Simple analytic proofs of some abstract versions of the Prime Number Theorem

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Nobody is forced to decide between vanilla ice cream and chocolate once for all, and it is even possible to mix the two, [...]. Paul Halmos

Introduction

The *Prime Number Theorem* (PNT) is considered by some experts one of the greatest achievements of mathematics, if not of mankind. Perhaps, this is an exaggeration; nevertheless each student of mathematics should have heard about it and part of its history during his or her studies and should have been confronted with at least an idea of its proof in the simplest form (i. e. without estimating error terms).

However, this was difficult to achieve for a long time since the analytic proofs of the PNT use quite heavy general machinery or at least highly specialized analytic considerations which are not part of the curriculum of standard courses in Real and Complex Analysis. The situation changed considerably in 1980 when D. J. Newman published his paper *Simple Analytic Proof of the Prime Number Theorem* [31] in the American Mathematical Monthly. By transferring his ingenious method from Dirichlet *series* to Laplace *transforms*, J. Korevaar finally presented in 1982 in Springer's *Mathematical Intelligencer* a proof which - besides standard facts on prime numbers and, of course, the Riemann ζ–function - was based only on the Cauchy *integral formula* [17]. So, if one is willing to spend some time on the ζ–function and at about one hour on the interrelation between the "prime counting" function π and some of the Chebyshev functions, one can - as I did from that time on - give a complete proof of the PNT as early as at the end of a first introductory lecture course on *Complex Analysis* that needs a very short and elementary digression into number theory only (see my manuscript *Funktionentheorie* on my homepage http://www.math.uni-hamburg.de/home/riemenschneider/funvorl1.pdf). Newman's method in Korevaar's simplification found its way into prominent publications (see e. g. Zagier [42] and Korevaar's own masterly survey article [18] on Tauber Theorems) and even into the textbook literature (see, e. g., Lang [26] and Jameson [14]).

Parallel to this story there is another one which has to do with my University and might be less known. When I told the number theorists Helmut Brückner and Helmut Müller about this new approach they informed me that our former colleague, the renowned algebraist Ernst Witt, had distributed in the sixties and seventies of the 20th century a proof of the PNT, even in an abstract setting, on just one typewritten page. I must confess that I had problems to understand the page immediately in full detail and lost interest in its content and finally the page itself.

In 1998, the collected papers of Witt appeared [41], edited by Ina Kersten, containing also a facsimile of the sheet mentioned before and a short extra sheet written by hand, followed by some remarks of Horst Leptin [28] who made clear that Witt's method is sound (in the classical, i. e. not generalized case) by invoking only elementary facts on L^2–theory besides the standard results on number theory and the ζ–function. However, to come to this conclusion, Leptin had to fill quite a lot of new lines in between the lines of Witt, and he remarked that there might be a gap in Witt's argument when applied to the generalized situation. It was H. Müller [30] who showed in 2002 that the gap can be bridged.

Unfortunately, Leptin missed the opportunity to put Witt's proof in the right (historic) perspective, especially by not mentioning that Witt used (consciously, as one may suspect) some version of a

TAUBER type theorem of IKEHARA and WIENER and ideas by several mathematicians for simplifying its proof.

I came back to this subject after my retirement when revising my texts on standard lecture courses. I thank H. MÜLLER and my student SOLVEJG GLATZ for several discussions which clarified the situation to the effect that WITT's arguments need - besides elementary complex analysis and the RIEMANN - LEBESGUE Lemma - only standard L^1–theory like the convergence theorems of B. LEVI and LEBESGUE and the TONELLI - FUBINI Theorem. One can even avoid the LEBESGUE–integral. Thus, the proof - even for the generalized version where one has to bring some more basic facts of FOURIER analysis into the game - can easily be understood by undergraduate students after a good introduction into Analysis of, say, 4 semesters, including standard Complex Analysis[1,2].

In this note, I will give a short survey on both approaches and try to convince the reader that one can combine the two (chocolate and vanilla) to eliminate number theory completely by standard (complex and real) analysis. I also include a simple proof of the IKEHARA - WIENER Theorem along these lines.

Acknowledgements. I am indebted to the (unknown) referee for constructive criticism and helpful remarks on two versions of this manuscript which enabled me, as I hope, to improve the text considerably by, e. g., eliminating a few mathematical errors and some historical inaccuracies. To be sure, solely the author is responsible for any kind of opinion stated expressis verbis in the text. I also would like to thank the ICTP in Trieste for its hospitality during my stay in June 2008 that gave me the opportunity to dive more deeply into the matter and to start the revision of the paper under such splendid conditions.

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1 The Prime Number Theorem in WITT's formulation

The "classical" Prime Number Theorem states that the prime counting function

$$\pi(x) := \#\{p \in \mathbb{P} : p \le x\} = \sum_{p \le x} 1 , \quad x \in \mathbb{R}_+ ,$$

where $\mathbb{P} \subset \mathbb{N}$ denotes the set of prime numbers, behaves asymptotically for $x \to \infty$ like $x/\log x$, i. e.[3]

$$\pi(x) \sim \frac{x}{\log x} \ (x \to \infty)^4 \quad \text{or} \quad \lim_{x\to\infty} \frac{\pi(x)\,\log x}{x} = 1 .$$

[1]In fact, our Physics students in Hamburg learn the necessary facts from Analysis already in the first 3 semesters of their 4 semester course in Mathematics. It is quite unfortunate that after the last University reform in Germany our Mathematics students will most probably not sufficiently be armed to follow the arguments in full detail.

[2]I shall call such arguments occasionally *simple* or *elementary* although both adjectives may have completely different meanings for number theorists in connection with the classical PNT. DIAMOND tries to explain the distinction in his highly recommendable survey article [8]: *The approach to the prime number problem proposed by Riemann, using a function of a complex variable generated by arithmetic data, came to be called* analytic. *On the other hand, direct real variable treatment of arithmetic data, such as the method of Chebyshev, came to be called* elementary. *[...] To avoid confusion, we shall (with apologies to Sherlock Holmes) use the word* simple *for "easy to understand". It will be seen that some elementary arguments are far from simple.*

[3]$\log x$ denotes in the present paper always the *natural* logarithm of x to the basis e.

[4]In the following we drop the symbol $(x \to \infty)$.

