields with the following property: for each pair of at most countably infinite (possibly empty) subsets A, B in $\mathbb{K}$ with $A < B$ (i. e. $a < b$ for all elements $a \in A$, $b \in B$) there exists an element $x \in \mathbb{K}$ satisfying $A < \{x\} < B$. Structures with this separation property have been introduced by Hausdorff [2]. An example of such a field is the field ${}^*\mathbb{K}$ of the so called *nonstandard* real numbers (cf. e. g. [4]). Applying the separation property to $A = \mathbb{N} \subset \mathbb{K}$, $B = \emptyset$, one can see at once that $\mathbb{K}$ is nonarchimedean. Setting, for an arbitrary sequence (x_k):

$$A = \{0\}, \quad B = \{|x_j - x_k| > 0 : j, k \in \mathbb{N}\},$$

and choosing ε with $A < \{\varepsilon\} < B$, it follows easily that every Cauchy sequence in $\mathbb{K}$ is trivial, in particular convergent. Thus, such η_1–fields are examples of Cauchy complete fields which do not possess nontrivial sequences converging to zero. Therefore, they obey not only, by trivial reasons, the weak principle of nested intervals, but also the strong one: if $I_j = [\,a_j,\, b_j\,]$ is a sequence of nested intervals and if, without loss of generality, $a_j < b_j$ for all j, one applies the separation principle to the sets

$$A = \{\, a_j : j \in \mathbb{N} \} \quad \text{and} \quad B = \{\, b_j : j \in \mathbb{N} \} \,.$$

Now we come to an example of a nonarchimedean ordered field which satisfies the weak principle of nested intervals and contains analytically nilpotent elements. We follow closely the presentation of Efimow [1]. Let $\mathbb{K}$ be an arbitrary ordered field. It is easily seen that the ring $\mathbb{L}$ of *formal Laurent series*

$$(+) \qquad p = a_0\, T^n + a_1\, T^{n+1} + a_2\, T^{n+2} + \cdots \,, \quad a_j \in \mathbb{K}\,,\ n \in \mathbb{Z}$$

forms a field since each formal power series $1 + a_1 T + a_2 T^2 + \cdots$ has an inverse. For $p \neq 0$ there exists a uniquely determined integer $n \in \mathbb{Z}$ such that the coefficient a_0 in the representation $(+)$ is different from 0. We call $n = n\,(p)$ the *order* of p (for $p = 0$ one has to put $n\,(p) = -\infty$). A Laurent series p is called *positive* if $n\,(p) > -\infty$ and $a_0 > 0$. The set P of positive elements satisfies the properties

$$P \cup (-P) \cup \{0\} = \mathbb{K}\,, \quad P \cap (-P) = \emptyset\,, \quad P + P \subset P \quad \text{and} \quad P \cdot P \subset P\,,$$

hence it defines an ordering on $\mathbb{L}$. As one can easily convince oneself,

$$\lim_{j \to \infty} T^j = 0\,.$$

So, $\mathbb{L}$ contains analytically nilpotent elements. Furthermore, $0 < T < a$ for all $a \in \mathbb{K}$, $a > 0$, and consequently $j < T^{-1}$ for all $j \in \mathbb{N}$. Thus, $\mathbb{L}$ is a nonarchimedean ordered field.

$\mathbb{L}$ satisfies for arbitrary $\mathbb{K}$ the weak axiom of nested intervals. This can be seen as follows: Let $I_j = [\,p_j,\, q_j\,]$ be a sequence of nested intervals in $\mathbb{L}$. Since one can replace it without loss of generality by a subsequence, one may assume that $p_j < p_{j+1} < q_{j+1} < q_j$ for all j (since otherwise the intersection $\bigcap I_j$ is not empty by trivial reasons). Put $n_j = n\,(p_j)$. This sequence has to be bounded from below; for if $\lim n_j = -\infty$ (without loss of generality we can again select a subsequence) and if, by the same argument, $n_j > n_{j+1}$ for all j, all leading coefficients $a_0^{(j)}$ of

$$p_j = a_0^{(j)}\, T^{n_j} + a_1^{(j)}\, T^{n_j+1} + \cdots \,, \quad j \geq 1$$

would have to be positive because of $p_{j+1} - p_j > 0$. But then, for given k, we would have $n_j < n\,(q_k)$ for sufficiently large j and thus

$$p_j - q_k > 0$$

which is impossible. By the same argument applied to $-q_j < -q_{j+1} < -p_k$ we get the boundedness of the sequence $n\,(q_j)$ from below. Therefore, it exists a number $m \in \mathbb{Z}$ such that we can write

$$p_j = a_0^{(j)}\, T^m + a_1^{(j)}\, T^{m+1} + \cdots$$

$$q_j = b_0^{(j)}\, T^m + b_1^{(j)}\, T^{m+1} + \cdots \,,$$

where the leading coefficients $a_0^{(j)}$ and $b_0^{(j)}$ are, however, allowed to be zero. Now, from $p_j < p_{j+1} < \cdots < q_{j+1} < q_j$, we deduce

$$(++) \qquad a_0^{(j)} \le a_0^{(j+1)} \le \cdots \le b_0^{(j+1)} \le b_0^{(j)} \ .$$

If, in addition, $\lim\,(q_j - p_j) = 0$, we must have, for large $j \ge j_0$, the inequality $q_j - p_j < T^{m+1}$ which implies $b_0^{(j)} = a_0^{(j)}$ for all $j \ge j_0$. We set

$$c_0 := a_0^{(j)} = b_0^{(j)} \ , \quad j \ge j_0 \ .$$

Proceeding by induction, one constructs an increasing sequence

$$j_0 < j_1 < j_2 < \cdots$$

with

$$c_k := a_k^{(j)} = b_k^{(j)} \ , \quad j \ge j_k \ .$$

The series $\sum_{k \ge 0} c_k \, T^{k+m}$ belongs to all intervals I_j.