If one instead of π uses the function

$$P(x) := \#\{ p \in \mathbb{P} : \log p \leq x \} ,$$

we have $P(x) = \pi(e^x)$, and therefore the PNT is equivalent to

$$(*) \qquad P(x) \sim e^x/x \quad \text{or} \quad \lim_{x\to\infty} x\, e^{-x} P(x) = 1 .$$

Remarks. 1. GAUSS conjectured already in 1792 or 1793 according to a letter to his former student ENCKE, dated December 24, 1849, that the prime counting function π behaves asymptotically like

$$\int_2^x \frac{du}{\log u} = \mathrm{li}(x) - \mathrm{li}(2)$$

with the "logarithmic integral"

$$\mathrm{li}(x) = \int_0^x \frac{du}{\log u} := \lim_{\varepsilon \searrow 0} \left(\int_0^{1-\varepsilon} \frac{du}{\log u} + \int_{1+\varepsilon}^x \frac{du}{\log u} \right)$$

(which in turn is by DE L'HOSPITAL's rule asymptotically equivalent to $x/\log x$). GAUSS' letter was published after RIEMANN's seminal paper [33] on the ζ–function which paved the way for the complex analytic proofs of the PNT in 1896 by HADAMARD [10] and independently by DE LA VALLÉE POUSSIN [36]. After many successful attempts to simplify the methods and to make them applicable to other problems, it was only more than 50 years later that ERDÖS [9] and SELBERG [35] gave the first so called *elementary* proofs of the PNT which use only elementary methods of *real* analysis avoiding the ζ–function altogether, but (as KOREVAAR puts it in [18]) do not *qualify as simple* at all.

2. Investigating the *absolute error term* for the PNT means to find estimates for the *difference*

$$\pi(x) - \int_2^x \frac{du}{\log u} .$$

There are many results in the literature addressing this problem, also in the abstract situation to be studied later. The interested reader may consult the standard literature like [21], [12], [34], [14] and the remarks at the end of Chapter 6, §1, p. 162 in [16].

WITT considers more generally an unbounded non decreasing ("isotonic") sequence

$$P : 0 < p_1 \leq p_2 \leq \cdots$$

of positive real numbers and the corresponding function

$$P(x) := \#\{ p \in P : p \leq x \} , \quad x \in \mathbb{R} .$$

Therefore, in the classical case, the function $P(x)$ is associated to the set $P := \log \mathbb{P}$, rather than to $\mathbb{P}$ itself. In the generalized context, we sometimes write $\pi(x) = P(\log x)$. Let now A denote the *additive* semigroup of $\mathbb{R}_+$ generated by P with the corresponding counting function $A(x)$ (which takes the "multiplicities" of elements in A into account, i. e. the fact that these elements will in general have finitely many representations as sums of elements in P). Then we can ask the following general question: *Given the asymptotic behaviour of $A(x)$ or of some kind of its average as $x \to \infty$ what can be said about the asymptotic behaviour of $P(x)$?* Any such result may be called an *abstract PNT*. See, e. g., the book [16] of KNOPFMACHER for many examples.

Remark. Studies in such *generalized* or *abstract* situations were started in 1937 by ARNE BEURLING in his mémoire *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés I* in *Acta Mathematica* [3]. Sticking more closely to the classical situation he defines the set of *generalized primes* by an isotonic sequence

$$\overline{P} : 1 < \overline{p}_1 \leq \overline{p}_2 \leq \cdots$$

of real numbers and the set of *generalized positive integers* (or BEURLING *numbers*) N as the *multiplicative* semigroup generated by $\overline{P}$. Transferring his investigations to the *additive* setting used in the present paper, his main result can be stated as follows: If the counting function $A(x)$ attached to the set $A := \log N$ satisfies condition

$$(B_\gamma) \qquad A(x) = \alpha\, e^x + O(x^{-\gamma}\, e^x) \text{ for some } \alpha > 0\,,\ \gamma > 3/2\,,$$

then the generalized PNT holds for the set $P := \log \overline{P}$ in the sense described above. DIAMOND [6] showed much later that $\gamma = 3/2$ does not suffice. In the multiplicative setting, the sequence $\overline{p}_1 < \overline{p}_2 < \cdots$ defined uniquely by

$$\int_1^{\overline{p}_j} (\log t)^{-1}(1 - \cos(\log t))\, dt = j$$

is a counterexample. BATEMAN and DIAMOND gave in [1] a proof of BEURLING's result which in many details is closer to LANDAU's proof of the PNT in [23] and at least indirectly related to WITT's page [41].

Having the arguments based on the RIEMANN ζ–function in the classical situation in mind it is quite reasonable to assume with WITT (see also Sections 2 and 3 where we analyze the consequences of this condition more closely):

$$(W_\beta) \qquad \int_0^\infty e^{-\beta x} |\, A(x) - \alpha\, e^x \,|\, dx < \infty \text{ for some } \alpha > 0\,,\ 0 < \beta < 1\,.$$

Remarks. 1. In the classical case, we have by the existence and uniqueness of the prime factorization in the ring $\mathbb{Z}$ of integers that

$$A = \{\log n\,,\ n \in \mathbb{N}^*\} = \log \mathbb{N}^*$$

and no multiplicities, such that, with the GAUSS bracket $[\,y\,] := \max_{n\in\mathbb{Z}} \{\, n \leq y\,\}\,,\ y \in \mathbb{R}\,,$

$$A(x) = [\,e^x\,]\,.$$

Consequently, the assumption (W_β) is satisfied with $\alpha = 1$ and any $0 < \beta < 1$.

2. A set P of generalized (additive) primes satisfies *Axiom A* according to [16] if one has

$$A(x) = \alpha\, e^{\delta x} + O(e^{\eta x}) \text{ for some } \alpha > 0\,, \quad 0 \leq \eta < \delta$$

for the counting function of the additive semigroup A generated by P. Restricting to the case $\delta = 1$ by an obvious homothety we see immediately that WITT's condition (W_β) is satisfied for all $\beta > \eta$.

3. If $\mathbb{K}$ is an algebraic number field of degree d, then Axiom A is satisfied for the set of logarithms of the norms of the prime ideals in its ring of integers $\mathbb{I}$:

$$P = \{\log N\,\mathfrak{p}\,,\ \mathfrak{p} \subset \mathbb{I}\}$$

with $\delta = 1$, $\eta = 1 - 1/d$ and a certain constant α depending on the field $\mathbb{K}$ (see WEBER [37]). Hence, the condition (W_β) is satisfied for any $\beta > 1 - \dfrac{1}{d}$.

Our goal in this survey is to present two simple and self contained proofs of the PNT in the following version (see [41]).

Theorem 1.1 *If the set P satisfies (W_β) for some $0 < \beta < 1$ then the* (generalized) PNT *holds in the form*

$$P(x) \sim \frac{e^x}{x} \quad \textit{or equivalently} \quad \pi(x) \sim \frac{x}{\log x}\,.$$

Remarks. 1. Due to the remarks above, this generalizes the classical PNT and the so called *Prime Ideal Theorem* of LANDAU [20].

2. If the set P of generalized primes satisfies *Axiom A*, then we can simply deduce from WITT's result the following form of the PNT (see Chapter 6, § 1 in [16]):

$$\pi(x) \sim \frac{x^\delta}{\delta \log x}\,.$$

Under this stronger assumption, one can moreover estimate the error term (see, e. g., MÜLLER [29] and BEKEHERMES [2]).