If $\mathbb{K}$, in particular, is an η_1–field, then $\mathbb{L}$ even obeys the strong principle of nested intervals. The first steps of the proof are the same as above including the inequality $(++)$. Now, one argues as follows: either $\lim \left(b_0^{(j)} - a_0^{(j)}\right) = 0$ and hence $c_0 = a_0^{(j)} = b_0^{(j)}$ for $j \ge j_0$ such that one can proceed with the next sequence of coefficients; or there exists an element c_0 with

$$a_0^{(j)} < c_0 < b_0^{(j)} \quad \text{for all} \quad j \ .$$

But then $p_j < c_0 T^m < q_j$ for all j.

If, on the other hand, $\mathbb{K}$ itself is archimedean, then the strong principle of nested intervals does not hold for the field $\mathbb{L}$ since, obviously, the intersection of the nested intervals

$$I_j = [\, jT, \, 1/j \,]$$

is void. (An analogous argument applies under the assumption that $\mathbb{K}$ has nontrivial sequences converging to zero).

In (1), the axiom of Archimedes can not be relaxed either. It is well known that each ordered field can be densely embedded as an ordered subfield into a Cauchy complete ordered field $\widetilde{\mathbb{K}}$. This implies immediately: If $\mathbb{K}$ contains analytically nilpotent elements so does $\widetilde{\mathbb{K}}$; $\mathbb{K}$ is nonarchimedean if and only if $\widetilde{\mathbb{K}}$ is nonarchimedean. Hence, there exist nonarchimedean Cauchy complete fields with analytically nilpotent elements. Moreover, by the same argument, there exists indeed a field satisfying all the properties (1) till (5), namely the Cauchy completion of the field $\mathbb{Q}$ of *rational numbers* (cf. the last section).

2 The second circle

The second circle deals essentially with the *axiom of suprema* and the *intermediate value theorem* and its logical relation to the notion of *connectedness*. In order to be able to carry out the following steps consistently one first has to develop the notion of a supremum of a nonempty subset of $\mathbb{K}$. Then, one has to introduce the notion of *open* sets in a $\mathbb{K}$–metric space X in the usual manner and at least to mention that each nonempty subset of such a space "inherits" a metric structure. *Connectedness* of a space means that it is impossible to decompose it into two disjoint non void open sets. A $\mathbb{K}$–metric space X is called *totally disconnected* if (besides the empty set) only the subsets with one element are connected. *Continuity* of mappings between such metric spaces will be introduced via the ε–δ–definition. It is easily seen that this is equivalent to the *sequential continuity* if the field $\mathbb{K}$ obeys the axiom $(*)$ (for the definition see the supplement to the first circle). The deeper reason for this statement lies simply in the fact that $\mathbb{K}$–metric spaces satisfy under the assumption of $(*)$ the *1. axiom of countability*. Thus, in this special case, subsets of such metric spaces are closed if and only if they are sequentially closed. The *axiom of Dedekind cuts* will be used in the usual formulation: If $\mathbb{K} = A \cup B$ with nonempty sets $A \leq B$ then there exists an element $c \in \mathbb{K}$ with $A \leq \{c\} \leq B$.

The second "semicircle" which however, due the first circle, does completely close up has the following shape: (4) $\Longrightarrow$ (6) $\Longrightarrow$ (7) $\Longrightarrow$ (8) $\Longrightarrow$ (9) $\Longrightarrow$ (10) $\Longrightarrow$ (11) $\Longrightarrow$ (12) $\Longrightarrow$ (2). The precise meaning is as follows:

(6) Existence of suprema for nonempty bounded sets

(7) $[0, 1]$ is connected

(8) $\mathbb{K}$ is not totally disconnected

(9) All intervals are connected

(10) Theorem of Bolzano (Intermediate value theorem)

(11) $\mathbb{K}$ is connected

(12) Dedekinds axiom of cuts

Remark. If one likes one can easily omit some of the items. Dedekind's axiom plays no role in the whole course, but should be considered for historical reasons.

Proof. (4) $\Longrightarrow$ (6) We reproduce the well known proof. Since the given set $A \subset \mathbb{K}$ is not empty and has an upper bound there exists at least one element $a_0 \in A$ and an upper bound $b_0 \in \mathbb{K}$ of A. If, by chance, $a_0 = b_0$, we are done; for then $a_0 = b_0$ is a *maximal* element of A, in particular a supremum. We thus may assume that the interval $I_0 = [\,a_0,\, b_0\,]$ doesn't consist of just one point. We now construct inductively a nested sequence of intervals $I_j = [\,a_j,\, b_j\,]$ with the following property: $I_j \cap A \neq \emptyset$, b_j is an upper bound of A, and the length of I_j is one half of the length of I_{j-1}. This is, besides the last requirement, fulfilled for I_0. If I_j is already constructed we look at the central point $m_j = (a_j + b_j)/2$ of the interval I_j. If m_j is an upper bound, we put $a_{j+1} = a_j,\, b_{j+1} = m_j$. In the other case we put $a_{j+1} = m_j,\, b_{j+1} = b_j$. In any case, we see that the interval $I_{j+1} = [\,a_{j+1},\, b_{j+1}\,]$ has the correct properties. Let K be the unique element in the intersection of these intervals. For arbitrary $a \in A$ we have $a \leq b_j$ for all j and hence $a \leq \lim\limits_{j \to \infty} b_j = K$. Consequently, K is an upper bound for A. If, conversely, $K' < K$, then there exists because of $\lim\limits_{j \to \infty} a_j = K$ an integer $k \in \mathbb{N}$ with $a_k > K'$. Since $I_k \cap A \neq \emptyset$, there exist elements $a \in A$ which are larger than K'. Hence, K' is not an upper bound for A. In other words: K is the smallest upper bound of A.