3. It should be emphasized that BEURLING's result is much stronger than WITT's because each condition (W_β), $0 < \beta < 1$, implies (B_γ) for all $\gamma > 0$ (see Section 2). In the Appendix to Section 5, we indicate how one can derive along the same way the abstract PNT under the weaker condition (B_γ) when $\gamma > 2$. Here, we follow the dissertation of BEKEHERMES [2].

WITT uses in [41] instead of $P(x)$ the function

$$F(x) = F_P(x) := e^{-x} \sum_{p \in P : np \le x} p = e^{-x} \sum_{p \in P} \left[\frac{x}{p} \right] p$$

and the following equivalence statement:

Proposition 1.2 *If $F(x) \to 1$ for $x \to \infty$, then the function P satisfies the* PNT *in the form* $(*)$, *and vice versa.*

This result is an immediate consequence of the inequalities due to WITT which are proven in its hand written note (see also LEPTIN's commentary [28]; however, similar statements are already contained in publications before WITT). For the convenience of the reader and to keep this article self contained, we repeat the arguments.

Proposition 1.3 *The functions F and P are related by the following inequalities ($x > 0$) :*

$$(**) \qquad F(x) \le x\, e^{-x} P(x) \le \frac{1}{p_1\, x}\, F(x - 2\log x) + \frac{x}{x - 2\log x}\, F(x)\,.$$

Proof. The left hand side of $(**)$ is obvious. For the other side, we follow WITT's (and LEPTIN's) conclusions almost *verbatim*: From

$$P(x) \le \sum_{p \in P} \left[\frac{x}{p} \right] \frac{p}{p_1} = \frac{1}{p_1}\, e^x\, F(x)$$

we deduce immediately after replacing x by $x - 2\log x$ that

$$x\, e^{-x}\, P(x - 2\log x) \le \frac{1}{p_1 x}\, F(x - 2\log x)\,.$$

Moreover, for $0 < y < x$, we have

$$y\,(P(x) - P(y)) = \sum_{\frac{y}{p} < 1 \le \left[\frac{x}{p}\right]} \frac{y}{p}\, p \le \sum_{p \in P} \left[\frac{x}{p} \right] p = e^x\, F(x)\,.$$

Substituting $y = x - 2\log x$ yields

$$x\, e^{-x}\,(P(x) - P(x - 2\log x)) \le \frac{x}{x - 2\log x}\, F(x)\,.$$

Addition of this inequality to the relevant one before gives the right hand side of $(**)$. □

Remark. In the classical case, we have

$$F(x) = e^{-x} \sum_{\substack{p \in P \\ p \le x}} \left[\frac{x}{p} \right] p = e^{-x}\, \psi(e^x)$$

with the CHEBYSHEV function

$$\psi(x) = \sum_{\substack{p \in \mathbb{P} \\ p \le x}} \left[\frac{\log x}{\log p} \right] \log p \le \pi(x)\, \log x \le x\, \log x\,.$$

It is well known - an elementary but by no means trivial fact proven by CHEBYSHEV (see, e. g., [21], [17]) - that more precisely

$$\psi(x) \le (4 \log 2)\, x$$

such that in this situation the function F is bounded (from above) - a fact we a priori do not know in the generalized case (but which is true under certain conditions like (W_β) as we will demonstrate later). Moreover,

$$\lim_{x\to\infty} \pi(x) \frac{\log x}{x} = \lim_{x\to\infty} \frac{\psi(x)}{x}$$

if either of these limits exists. Hence, the condition $F(x) \to 1$ implies directly the PNT in the classical case without use of WITT's inequalities above.

2 TAUBER Theorems and the classical PNT

All (elementary and non elementary) analytic proofs of the PNT use some theorems of so called TAUBER type. One of the standard ways is to write the logarithmic derivative of the RIEMANN ζ–function as a MELLIN transform involving the CHEBYSHEV function ψ:

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \psi(x)\, x^{-(s+1)}\, dx$$

(see Theorem 3.2 and the following Remark 1). Here we write $s = \sigma + i\,t$ for a complex number as usual in number theory since RIEMANN. Then, the PNT follows in the form

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

without any further considerations on the function ψ using the well known fact that the function

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}$$

has a holomorphic extension to the open half plane $\mathrm{Re}\, s > 0$ from the following TAUBER Theorem.

Theorem 2.1 (Ikehara - Wiener) *Let the real valued function* f *be non negative and non decreasing on* $[1, \infty)$ *and suppose that the* MELLIN *transform*

$$g(s) := s \int_1^\infty f(x)\, x^{-(s+1)}\, dx \left('' = '' f(1) + \int_1^\infty x^{-s}\, df(x) \right)$$

exists for $\mathrm{Re}\, s > 1$. *Suppose, moreover, that for some constant* α, *the (holomorphic) function*

$$g(s) - \frac{\alpha}{s-1}$$

has a continuous *extension to the closed half plane* $\mathrm{Re}\, s \ge 1$. *Then*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \alpha\,.$$

In particular, the assumptions can only be satisfied if $\alpha \ge 0$.

Remark. According to KOREVAAR's survey article [18], p. 482, IKEHARA applied WIENER's (his mentors) TAUBERIAN theory of 1928 [38] to obtain a proof of this result in [11], 1931, thus succeeding in removing earlier extra growth conditions. He adds: *Most subsequent proofs and extensions have benefited from Wiener's version* ([39]) *of the proof and from Bochner's simplification* ([4]) *of it*. He, moreover, mentions in this context papers by LANDAU [23], INGHAM [13], WIENER and PITT [40] as well as expositions in books by several authors to which I would like to add the one of PITT [32].

Of course, having CHEBYSHEV's bound $\psi(x) \le C\, x$ and the analytic properties of the RIEMANN ζ–function mentioned above at our disposal, we need only a much weaker version of this deep result.

Corollary 2.2 (Poor man's Ikehara - Wiener Theorem) *Let f be non negative, non decreasing and $O(x)$ on $[1, \infty)$ such that the* MELLIN *transform*

$$g(s) = s \int_1^\infty f(x)\, x^{-(s+1)}\, dx$$

is well defined and holomorphic on $\operatorname{Re} s > 1$*. Suppose that for some real constant α, the function*

$$g(s) - \frac{\alpha}{s-1}$$

can be continued holomorphically *to a neighborhood of every point on the line* $\operatorname{Re} s = 1$*. Then*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \alpha\,.$$

This, in turn, is a simple corollary to the following

Theorem 2.3 (Karamata - Ingham) *Let F be bounded on $[0, \infty)$ and integrable over any finite subinterval such that the* LAPLACE *transform*

$$G(z) = \int_0^\infty F(t)\, e^{-zt}\, dt$$

is well defined and holomorphic on the open half plane $\operatorname{Re} z > 0$*. Suppose that $G(z)$ can be continued holomorphically to a neighborhood of every point on the imaginary axis. Then*

$$\int_0^\infty F(t)\, dt$$

exists as an improper integral and equals the value of (the extension of) G at the origin.