(6) $\Longrightarrow$ (7) Let U_0, U_1 be open sets in $\mathbb{K}$ with $I_0 \cup I_1 = I = [\,0,\, 1\,]$ and $I_0 \cap I_1 = \emptyset$ where $I_j := I \cap U_j,\ j = 0, 1$. Assume without loss of generality that $0 \in I_0$ and define $A := \{a \in I : [\,0,\, a\,] \subset I_0\}$. Since $0 \in A$ the set A is not empty, and it is bounded from above because of $A \subset I$. Hence the supremum α of A exists. We have to show that α belongs to A and that $\alpha = 1$; for then $I = I_0$ and $I_1 = \emptyset$. Suppose to the contrary that $\alpha \notin A$. Then $\alpha \notin I_0$ since otherwise we could conclude from the openness of U_0 that a whole interval $(\,\alpha - \varepsilon,\, \alpha\,]$ with positive ε would be contained in I_0 and also $[\,0,\, \alpha\,] = [\,0,\, a\,] \cup [\,a,\, \alpha\,]$ for suitable $a \in A$ with $\alpha - \varepsilon < a < \alpha$. Contradiction! So, we must have $\alpha \in I_1$ which, again, is impossible due to the openness of U_1. Hence, $\alpha \in A$ and $[\,0,\, \alpha\,] \subset I_0$. But, finally, if $\alpha < 1$ we can find, again by invoking the openness of U_0, an element $\alpha' > \alpha$ with $[\,0,\, \alpha'\,] \subset I_0$ which contradicts the definition of α.

(7) $\Longrightarrow$ (8) is absolutely trivial.

(8) $\Longrightarrow$ (9) Assume first that there exists a non trivial connected *interval* $I = [\,a,\, b\,]$. For $c \leq d$, the affine mapping

$$\alpha\,(x) := c + \frac{d - c}{b - a}\,(x - a)\,, \quad x \in \mathbb{K}$$

is continuous and maps I onto the interval $[\,c,\, d\,]$. Since due to a simple lemma continuous images of connected sets are connected the latter interval has the desired connectedness property, too. For an arbitrary interval J one concludes as follows: If J is not connected it is possible to write J as a disjoint union of non empty sets J_0, J_1 which are the intersection of J with open sets in $\mathbb{K}$. Choose $c \in J_0$ and $d \in J_1$ and assume without loss of generality that $c < d$. From our assumption on J it follows however that the interval $[\,c,\, d\,]$ is not connected in sharp contrast to the first part of our reasoning. Hence we only have to defend our first assumption. Let $Z \subset \mathbb{K}$ be a non trivial connected subset, and let a, b be elements in Z with $a < b$. We put $I = [\,a,\, b\,]$. If there would exist a $c \in I$ not belonging to Z we would have $Z = \{\,x \in Z : x < c\,\} \cup \{\,x \in Z : x > c\,\}$ in contradiction to the connectedness of Z; thus $I \subset Z$. If I were not connected we could find a continuous function f on I which admits precisely the values 0 and 1: if we write I as a non trivial disjoint union $I_0 \cup I_1$ with $I_j = I \cap U_j$, U_j open in $\mathbb{K}$, the function

$$f(x) := \begin{cases} 0, & x \in I_0 \\ 1, & x \in I_1 \end{cases}$$

is continuous on I. But then, the function f defined on Z by $f(x) := f(a)$, $x \le a$, $x \in Z$, $f(x) := f(b)$, $x \ge b$, $x \in Z$ is continuous and thus Z can't be connected since the image set $f(Z) = \{0, 1\}$ is not connected.

(9) $\Longrightarrow$ (10) Let $I = [a, b] \subset \mathbb{K}$ be an arbitrary closed interval and $f : I \to \mathbb{K}$ a continuous function. Assume without loss of generality that $f(a) < f(b)$ and choose c with $f(a) < c < f(b)$. Then the sets $I_0 := \{x \in I : f(x) < c\}$ and $I_1 := \{x \in I : f(x) > c\}$ are open in I, disjoint and not empty. If $f(x) \neq c$ for all $x \in I$, then $I = I_0 \cup I_1$. Contradiction!

(10) $\Longrightarrow$ (11) If $\mathbb{K}$ is not connected we can write $\mathbb{K} = U_0 \cup U_1$ with nonempty disjoint open sets U_0, U_1. Choose elements $a \in U_0$ and $b \in U_1$ and suppose $a < b$. Then, if we put $I_j := [a, b] \cap U_j$, $j = 0, 1$, the function

$$f(x) := \begin{cases} -1, & x \in I_0 \\ 1, & x \in I_1 \end{cases}$$

is continuous on I, but doesn't take on the value 0 lying in between $f(a) = -1$ and $f(b) = 1$.

(11) $\Longrightarrow$ (12) Let A, B be nonempty subsets of $\mathbb{K}$ with $A \cup B = \mathbb{K}$ and $A \le B$. If A and B would be open we could conclude that $A \cap B = \emptyset$ (otherwise there are elements $a \in A, b \in B$ with $a > b$), contradicting the connectedness of $\mathbb{K}$. Hence, at least one of the sets is not open, say A. Then there exists an element $c \in A$ such that $(c - \varepsilon, c + \varepsilon) \not\subset A$. Since all elements which are smaller than c belong to A, we necessarily have $[c, c + \varepsilon) \cap B \neq \emptyset$ for all $\varepsilon > 0$. This immediately implies $A \le \{c\} \le B$.

(12) $\Longrightarrow$ (2) Let $a_j \in \mathbb{K}$ be an increasing bounded sequence. We put

$$A := \{x \in \mathbb{K} : x \le a_j \text{ for some } j\}, \quad B := \{x \in \mathbb{K} : x > a_j \text{ for all } j\}.$$

Let c be determined by $A \le \{c\} \le B$. Due to our assumption, $a_j \le c$ for all $j \in \mathbb{N}$. If $\varepsilon > 0$ is arbitrarily given, we find j_0 such that $a_{j_0} > c - \varepsilon$ (otherwise, we would have $c - \varepsilon/2 \in B$). Since the sequence a_j is increasing it converges to c. □

We add still another characterization via the notion of *pathwise connectedness*. A $\mathbb{K}$–metric space X is called *pathwise connected* if to each pair of points $x_0, x_1 \in X$ there exists a continuous mapping $\gamma : I \to X$, $I := [0, 1] \subset \mathbb{K}$ (a *path*) with $\gamma(0) = x_0$, $\gamma(1) = x_1$. We claim that the former axioms are equivalent to

(13)