The problem with the last theorem (see KARAMATA [15], theorem B, and INGHAM [13], theorem III) is that f. i. INGHAM used WIENER theory to derive it (and even some stronger results). The breakthrough of NEWMAN in [31] lies in the fact that he replaced such deep methods in the proof of it, as KOREVAAR puts it, *by an ingenious application of complex integration theory, involving nothing more difficult than* CAUCHY*'s integral formula, together with suitable estimates*.

We will present the modified method of proof by KOREVAAR in the appendix to this section. Here, it might be in order to mention that it is quite elementary to go from the KARAMATA - INGHAM result to Corollary 2.2. The main point is that, given f, one has to put

$$F(t) := e^{-t} f(e^t) - \alpha\,.$$

Since by assumption $f(x) \le Cx$, F is bounded and locally integrable. The LAPLACE transform of F is easily been calculated:

$$\begin{aligned} G(z) &= \int_0^\infty \{ e^{-t} f(e^t) - \alpha \}\, e^{-zt}\, dt \quad , \qquad x = e^t \\ &= \int_1^\infty f(x)\, x^{-(z+2)}\, dx - \frac{\alpha}{z} = \frac{1}{z+1} \left\{ g(z+1) - \frac{\alpha}{z} - \alpha \right\}\,. \end{aligned}$$

This transform can, by assumption on g, be holomorphically extended into a neighborhood of $\operatorname{Re} z = 0$, such that the integral

$$\int_0^\infty F(t)\, dt = \int_0^\infty \{ e^{-t} f(e^t) - \alpha \}\, dt = \int_1^\infty \frac{f(x) - \alpha x}{x^2}\, dx$$

exists due to Theorem 2.3. Therefore, the problem is reduced to the following very elementary theorem of TAUBER type.

Lemma 2.4 *Let* $f : [1, \infty) \to \mathbb{R}$ *be a non negative, non decreasing function and suppose that for some* α *the improper integral*

$$\int_1^\infty \frac{f(x) - \alpha x}{x^2}\, dx$$

exists. Then α *is real, non negative, and* $f(x) \sim \alpha\, x$ *.*

Proof. Clearly, α must necessarily be real and non negative. Assume, for the moment, that we have $\limsup f(x)/x > \alpha\ (\geq 0)$. Then there exists $\delta > 0$ such that

$$f(y) > (\alpha + 2\delta)\, y$$

for arbitrarily large y. This would imply

$$f(x) \geq f(y) > (\alpha + 2\delta)\, y > (\alpha + \delta)\, x$$

for all x with $y < x < \rho\, y$, where $\rho := (\alpha + 2\delta)(\alpha + \delta)^{-1} > 1$. Hence

$$\int_y^{\rho y} \frac{f(x) - \alpha x}{x^2}\, dx > \int_y^{\rho y} \frac{\delta}{x}\, dx = \delta \log \rho = C_0 > 0\,,$$

such that the improper integral can not converge according to the Cauchy criterion in contradiction to the assumption. - One treats exactly in the same manner the other case $\liminf f(x)/x < \alpha$, in which necessarily $\alpha > 0$. □

Remarks. 1. A slightly different proof of the preceding lemma can be found in Jameson [14], pp. 130–131, Proposition 3.4.1.

2. In his survey [18] on Tauber *theory*, Korevaar asks at the end the following question: *Newman's contour integration method [· · ·] provides a beautiful proof for the prime number theorem. Nevertheless the Wiener-Ikehara way to the PNT [· · ·] is somewhat more direct. Is it possible to derive the Wiener-Ikehara theorem by simple complex analysis?* Meanwhile he has answered the question himself affirmatively in [19] under certain more restrictive assumptions on f which, nevertheless, yield the classical PNT.

One can use Lemma 2.4 and a generalization to shed more light on Witt's assumptions (W_β). By the substitution $x = \log t$ and with the non negative, non decreasing function $N(t) = A(\log t)$, $t \geq 1$, this condition is equivalent to the existence of the integral

$$\int_1^\infty \frac{N(t) - \alpha\, t}{t^{1+\beta}}\, dt\,.$$

By assumption, we have $1 + \beta < 2$, and therefore Lemma 2.4 implies $N(t) \sim \alpha\, t$ or, differently stated, the

Corollary 2.5

$$\lim_{x\to\infty} e^{-x} A(x) = \alpha\,.$$

However, assuming a stronger condition as in Lemma 2.4 leads also to a stronger conclusion. We claim the following result[5]:

Corollary 2.6 *Under the assumption* (W_β), $0 < \beta < 1$, *it follows for all* $\gamma > 0$ *that*

$$\lim_{t\to\infty} (N(t) - \alpha\, t)\, \frac{\log^\gamma t}{t} = 0 \quad \textit{or} \quad \lim_{x\to\infty} (A(x) - \alpha\, e^x)\, \frac{x^\gamma}{e^x} = 0\,.$$

In other words :

$$A(x) = \alpha\, e^x + o(x^{-\gamma}\, e^x)\,.$$

In particular, A *satisfies* Beurling*'s condition* (B_γ) *for all* $\gamma > 0$ *.*

[5]Its proof has been communicated to me by the referee.

Proof. We follow the same strategy as in the proof Lemma 2.4. Assume that (W_β) is satisfied for some β, $0 < \beta < 1$, and assume moreover that for some $\gamma > 0$ we have

$$L := \limsup_{t\to\infty} (N(t) - \alpha t) \frac{\log^\gamma t}{t} > 0 .$$

Then, there exist arbitrarily large y such that

$$N(y) - \alpha y \geq \frac{L}{2} \frac{y}{\log^\gamma y} .$$

Put now

$$y_L := y + \frac{L}{4\alpha} \frac{y}{\log^\gamma y} > y .$$

Since the function N is non decreasing, it follows for all t between y and y_L that

$$N(t) - \alpha t \geq N(y) - \alpha t \geq \frac{L}{2} \frac{y}{\log^\gamma y} - \alpha (t - y) \geq \frac{L}{4} \frac{y}{\log^\gamma y} .$$

Therefore,

$$\int_y^{y_L} |N(t) - \alpha t| \, t^{-(\beta+1)} \, dt \geq \frac{1}{\alpha} \left(\frac{L}{4} \frac{y}{\log^\gamma y} \right)^2 y_L{}^{-(\beta+1)} \geq C_{\beta,\gamma} \frac{y^{1-\beta}}{\log^{2\gamma} y} .$$