Pathwise connected $\mathbb{K}$–metric spaces are connected

This can be seen via (7) $\Longleftrightarrow$ (13) Let X be pathwise connected and suppose $X = U_0 \cup U_1$ with nonempty open subsets $U_0,\, U_1$ having no points in common. Choose $x_0 \in U_0,\, x_1 \in U_1$ and a continuous *path* $\gamma : I \to X$ with $\gamma(0) = x_0,\, \gamma(1) = x_1$. Then the preimages $I_j := \gamma^{-1}(U_j),\, j = 0, 1$, are disjoint open subsets of I with $I = I_0 \cup I_1$. Contradiction! On the other hand the unit closed interval I is pathwise connected since for all $a,\, b \in I$ the image of the continuous path $\gamma : I \to I$ with $\gamma(t) := a + t(b-a)$ is contained in I. $\square$

Remark. Since the existence of suprema implies the axiom of Archimedes as well as the principles of nested intervals, the axiom (6) can be replaced by the following statement.

$$(6)' \quad \boxed{\begin{array}{c}\text{To each nonempty bounded set } A \subset \mathbb{K} \\ \text{it exists an increasing sequence } (a_j) \\ \text{with } a_j \in A \text{ and } \lim a_j = \sup A\end{array}}$$

The third circle

In the following, the field $\mathbb{K}$ is still ordered, and $f : I := [\,a,\, b\,] \to \mathbb{K}$ denotes any function. For the notion of *differentiability* at a point $c \in I$ we use the *difference quotient*

$$(\Delta f)\,(x) \;=\; \frac{f\,(x) \,-\, f\,(c)}{x \,-\, c}\,, \quad x \in I\,, \quad x \neq c$$

and demand that we can find a (necessarily uniquely determined) number $A \in \mathbb{K}$ which we call the (first) *derivative* or *differential quotient* $f'\,(c)$ of f at c such that there exists to each given $\varepsilon > 0$ a $\delta > 0$ with

$$|\,(\Delta f)\,(x) \,-\, A\,| \,<\, \varepsilon$$

for all $x \in I$ with $0 \,<\, |\,x \,-\, c\,| \,<\, \delta$.

Remark. If the field fulfills the axiom $(*)$ then the definition of the derivative coincides with the following (a priori *formal*) one:

$$f'\,(c) \;:=\; \lim_{\substack{x\to c\\ x\neq c}} \frac{f\,(x) \,-\, f\,(c)}{x \,-\, c}\,,$$

where the existence of the limit means that

$$A \;:=\; \lim_{j\to\infty} \frac{f\,(x_j) \,-\, f\,(c)}{x_j \,-\, c}$$

exists for all sequences (x_j) with $x_j \in I$, $x_j \neq c$ and $\lim x_j = c$ and that A is independent of the special choice of the sequence (x_j). This definition is obviously meaningful only if there are such sequences in $\mathbb{K}$, that is only if $\mathbb{K}$ satisfies $(*)$.

It is immediately seen that differentiable functions are continuous: By the definition, there is to each given $\varepsilon > 0$ a $\delta > 0$ with

$$|\,f\,(x) \,-\, f\,(c)\,| \,<\, (\varepsilon \,+\, |\,f'(c)\,|)\,|\,x \,-\, c\,|$$

for all $x \in I$ with $x \neq c$ and $|x - c| < \delta$.

We say that f satisfies condition $(+)_n$, $n \in \mathbb{N}$ fixed, if f is n–times continuously differentiable on $I := [a, b]$ and if the n–th derivative $f^{(n)}$ is differentiable at least on the open interval (a, b). In the axioms stated below the meaning of the various theorems is as follows.

(15) Theorem of Rolle. Let $f : [a, b] \to \mathbb{K}$ be a function with $(+)_0$ and $f(a) = f(b)$, then there exists an element $\xi \in (a, b)$ such that $f'(\xi) = 0$.

(16) Generalized (or 2.) mean value theorem. Let $f, g : [a, b] \to \mathbb{K}$ be functions satisfying $(+)_0$ and $g'(x) \neq 0$ for all $x \in (a, b)$, then $g(a) \neq g(b)$, and it exists an element $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} .$$

(17) Mean value theorem. If $f : [a, b] \to \mathbb{K}$ satisfies the condition $(+)_0$ then there exists an element $\xi \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) .$$

(18) Theorem of Taylor expansion. If $f : [a, b] \to \mathbb{K}$ fulfills $(+)_n$, then, to each $x \in (a, b]$, there exists ξ with $a < \xi < x$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x) ,$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1}$$

(remainder term in Lagrange form).

(19) If f satisfies $(+)_n$ and $f^{(n+1)} = 0$ on (a, b) then f is a polynomial function of degree less or equal to n, i. e. a function of type

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n , \quad a_j \in \mathbb{K} .$$

(20) If f satisfies $(+)_0$ and $f' = 0$ on (a, b) then f is constant.

The third "circle" consists of the implications (4) $\Longrightarrow$ (14) $\Longrightarrow$ (15) $\Longrightarrow$ (16) $\Longrightarrow$ (17) $\Longrightarrow$ (18) $\Longrightarrow$ (19) $\Longrightarrow$ (20) $\Longrightarrow$ (7) with:

(14) Continuous functions on $[a, b]$ assume their maximum

(15) Theorem of Rolle

(16) $\boxed{\text{Generalized mean value theorem}}$

(17) $\boxed{\text{Mean value theorem}}$

(18) $\boxed{\text{Taylor expansion (with Lagrange remainder term)}}$

(19) $\boxed{\begin{array}{c}\text{Characterization of polynomial functions}\\ \text{by } f^{(n+1)} \equiv 0\end{array}}$

(20) $\boxed{f' \equiv 0 \implies f = \text{const.}}$

Proof. (4) $\Longrightarrow$ (14) Together with (4) also (5), (6) and (6)$'$ are fulfilled. Let $K := \sup f(I) \in \mathbb{K}$ if the set $f(I)$ is bounded from above; if not we put $K = \infty$. Certainly, we can find a sequence $x_j \in I$ with $\lim_{j \to \infty} f(x_j) = K$. Due to (5), we may assume (after going over to a subsequence) that the limit $c \in I$ of the sequence (x_j) exists. Since f is (sequentially) continuous we immediately derive

$$K = \lim_{j \to \infty} f(x_j) = f(c)\,.$$

Hence, the function f assumes its maximum at the place c.