But the right hand side goes to infinity such that the integral in the definition of (W_β) can not exist by CAUCHY's criterion. Hence, (W_β) implies

$$L = \limsup_{t\to\infty} \frac{N(t) - \alpha t}{t/\log^\gamma t} \leq 0$$

for all $\gamma > 0$. By the same method, one can prove that

$$L' = \liminf_{t\to\infty} \frac{N(t) - \alpha t}{t/\log^\gamma t} \geq 0 ,$$

since, otherwise, the integrals

$$\int_y^{y_{L'}} |N(t) - \alpha t| \, t^{-(\beta+1)} \, dt , \quad y_{L'} := y - \frac{L'}{4\alpha} \frac{y}{\log^\gamma y} > y ,$$

tend with y to infinity. □

Appendix: KOREVAAR's proof of the KARAMATA - INGHAM Theorem

KOREVAAR's approach to Theorem 2.3 is well documented in literature (see [17], [18], [26] and [42]). However, for the convenience of the reader and for the sake of completeness, we insert here a proof, even for a more general result, which was mentioned already in [17].

Theorem 2.7 *Let F be bounded on $[0, \infty)$ and integrable over any finite subinterval such that the* LAPLACE *transform*

$$G(z) = \int_0^\infty F(t) \, e^{-zt} \, dt$$

is well defined and holomorphic on the open half plane $\operatorname{Re} z > 0$. *Suppose that $G(z)$ can be continued* continuously *to the closed half plane* $\operatorname{Re} z \geq 0$ *and that the limit*

$$\lim_{\substack{z\to 0 \\ \operatorname{Re} z \geq 0}} \frac{G(z) - G(0)}{z}$$

exists. Then

$$\int_0^\infty F(t)\, dt$$

exists as an improper integral and equals the value of (the extension of) G at the origin.

Proof. Dividing F by a constant we may assume that $|F(t)| \leq 1$. We set

$$G_\lambda(z) := \int_0^\lambda F(t)\, e^{-zt}\, dt\,, \quad 0 < \lambda < \infty\,.$$

G_λ is an *entire* function, i. e. holomorphic on $\mathbb{C}$, and it suffices to show

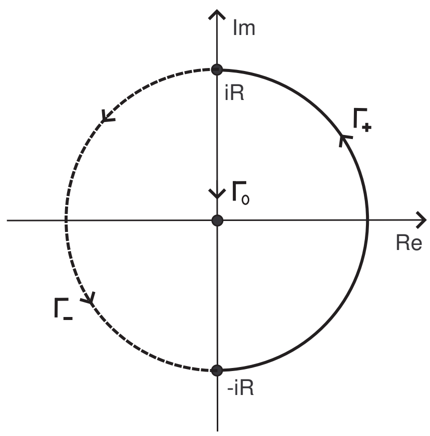
$$G_\lambda(0) = \int_0^\lambda F(t)\, dt \underset{\lambda \to \infty}{\to} G(0)\,.$$

Replacing F on the interval $[0, 1]$ by $\widetilde{F} := F - G(0)$ replaces the values $G(z)$ of the original transform G by

$$\widetilde{G}(z) = G(z) - G(0)\, H(z) \quad \text{with} \quad H(z) = \int_0^1 e^{-zt}\, dt = -\left.\frac{e^{-zt}}{z}\right|_{t=0}^{1} = \frac{1 - e^{-z}}{z}\,.$$

The function $H(z)$ is holomorphic on $\mathbb{C}^*$ with a resolvable singularity and value 1 at the origin. Therefore, also the limit for $\widetilde{G}$ instead of G exists, such that we may assume without loss of generality that $G(0) = 0$.

We next apply Newman's contour integration method; for this, let Γ be a positively oriented circle about the origin with suitable large radius R, and denote by Γ_+ the semicircle $\Gamma \cap \{\operatorname{Re} z \geq 0\}$ and Γ_- the part in the left half plane. Further, let Γ_0 be the oriented segment of the imaginary axes from iR to $-iR$.



By Cauchy's integral formula, we have

$$G_\lambda(0) = G_\lambda(0)\, e^0 = \frac{1}{2\pi i} \int_\Gamma \frac{G_\lambda(z)\, e^{\lambda z}}{z}\, dz = \frac{1}{2\pi i} \int_\Gamma G_\lambda(z)\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz\,.$$

The function

$$G(z)\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right)$$

is, by the assumptions, holomorphic in $\mathrm{Re}\, z > 0$ and continuous on $\mathrm{Re}\, z \geq 0$. Hence, by CAUCHY's Theorem and a simple limit argument,

$$\frac{1}{2\pi i} \int_{\Gamma_+ \cup \Gamma_0} G(z)\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz = 0 = G(0) .$$

Addition of both identities results in a formula which expresses the value $2\pi i\,(G(0) - G_\lambda(0))$ as a sum of the following three integrals (up to sign):

$$\text{(I)} \int_{\Gamma_+} (G(z) - G_\lambda(z))\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \,, \quad \text{(II)} \int_{\Gamma_-} G_\lambda(z)\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \,,$$

$$\text{(III)} \int_{\Gamma_0} G(z)\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \,.$$

It remains to show that each of the three integrals becomes small for large $\lambda \geq \lambda_0$ if the radius R is suitably chosen. We do this for each type separately. First, notice that for $z \in \Gamma$, i. e. $|z| = R$, we have

$$\frac{1}{z} + \frac{z}{R^2} = \frac{z + \overline{z}}{R^2} = \frac{2x}{R^2} \,, \quad x = \mathrm{Re}\, z \,.$$

Hence, if $z \in \Gamma_+$ and I_λ denotes the integrand of the integral (I),

$$|I_\lambda(z)| \leq e^{\lambda x} \frac{2x}{R^2} \left| \int_\lambda^\infty F(t)\, e^{-zt}\, dt \right| \leq e^{\lambda x} \frac{2x}{R^2} \int_\lambda^\infty e^{-xt}\, dt = \frac{2}{R^2} \,,$$

since the last integral is equal to $x^{-1}\, e^{-\lambda x}$. This simple result is the main reason for replacing the factor $1/z$ by the expression $e^{\lambda z}\,(1/z + z/R^2)$; the trick goes back to CARLEMAN. This immediately implies

$$\left| \int_{\Gamma_+} I_\lambda(z)\, dz \right| \leq \frac{2}{R^2}\, \pi R = \frac{2\pi}{R} \,,$$

such that this part of the integral is independently of λ smaller than $2\pi\varepsilon/3$ if $R > 3/\varepsilon$. Similarly, we have on Γ_-

$$|G_\lambda(z)| = \left| \int_0^\lambda F(z)\, e^{-zt} dt \right| \leq \int_0^\lambda e^{-xt} dt \leq - \left. \frac{e^{-xt}}{x} \right|_{t=0}^{\lambda} = \frac{1 - e^{-\lambda x}}{x} < \frac{e^{\lambda |x|}}{|x|}$$

which implies as before that independently of λ the integral of type (II) behaves like $1/R$ for $R \to \infty$. Thus, this part of the integral is again independently of λ for sufficiently large R in absolute value smaller than $2\pi\varepsilon/3$.

It remains to investigate the behaviour of the integral (III)

$$\int_{\Gamma_0} G(z)\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz = -i \int_{-R}^{R} G(it)\, e^{i\lambda t} \left(\frac{1}{it} + \frac{it}{R^2} \right) dt$$

for fixed R when $\lambda \to \infty$. By assumption, the function $\varphi(t) := G(it)\,(1/it + it/R^2)$ is continuous on $\mathbb{R}$. Hence, the integral (III) as a function of λ can be interpreted as the FOURIER transform of an L^1–function on $\mathbb{R}$. By the RIEMANN - LEBESGUE Lemma (cf. Corollary 2.8), the integral tends to 0 with $\lambda \to \infty$. Hence, if for given $\varepsilon > 0$ the radius R is chosen so large that the integrals (I) and (II) are smaller than $2\pi\varepsilon/3$ for all λ, there exists to this R a λ_0 such that for all $\lambda \geq \lambda_0$ the third integral is less than $2\pi\varepsilon/3$, too. Hence, $|G(0) - G_\lambda(0)| < \varepsilon$ for all $\lambda \geq \lambda_0$. □

Remark. It is easy to avoid here the direct use of the RIEMANN - LEBESGUE Lemma via integration by parts ([18], [42]). In any case, we need only a special case of this Lemma. Since we shall apply this version once more in Section 4, we insert a proof.

Lemma 2.8 (Riemann - Lebesgue) *Let* $\varphi : \mathbb{R} \to \mathbb{C}$ *be a continuous function. Then, for all real* $a < b$,

$$\lim_{\lambda \to \pm\infty} \int_a^b \varphi(t)\, e^{i\lambda t}\, dt = 0 .$$

Proof. We may restrict to the case $\lambda \to +\infty$ (otherwise substitute $-t$ for t). To any $\varepsilon > 0$ there exists a continuous function φ_0 with compact support contained, say, in the interval $[a - 1, b + 1]$, such that $\| \varphi_0 - \widetilde{\varphi} \|_{L^1} < \varepsilon$, where $\widetilde{\varphi}$ denotes the trivial extension to $\mathbb{R}$ of the restriction $\varphi_{|[a,b]}$. Since the factor $e^{i\lambda t}$ is of absolute value 1, we can assume without loss of generality that φ itself has already compact support, and we have to show that

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} \varphi(t)\, e^{i\lambda t}\, dt = 0 .$$

Now, for any $c \in \mathbb{R}$,

$$I_\lambda := \int_{-\infty}^{\infty} \varphi(t)\, e^{i\lambda t}\, dt = \int_{-\infty}^{\infty} \varphi(t + c)\, e^{i\lambda (t+c)}\, dt .$$

In particular, setting $c := \pi/\lambda$, we get

$$2\, I_\lambda = \int_{-\infty}^{\infty} \varphi(t)\, e^{i\lambda t}\, dt - \int_{-\infty}^{\infty} \varphi(t + \pi/\lambda)\, e^{i\lambda t}\, dt = \int_{-\infty}^{\infty} \{\varphi(t) - \varphi(t + \pi/\lambda)\}\, e^{i\lambda t}\, dt .$$

By uniform continuity, the expression in brackets tends uniformly to zero for $\lambda \to \infty$ and vanishes on a fixed compact set for all $\lambda \ge \pi$. Thus, $2\, I_\lambda \to 0$ as $\lambda \to \infty$. □

The generalized version of the Karamata - Ingham Theorem 2.7 leads, of course, to a version of the Ikehara - Wiener Theorem which is less restrictive than Corollary 2.2 but still has a simple complex analytic proof.

Corollary 2.9 *Let* f *be non negative, non decreasing and* $O(x)$ *on* $[1, \infty)$ *such that the* Mellin *transform*

$$g(s) = s \int_1^{\infty} f(x)\, x^{-(s+1)}\, dx$$

is well defined and holomorphic on $\mathrm{Re}\, s > 1$. *Suppose that for some real constant* α, *the function*

$$h(s) := g(s) - \frac{\alpha}{s - 1}$$

can be continued continuously *to a neighborhood of every point on the line* $\mathrm{Re}\, s = 1$, *and suppose moreover that the limit*

$$\lim_{\substack{s \to 1 \\ \mathrm{Re}\, s \ge 1}} \frac{h(s) - h(1)}{s - 1}$$

exists. Then

$$\lim_{x \to \infty} \frac{f(x)}{x} = \alpha .$$

3 A short introduction to the ζ - function

We present here a very fast introduction to the ζ–function ζ_A associated to an abelian semigroup as in Section 1 and its connection to the function F_P. The main (elementary) result is contained in the following theorem.

Theorem 3.1 *The integral resp. the infinite series resp. the infinite product*

$$\int_0^\infty e^{-sx} A(x)\, dx \ , \quad \sum_{a \in A} e^{-sa} \ , \quad \sum_{p \in P} e^{-sp} \ , \quad \prod_{p \in P} \frac{1}{1 - e^{-sp}}$$

converge for the same real values $s = \sigma > 0$. If they converge for some real $\sigma_1 > 0$, they converge absolutely and uniformly for all s with Re $s = \sigma \geq \sigma_1$*, and the identities*

$$(+) \qquad s \int_0^\infty e^{-sx} A(x)\, dx = \sum_{a \in A} e^{-sa} = \prod_{p \in P} \frac{1}{1 - e^{-sp}} = \exp\Big(\sum_{p \in P} \sum_{n=1}^{\infty} \frac{e^{-snp}}{n} \Big) \ .$$

hold in this closed half plane.

Sketch of proof (for more details, see e. g Knopfmacher [16] or Lemma 2A in [1]). The statements about *absolute* and *uniform* convergence for the integral resp. the series follow from

$$|\, e^{-sa} \,| = e^{-\sigma a} \leq e^{-\sigma_1 a} \quad \text{for all} \quad \mathrm{Re}\, s := \sigma \geq \sigma_1 \ .$$

If one denotes by $a_0 := 0 < a_1 = p_1 < a_2 < \cdots$ the elements of the semigroup A, then, $A(x) = 0$ for $x < a_1$ and

$$s \int_0^\infty e^{-sx} A(x)\, dx = s \int_0^\infty e^{-sx} \sum_{a \in A, a \leq x} 1\, dx = \sum_{a \in A} s \int_a^\infty e^{-sx}\, dx = \sum_{a \in A} e^{-sa} \ .$$

Since P is a subset of A, we have automatically with $\sum_{a \in A} e^{-sa}$ the (absolute and locally uniform) convergence of the series $\sum_{p \in P} e^{-sp}$, which, by standard results of infinite products, is equivalent to the absolute (and locally uniform) convergence of the infinite product

$$\prod_{p \in P} (1 - e^{-sp}) \neq 0 \quad \text{and hence of} \quad \prod_{p \in P} (1 - e^{-sp})^{-1} \ .$$

Developing $(1 - e^{-sp})^{-1}$ via the geometric series it is clear by the definition of the semigroup A that, at least formally, the infinite product in (+) is equal to the infinite sum in (+). It is not so difficult to make this formal argument rigorous.