(14) $\Longrightarrow$ (15) We deduce first from the condition (14) that the field $\mathbb{K}$ is archimedean ordered; or to put it the other way around: for a nonarchimedean field $\mathbb{K}$ the statement (14) is wrong. In that case there exist "infinitely small" positive elements in $\mathbb{K}$, i. e. elements $\varepsilon > 0$ with $\varepsilon < 1/n$ for all $n \in \mathbb{N}^*$, and consequently "infinitely large" elements K which are larger than any $n \in \mathbb{N}$. The sets

$$U_m := \{x \in \mathbb{K} : x = m \text{ or } |x - m| \text{ infinitely small}\}\,, \quad m \in \mathbb{N}^*\,,$$

are open (and pairwise disjoint) since together with ε, ε' also $\varepsilon + \varepsilon'$ is infinitely small. Namely, if $x \in U_m$ and $|x - x'| < \varepsilon$ for any infinitely small positive ε, then also $x' \in U_m$. We claim moreover that the (necessarily open) union of the sets U_m is *closed*, i. e. that the complement

$$U_0 := \mathbb{K} \setminus \bigcup_{m \in \mathbb{N}^*} U_m$$

is open. If $x \in U_0$ and $\varepsilon > 0$ is infinitesimally small we immediately see as above that the whole $(x - \varepsilon, x + \varepsilon)$ is contained in U_0.

Now, $\bigcup_{m \in \mathbb{N}} U_m$ is an open covering of $[0, K]$, K infinitely large, from which no element can be removed. The function

$$f(x) := \begin{cases} m, & x \in U_m \\ 0, & \text{otherwise} \end{cases}$$

is continuous but assumes no maximum on the interval $[0, K]$.

We next show that the theorem of Rolle is valid. By assumption, the function f is continuous on $[a, b] \subset \mathbb{K}$, hence assumes its maximum and minimum. We may suppose that one of these extremal places ξ lies in the open interval (a, b) since otherwise $f(x) \equiv f(a)$ and $f' = 0$. Without loss of generality we assume moreover that f has a maximum at ξ. Then we have for the difference quotient

$$\frac{f(x) - f(\xi)}{x - \xi} \begin{cases} \geq 0, & x < \xi \\ \leq 0, & x > \xi . \end{cases}$$

From the definition of the differential quotient it follows for all positive ε and all $x > \xi$ which are lying sufficiently close to ξ :

$$f'(\xi) < \varepsilon + (\Delta f)(x) \leq \varepsilon .$$

Hence, $f'(\xi) \leq 0$. Correspondingly, we get with elements $x < \xi$ that $f'(\xi) \geq 0$.

(15) $\Longrightarrow$ (16) Introduce the new function

$$F(x) := (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)) .$$

F satisfies $(+)_0$ and $F(a) = F(b) = 0$. Hence we find $\xi \in (a, b)$ with $F'(\xi) = 0$, and therefore

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)) .$$

But $g(a) \neq g(b)$ since otherwise we find $\eta \in (a, b)$ with $g'(\eta) = 0$, in contradiction to our assumption on g.

(16) $\Longrightarrow$ (17) Look at the function $g(x) = x$.

(17) $\Longrightarrow$ (18) To each $a < x \leq b$ there exists a uniquely determined element $\rho(x) \in \mathbb{K}$ such that

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x - a)^j + \rho(x)\frac{(x - a)^{n+1}}{(n+1)!} .$$

Now, for fixed x with $a < x \leq b$, put

$$F(t) := f(x) + \sum_{j=0}^{n} \frac{f^{(j)}(t)}{j!}(x - t)^j + \rho(x)\frac{(x - t)^{n+1}}{(n+1)!} .$$

By our assumptions on f the function F is continuous for all $t \in [a, x]$ and differentiable on (a, x). Furthermore, $F(x) = 0$ and $F(a) = 0$ according to the definition of $\rho(x)$. Hence, we find an element ξ between a and x such that $F'(\xi) = 0$. By a simple calculation, we get

$$\begin{aligned} F'(t) &= f'(t) + \sum_{j=1}^{n} \left\{ \frac{f^{(j+1)}(t)}{j!}(x - t)^j - \frac{f^{(j)}(t)}{(j-1)!}(x - t)^{j-1} \right\} - \rho(x)\frac{(x - t)^n}{n!} \\ &= \frac{f^{(n+1)}(t)}{n!}(x - t)^n - \rho(x)\frac{(x - t)^n}{n!} . \end{aligned}$$

Since $a < \xi < x$ it follows that $\rho(x) = f^{(n+1)}(\xi)$.

(18) $\Longrightarrow$ (19) We have $R_{n+1}(x) = 0$ for all $x > a$ by assumption. Hence

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j, \quad x \in [a, b].$$

(19) $\Longrightarrow$ (20) Trivial since polynomial functions of degree ≤ 0 are constants.

(20) $\Longrightarrow$ (7) If $[0, 1]$ is not connected there exist locally constant functions which are not constant. □

Remark. The trick in the proof of the conclusion (17) $\Longrightarrow$ (18) is taken from Heuser [3].

The preceding axioms are equivalent to either of the following assertions. Here a function $f : I \to \mathbb{K}$, I an arbitrary interval, is called *convex* if for all $a, b \in I$, $a < b$:

$$f(x) \leq f(a) + (x-a)\frac{f(b) - f(a)}{b-a}, \quad x \in [a, b].$$

$$\text{(21)} \quad \boxed{f' \geq 0 \Longrightarrow f \text{ increasing}}$$

$$\text{(22)} \quad \boxed{f'' \geq 0 \Longrightarrow f \text{ convex}}$$

Obviously, (21) can be derived from the mean value theorem (17). On the other hand, if (21) is fulfilled, we conclude from $f' = 0$ that $f' \geq 0$ and $(-f)' \geq 0$ and hence that f and $-f$ are increasing. Hence, f must be constant; in other words: (21) implies (20). We don't repeat the standard proof for (22) using only former results. If, finally, (22) is fulfilled and f is a differentiable function with $f' = 0$ then f is twice differentiable with $f'' = 0$. Due to (22) f as well as $-f$ are convex. Therefore, f is an affine linear function: $f(x) = dx + c$. Because of $f' = 0$, necessarily $d = 0$ and $f = c$ is constant. Hence, (22) implies (20). □.

Remark. If one demands the validity of e. g. the intermediate value theorem or the mean value theorem only for *polynomials* instead of continuous and differentiable functions one is lead to the much larger class of the *real closed fields* invented and thoroughly studied by Emil Artin and Otto Schreier.

The fourth circle

The fourth circle comprises *compactness statements* in the sense of properties of open coverings of intervals $[a, b] \subset \mathbb{K}$. Notice that there exists a classification of all (not necessarily ordered) *locally compact* fields. *Lebesgues' Lemma* is used in the following formulation: To each open covering

$$[a, b] \subset \bigcup_{\iota \in I} U_\iota$$

it exists $\delta > 0$ (sometimes called the *Lebesgue number of the covering*) such that to each pair of elements $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| \leq \delta$ there exists $\iota \in I$ such that $x_1, x_2 \in U_\iota$. We prove the following implications: (4) $\Longrightarrow$ (23) $\Longrightarrow$ (24) $\Longrightarrow$ (25) $\Longrightarrow$ (26) $\Longrightarrow$ (27) $\Longrightarrow$ (2).

(23) $[0, 1]$ is compact

(24) $\mathbb{K}$ is archimedean
&
Lebesgues' Lemma

(25) $\mathbb{K}$ is archimedean
&
Continuous functions on $[a, b]$
are uniformly continuous

(26) Continuous functions on $[a, b]$
can arbitrarily closely approximated
by step functions

(27) $\mathbb{K}$ satisfies $(*)$
&
Continuous functions on $[a, b]$ are bounded

Remark. It is plain that together with $[0, 1]$ every closed interval $[a, b] \subset \mathbb{K}$ is compact.

Remark. I didn't find the statements (24), (25), (26) and (27) explicitly formulated as axioms of the real numbers in the literature. It is clear that (27) can also be replaced by

(28) Continuous functions on $[a, b]$
are bounded by a natural number n

since one easily deduces from it that $\mathbb{K}$ is archimedean. (For positive a the function $f(x) = ax$ is continuous on the interval $[0, 1]$; consequently, $a = f(1) \leq n$ with suitable $n \in \mathbb{N}$). One can find this axiom in Steiner [7].

Proof. We first convince ourselves that the condition (23) resp. (26) implies the axiom of Archimedes. In other words: in a nonarchimedean field $\mathbb{K}$ neither (23) nor (26) holds. In the case of (23) we have this already seen in the proof of the implication (14) $\Longrightarrow$ (15) where we constructed an interval $[0, K]$ together with a countably infinite open covering from which no element can be deleted. Thus, this interval - and also the standard interval $[0, 1]$ - is not compact.

If (26) holds true we find for the function $f(x) = x$ on $[0, 1]$ and any infinitely small $\varepsilon > 0$ a subdivision $a_0 = 0 < a_1 < \cdots < a_n = 1$ and elements $c_j \in \mathbb{K}$ such that

$$|x - c_j| < \varepsilon , \quad a_j \leq x \leq a_{j+1} .$$

From this one deduces that all a_j and all c_j must be infinitely small which is impossible because of $a_n = 1$.

We are now in the position to prove these implications. (4) $\Longrightarrow$ (23). Let $\cup_{\iota \in I} U_\iota \supset [0, 1]$ be an open covering from which one can't select a finite subcovering. This remains true for at least one of the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Inductively proceeding in the same manner we find a nested sequence of intervals $I_0 \supset I_1 \supset I_2 \supset \cdots$ such that no I_j can be covered by finitely many U_ι. But choosing $x_0 \in \cap_{j=0}^{\infty} I_j$ there exists a $\iota_0 \in I$ such that $x_0 \in U_{\iota_0}$ and therefore also a j_0 with $I_{j_0} \subset U_{\iota_0}$. Contradiction!

(23) $\Longrightarrow$ (24) Each set U_ι is a union of intervals $\{|x - x_0| < \delta_0\}$ with

$$\{|x - x_0| < 2\delta_0\} \subset U_\iota .$$

Since $[a, b]$ is compact it can be covered by finitely many of such intervals:

$$[a, b] \subset \bigcup_{j=1}^{n} \{|x - x_j| < \delta_j\} , \quad \{|x - x_j| < 2\delta_j\} \subset U_{\iota_j} .$$

Put $\delta := \min(\delta_1, \ldots, \delta_n)$; let $x \in [a, b]$, hence $|x - x_j| < \delta_j$ for some j, and let x' be another element with $|x - x'| \leq \delta$. Then, $|x' - x_j| \leq |x' - x| + |x - x_j| < \delta + \delta_j \leq 2\delta_j$, in particular $x, x' \in U_{\iota_j}$.

(24) $\Longrightarrow$ (25) We show instead: To every continuous function $f : [a, b] \to \mathbb{K}$ and every $\varepsilon > 0$ it exists a finite (even an equidistant) subdivision of $[a, b] : a_0 = a < a_1 < \cdots < a_n = b$ such that for all x_j, ξ_j with $a_j \leq x_j, \xi_j \leq a_{j+1}$ one has:

$$|f(x_j) - f(\xi_j)| < \varepsilon .$$

This is, in fact, equivalent to the uniform continuity of f under the additional assumption of the axiom of Archimedes. Let $\varepsilon > 0$ be given. Then, to each $x_0 \in [a, b]$ there exists a $\delta_0 = \delta(x_0)$ with the property that from $|x - x_0| < \delta_0$, $x \in [a, b]$ one can conclude that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$. Let $\delta > 0$ be the Lebesgues number of the covering

$$\bigcup_{x_0 \in [a,b]} \{|x - x_0| < \delta_0\} ,$$

and choose $n \in \mathbb{N}$ in such a way that

$$\frac{1}{n}(b-a) \leq \frac{\delta}{2}.$$

Then our claim is true for the subdivision $a_j = a + \frac{j}{n}(b-a)$, $j = 0, 1, \ldots, n$.

(25) $\Longrightarrow$ (26) Trivial because of the equivalent formulation used in the preceding step.

(26) $\Longrightarrow$ (27) Trivial because of the remark at the beginning of the proof.