The last equality is a consequence of the continuity of the exponential function and the Taylor expansion of the main branch of the logarithm. □

Definition. If $\sigma_0 := \inf \sigma_1$ over all elements σ_1 as in Theorem 3.1 exists as a real number, we define the function ζ_A on the open half plane Re $s > \sigma_0$, the *ζ–function associated to* A, by one of the identical expressions in (+). In the special case $P = \log \mathbb{P}$, this is obviously Riemann's ζ–function

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \ .$$

Due to the statements of Theorem 3.1, ζ_A is as a locally uniform limit of holomorphic functions holomorphic itself, and it has no zeros due to the product expansion which we call, as in the classical situation, its Euler *product (expansion)*.

Remark. Assuming (B_γ) with $\gamma > 1$ for the semigroup A (in particular (W_β) for any $0 < \beta < 1$), it follows from (+++) in the Appendix to Section 5, that $\sigma_0 = 1$.

Using the Euler product expansion of the function ζ_A as in the classical case, one finds easily the relation between its logarithmic derivative and the Mellin *transform* of the function $F = F_P$.

Theorem 3.2

$$g(s) := -\frac{\zeta_A'(s)}{\zeta_A(s)} = s \int_0^\infty e^{-(s-1)x} F_P(x)\, dx \ .$$

Proof. From the EULER product expansion, one deduces immediately

$$g(s) = \sum_{p \in P} \frac{p\, e^{-sp}}{1 - e^{-sp}} = \sum_{p \in P} \left(p \sum_{n=1}^{\infty} e^{-snp} \right) .$$

On the other hand, $F(x) = \sum_{p \in P} F_p(x)$ with $F_p(x) = e^{-x} \left[\frac{x}{p} \right] p$ for fixed $p \in P$, and

$$\int_{\mathbb{R}_+} e^{-(s-1)x} F_p(x)\, dx = \int_{\mathbb{R}_+} e^{-sx}\, p \sum_{n \le x/p} 1\, dx = p \sum_{n=1}^{\infty} \int_{np}^{\infty} e^{-sx}\, dx = \frac{p}{s} \sum_{n=1}^{\infty} e^{-snp} .$$

Adding up over all $p \in P$ leads to the correct result. □

Remarks. 1. Replacing x by $\log x$, the formula in Theorem 3.2 becomes

$$g(s) = s \int_1^{\infty} F(\log x)\, x^{-s}\, dx = s \int_1^{\infty} (x\, F(\log x))\, x^{-(s+1)}\, dx .$$

Now, $x\, F(\log x)$ is non negative and non decreasing. So, also the generalized Prime Number Theorem 1.1 follows from the IKEHARA - WIENER Theorem 2.1 under the assumption that $\sigma_0 = 1$ and the function

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}$$

has a continuous extension to the line $\mathrm{Re}\, s = 1$.

2. If we want a *simple* proof (say via Corollary 2.2) we have not only to show that the function above can *holomorphically* be extended, but also that F is a priori bounded (which somehow was taken for granted by WITT [41]). We will give in the next section an elementary *analytic* argument based on MÜLLER's paper [30] to the effect that our standard assumptions indeed have $F \le M$ as a consequence thus proving the PNT under WITT's conditions. (However, it should be noted that DIAMOND in [7] showed that "Beurling generalized prime numbers satisfy Chebyshev estimates" (hence also boundedness of F) under the much weaker condition (B_γ), $\gamma > 1$). By the same arguments we will see that already the assumptions of Theorem 2.1 imply $f = O(x)$. This opens the way for a simple but not entirely complex analytic proof of the IKEHARA - WIENER Theorem (Section 5); see also KOREVAAR [18] and PITT [32], Chapter 6.1.

Under WITT's condition (W_β), all ingredients for such a simple proof (besides boundedness) are more or less for free. By assumption, the integral

$$\int_0^{\infty} e^{-sx} (A(x) - \alpha\, e^x)\, dx$$

converges absolutely and uniformly for all $\mathrm{Re}\, s = \sigma \ge \sigma_1 > \beta$ and therefore defines by the WEIERSTRASS Regularity Theorem a holomorphic function in the open half plane $\sigma > \beta$ containing 1. Moreover, for $\sigma > 1$,

$$\frac{\alpha\, s}{s-1} + s \int_0^{\infty} e^{-sx} (A(x) - \alpha\, e^x)\, dx = s \int_0^{\infty} e^{-sx} A(x)\, dx .$$

Consequently, ζ_A has a holomorphic extension to the punctured half plane $\{\mathrm{Re}\, s = \sigma > \beta\} \setminus \{1\}$ and a pole of order one at the point $s = 1$ with residue α.

To complete this first part of the simple proof of the PNT we have to show that the logarithmic derivative

$$g(s) := -\frac{\zeta'(s)}{\zeta(s)}$$

is holomorphic in a neighborhood of the closed half plane $\sigma \geq 1$ (except at $s = 1$ where it has a pole of order 1 with residue 1). For this, we need the following generalization of MERTENS result[6]:

Lemma 3.3 ζ *does not vanish on the real line* Re $s = 1$ *.*

Proof. For complex numbers z with $|z| = 1$ one has $\overline{z} = z^{-1}$ and therefore the identity

$$\Big| \sum_{\ell=0}^{m-1} z^\ell \Big|^2 = \Big(\sum_{\ell=0}^{m-1} z^\ell \Big) \Big(\sum_{\ell'=0}^{m-1} \overline{z}^{\ell'} \Big) = \sum_{\ell,\ell'=0}^{m-1} z^{\ell-\ell'} = \sum_{|k|<m} (m - |k|)\, z^k \, .$$