(27) $\Longrightarrow$ (2) We assume that the field $\mathbb{K}$ satisfies $(*)$ but not (2), and construct a continuous function f on an interval $[0, K]$ which is unbounded. Since (2) is not true there exists a (without loss of generality strongly) increasing sequence $a_0 < a_1 < \cdots$ having an upper bound which is not convergent. We define

$$I_0 = \{x \in \mathbb{K} : x \leq a_0\}, \quad I_j = \{x \in \mathbb{K} : a_{j-1} \leq x \leq a_j\}, \quad j \geq 1,$$

and $A = \cup_{j=0}^{\infty} I_j$. Obviously, A is also the union of the open intervals $(-\infty, a_j)$, hence open itself. However, due to the nonexisting limit of the sequence $(a_j)_{j\in\mathbb{N}}$ it is easily seen that the set A is sequentially closed in $\mathbb{K}$ and hence, due to $(*)$, even closed.

Namely, if (b_k) is a sequence in A converging to $b \in \mathbb{K}$ then there are two possibilities: Either there exists a j such that $b_k \leq a_j$ for almost all k. Then $b \leq a_j$ and consequently $b \in A$. Or for each j there are infinitely many k with $b_k > a_j$ and hence $b \geq a_j$ for all j. Choose now to an arbitrarily given $\varepsilon > 0$ an index k such that $b_k > b - \varepsilon$ and a j_0 with $b_k \leq a_{j_0}$. Then we have for all $j \geq j_0$:

$$0 \leq b - a_j \leq b - a_{j_0} \leq b - b_k < \varepsilon$$

in contradiction to the assumption that the sequence (a_j) does not converge.

Therefore, the set

$$\mathbb{K} \setminus A = \{x \in \mathbb{K} : x > a_j \text{ for all } j\}$$

is open (and by assumption nonempty). Let furthermore $c_j \in \mathbb{K}$ be a sequence with $c_0 = 0 < c_1 < c_2 < \cdots$ and $\lim_{j\to\infty} c_j = \infty$ (existing again due to $(*)$). Then, we define $f : \mathbb{K} \to \mathbb{K}$ via

$$f(x) := \begin{cases} 0, & x \in I_0 \\ c_{j-1} + \dfrac{x - a_{j-1}}{a_j - a_{j-1}}(c_j - c_{j-1}), & x \in I_j, \quad j \geq 1 \\ 0, & x \notin I. \end{cases}$$

f is obviously continuous, but unbounded on every interval $[0, K]$ with $K \in \mathbb{K} \setminus A$. □

The fifth circle

The fifth circle contains (Riemanns) *integration theory* and the *main theorem of differential and integral calculus*. We prove the equivalence of each of the following statements to a former one; here f always denotes a continuous function $f : [a, b] \to \mathbb{K}$. The main theorem asserts that *each continuous function* $f : [a, b] \to \mathbb{K}$ *admits an antiderivative, and two antiderivatives differ only by an additive constant.*

$$\text{(29)} \quad \boxed{\begin{array}{c} \mathbb{K} \text{ satisfies } (*) \\ \& \\ \text{For continuous } f \text{ exists the upper integral} \\ \displaystyle\int_a^{*b} f(x)\,dx \end{array}}$$

$$\text{(30)} \quad \boxed{\begin{array}{c} \mathbb{K} \text{ satisfies } (*) \\ \& \\ \text{For continuous } f \text{ exists the Riemann integral} \\ \displaystyle\int_a^{b} f(x)\,dx \end{array}}$$

$$\text{(31)} \quad \boxed{\begin{array}{c} \mathbb{K} \text{ satisfies } (*) \\ \& \\ \text{For continuous } f \geq 0 \text{ exists an antiderivative,} \\ \text{and each antiderivative is increasing} \end{array}}$$

$$\text{(32)} \quad \boxed{\begin{array}{c} \mathbb{K} \text{ satisfies } (*) \\ \& \\ \text{Main theorem of differential and integral calculus} \end{array}}$$

Proof. From the axiom (26) follows that to each continuous function $f : [a, b] \to \mathbb{K}$ there exist step functions $\varphi, \psi : [a, b] \to \mathbb{K}$ with $\varphi \leq f \leq \psi$. Hence, the set

$$\left\{ \int_a^b \psi(x)\,dx : f \leq \psi \text{ step function} \right\}$$

is nonempty and bounded from below by

$$\int_a^b \varphi(x)\,dx$$

for any step function $\varphi \leq f$. Thus, because of (6), there exists

$$\int_a^{*b} f(x)\,dx = \inf\left\{\int_a^b \psi(x)\,dx : f \leq \psi,\ \psi \text{ step function}\right\}.$$

If, conversely, (29) or (30) holds then, necessarily, every continuous function $f : [a, b] \to \mathbb{K}$ must be bounded (from above). Hence, (27) is fulfilled. In particular - due to (26) - we find to each continuous function $f : [a, b] \to \mathbb{K}$ and every $\varepsilon > 0$ step functions φ, ψ with $\varphi \leq f \leq \psi$ and $\sup_{[a,b]}(\psi - \varphi) \leq \dfrac{\varepsilon}{b-a}$. Consequently,

$$0 \leq \int_a^b \psi(x)\,dx - \int_a^b \varphi(x)\,dx \leq \varepsilon,$$

and with (29) we conclude that

$$\int_a^{*b} f(x)\,dx = \int_{*a}^b f(x)\,dx,$$

hence (30).

From (30) to (31) one argues as follows: By assumption, the integral

$$F(x) := \int_a^x f(\xi)\,d\xi$$

exists for all $x \in [a, b]$. The well known proof shows that F is differentiable with $F' = f$. Since the integral is additive we get for $x_1 < x_2$:

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x)\,dx = \int_{*x_1}^{x_2} f(x)\,dx \geq 0$$

since $\varphi : [x_1, x_2] \to \mathbb{K}$ with $\varphi(x) = 0$ is a step function such that $\varphi \leq f|[x_1, x_2]$. If G is another antiderivative we have $(G - F)' = G' - F' = f - f = 0$ and, because of (20), $G - F =$ const. In particular, G is increasing, too.

(31) $\Longrightarrow$ (32) Every continuous function f can be written as $f = f_+ - f_-$ with the continuous functions

$$f_+ = \max(f, 0), \quad f_- = -\min(f, 0).$$

Since f_+ and f_- are nonnegative they admit antiderivatives F_+ and F_- resp., and $F := F_+ - F_-$ is an antiderivative of f. Let G be another antiderivative of f; then $H := G - F$ is an antiderivative of 0. Hence $H' = 0 \geq 0$ and $(-H)' = 0 \geq 0$. Due to (31), H as well as $-H$ must be increasing functions which implies $H =$ const.

Finally, if (32) holds true then $f' = 0 = 0'$ implies $f =$ const. This is axiom (20). □

Other characterizations of the real numbers

Up to now, we didn't use the fact that the axioms considered so far characterize in fact a (up to isomorphisms of ordered fields) uniquely determined field. The *existence* can be derived e. g.

from the general theory of Cauchy completions for $\mathbb{K}$–metric spaces thus using axiom (1) as a guiding principle for the construction (see e. g. [6] for more details). Needless to say that every course treating the subjects of this note should emphasize the (hi)story of constructing the reals and the role played also by the other axioms, especially by (2), (3) and (12).

As we already mentioned earlier (at the end of the first circle) each archimedean ordered field can be embedded (as an ordered subfield) into such an archimedean ordered Cauchy complete field. This follows easily e. g. from the proposition which we proved there and axiom (2) and implies the just formulated *uniqueness* statement. Thus, archimedean ordered Cauchy complete fields can also be characterized by the axiom of *maximality*. (On the other hand, the field $\mathbb{Q}$ of *rational* numbers is the *minimal* archimedean ordered field). Notice that we didn't use this fact either in the previous circles.

(33) $\boxed{\begin{array}{c}\mathbb{K} \text{ is archimedean}\\ \& \\ \text{Each archimedean ordered field}\\ \text{can be embedded into } \mathbb{K}\end{array}}$

Another manifestation of the uniqueness is the *axiom of g–adic expansion* which we also claim to be equivalent to the other ones.

(34) $\boxed{\text{g–adic expansion}}$

By this, we understand more precisely the following assertion: *For each natural number $g \geq 2$, the g–adic series*

$$\sum_{\ell \leq k} a_k g^{-k}\,, \quad \ell \in \mathbb{Z}\,, \quad a_k \in \{0,\, 1, \ldots, g-1\}$$

are convergent in $\mathbb{K}$, and each nonnegative element in $\mathbb{K}$ can be expanded into such a series.

Proof. The partial sums

$$x_j := \sum_{\ell \leq k \leq j} a_k g^{-k}$$

of such a g–adic series form an increasing sequence. Under the assumption of (2) we get, using (3), that $(g^{-k})_{k\in\mathbb{N}}$ is a sequence converging to zero. Hence

$$x_j \leq \sum_{\ell \leq k} (g-1)\, g^{-k} = g^{\ell}\,(g-1)\,\frac{1}{1 - g^{-1}} = g^{\ell+1}$$

and the given g–adic series is convergent. The representation of any positive element by means of such a series is a simple consequence of the axiom of Archimedes as we have seen at the end of the first circle.

Conversely, if every g–adic series is convergent then it follows from the convergence of the special series $\sum_k g^{-k}$ that the element $1/g$ is analytically nilpotent. Therefore, the sequence

$(g^k)_{k\in\mathbb{N}}$ of positive integers in $\mathbb{K}$ is unbounded; hence $\mathbb{K}$ is archimedean and can be embedded into the maximal archimedean field $\widetilde{\mathbb{K}}$. Since all positive elements of $\widetilde{\mathbb{K}}$ can be developed into a g–adic series the fields $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ must coincide. □

Using (33) we can deduce further equivalent axioms for the real numbers in terms of statements on absolutely convergent series like the *criterion of majorants*, the *associativity* of absolutely convergent series and the conclusion „absolute convergence $\Longrightarrow$ convergence".

Warning This is not true with respect to *commutativity* of infinite series. This clearly holds in *any* archimedean ordered field due to (33).

Thus, we claim that the following axioms are equivalent to the former ones.

(35)

$\mathbb{K}$ is archimedean
&
Criterion of majorants

(36)

$\mathbb{K}$ is archimedean
&
Associativity for absolutely convergent series

(37)

$\mathbb{K}$ is archimedean
&
Absolutely convergent series are convergent

Proof. By assumption, $\mathbb{K}$ is an ordered subfield of the maximal field $\widetilde{\mathbb{K}}$. It remains to show that in case $\mathbb{K}\neq\widetilde{\mathbb{K}}$ neither of the assertions (35), (36) and (37) are true. For this it suffices to construct a series $\sum_{k=0}^{\infty} a_k$, $a_k\in\mathbb{K}$ which is not convergent in $\mathbb{K}$ such that $\sum_{k=0}^{\infty} |\,a_k\,|$, nevertheless, is convergent in $\mathbb{K}$. (In case of (36) a counter example is given by the double series $(a_{jk})_{j,k\in\mathbb{N}}$ with $a_{0k}=a_k$, $a_{1k}=-a_k$, $a_{jk}=0$ otherwise).

So, let $\mathbb{K}\neq\widetilde{\mathbb{K}}$, and take an element $a\in[\,0,\,1\,]\subset\widetilde{\mathbb{K}}$ which is not contained in $\mathbb{K}$. Write a as a dual number:

$$a=\sum_{k=1}^{\infty}\frac{c_k}{2^k}\,,\qquad c_k\in\{0,\,1\}\,.$$

Since for all $j<\ell$

$$\frac{0}{2^j}+\frac{0}{2^{j+1}}+\cdots+\frac{0}{2^{\ell-1}}+\frac{1}{2^\ell}=\frac{1}{2^j}-\frac{1}{2^{j+1}}-\cdots-\frac{1}{2^\ell}$$

we can write a also as

$$a = \sum_{k=1}^{\infty} a_k \quad \text{with} \quad a_k = \pm \frac{1}{2^k} \in \mathbb{Q} \subset \mathbb{K} \, .$$

The series $\sum |a_k|$ is convergent to 1 in $\mathbb{Q}$ (and thus also in $\mathbb{K}$). $\square$

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