This yields with $z = e^{-inpt}$ for $\sigma > 1$ the relation

$$(++) \qquad \prod_{|k|<m} \zeta(\sigma + kit)^{m-|k|} = \exp \Big(\sum_{\substack{n=1 \\ p\in P}}^{\infty} \frac{e^{-\sigma np}}{n} \Big| \sum_{\ell=0}^{m-1} e^{i\ell npt} \Big|^2 \Big) \geq 1 \, ,$$

in particular, with $m = 3$, that $|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)^2| \geq 1$. But then, it is impossible - since $\zeta^3(s)$ has a pole of order 3 at $s = 1$ - that

$$\lim_{\sigma \searrow 1} \zeta(\sigma + it) = 0 \quad \text{for} \quad t \neq 0 \, . \qquad \square$$

Remarks. 1. This very elegant proof can be found on WITT's page (see also [1], Lemma 8B). Obviously, it works under the a priori weaker assumption that the generalized ζ–function $\zeta(s) := \sum_{a\in A} e^{-sa}$ satisfies what we will call the *standard assumptions*, i. e. it exists (and hence is holomorphic) on the open half plane $\sigma > \sigma_0 = 1$, and

$$\zeta(s) - \frac{\alpha}{s-1}$$

can holomorphically be extended to an open neighborhood of the closed half plane $\sigma \geq 1$ for some $\alpha \neq 0$. In fact, one can use (++) to show that Lemma 3.3 remains valid under the weaker condition (B_γ), $\gamma > 3/2$ (see [1] and the Appendix to Section 5).

2. Another short proof of Lemma 3.3 in the classical case - based on much longer presentations by HADAMARD, DE LA VALLÉE POUSSIN and MERTENS - has been given by ZAGIER [42] (see also DIAMOND [8]).

4 WITT's approach and boundedness of the function F

Our goal now is to show that the function F is always bounded (from above) under the assumption (W_β) (or the weaker "standard" assumptions mentioned in Remark 1 after Lemma 3.3). This can easily be deduced from WITT's approach to the PNT (with its amelioration in [30]). - In fact, we want to show much more, namely:

Theorem 4.1 *Let the real valued function f be non negative and non decreasing on $[1, \infty)$ and suppose that the* MELLIN *transform*

$$g(s) := s \int_1^\infty f(x)\, x^{-(s+1)}\, dx$$

exists for Re $s > 1$. *Suppose, moreover, that for some constant α, the holomorphic function*

$$g(s) - \frac{\alpha}{s-1}$$

has a continuous *extension to the closed half plane* Re $s \geq 1$. *Then, f is $O(x)$.*

[6] As to the RIEMANN *hypothesis* (RH), there are positive and negative examples, even for isotonic sequences which behave similar to the logarithms of the natural numbers; for a more recent paper, see f. i. [43]. This indicates that the RH depends also deeply on the *additive* structure of $\mathbb{N}$.

Corollary 4.2 *In the poor man's* Ikehara–Wiener Theorem 2.2 *and in* Corollary 2.9 *the assumption* $O(x)$ *for* f *may be dropped.*

Question. If one does not like to mix chocolate and vanilla ice cream it is necessary to prove the statement of the preceding corollary exclusively by standard complex analysis. Is this possible?

We write as before $F(x) = e^{-x} f(e^x)$ such that after substitution, as in Theorem 3.2,

$$g(s) = s \int_0^\infty e^{-(s-1)x} F(x)\, dx \,.$$

We then have to conclude that under the assumptions of Theorem 4.1 the function F is bounded. For this, we will first show the boundedness of a certain convolution $F * \Phi$ and need therefore a kind of converse which is implicit in MÜLLER's paper [30].

Lemma 4.3 *Let* Φ *be a non negative integrable function on* $\mathbb{R}$ *with*

$$J := \int_{-a}^{a} \Phi(x)\, dx > 0$$

for some $a > 0$, *and suppose that the convolution* $F * \Phi$ *exists and is bounded. Then* F *is bounded.*

Remark. Note that, in turn, $F \leq M$ implies the existence (and boundedness) of the convolution $F * \Phi$ since F is by assumption locally integrable.

Proof. The main point of the argument is the monotonicity of the function $e^x F(x)$. Take any $x > 0$ and $y \in [-a, a]$. Then

$$e^{x-y} F(x - y) \geq e^{x-a} F(x - a) \quad \text{and} \quad e^{y-x} \geq e^{-(x+a)} \,.$$

From this we deduce that

$$\int_{-a}^{a} e^{x-y} F(x - y)\, e^{y-x}\, \Phi(y)\, dy \geq e^{x-a} F(x - a)\, e^{-(x+a)} \int_{-a}^{a} \Phi(y)\, dy = J\, e^{-2a} F(x - a) \,.$$

But the left hand side is not greater than the convolution $(F * \Phi)(x)$. This implies the result. □

Theorem 4.4 *Let the real valued function* f *be non negative and non decreasing on* $[1, \infty)$ *and suppose that the* MELLIN *transform*

$$g(s) := s \int_1^\infty f(x)\, x^{-(s+1)}\, dx$$

exists for $\operatorname{Re} s > 1$. *Suppose, moreover, that for some constant* α, *the holomorphic function*

$$g(s) - \frac{\alpha}{s - 1}$$

has a continuous *extension to the closed half plane* $\operatorname{Re} s \geq 1$. *Let further* $\Phi \geq 0$ *be the* FOURIER *transform of an even, non negative continuous function* φ *with compact support* $T \subset \mathbb{R}$. *Put* $F(x) := e^{-x} f(e^x)$ *for* $x \geq 0$ *and extend this function trivially to* $\mathbb{R}$. *Then the convolution* $F * \Phi$ *exists, and*

$$\lim_{x \to \infty} (F * \Phi)(x) = \alpha \int_{\mathbb{R}} \Phi(y)\, dy \,.$$

Proof. Let us first remark that by definition the MELLIN transform g has real values for real $s = \sigma$. Hence, the constant α must be real. Moreover, the assumption on φ implies that its FOURIER transform Φ is real–valued and also even:

$$\sqrt{2\pi}\,\Phi(x) = \int \varphi(t)\,e^{-ixt}\,dt = \int \varphi(-t)\,e^{-ixt}\,dt = \int \varphi(t)\,e^{ixt}\,dt = \int \varphi(t)\,\frac{e^{ixt} + e^{-ixt}}{2}\,dt$$
$$= \int \varphi(t)\,\cos(xt)\,dt\,.$$

WITT starts implicitly with the following two–dimensional integral

$$I_\sigma = I_\sigma(x) := \int_{\mathbb{R}\times\mathbb{R}} e^{ixt}\,\varphi(t)\,e^{-sy}\,F(y)\,dt\,dy\,, \quad \text{where } s = \sigma + it\,,\ \sigma > 0 \text{ fixed}\,.$$

Note that the absolute value of the integrand is equal to

$$\varphi(t)\,e^{-\sigma y}\,F(y)\,,$$

and we have by our assumption of g being a MELLIN transform of f that

$$g(s) = s\int_0^\infty e^{-(s-1)x}F(x)\,dx\,, \quad \sigma > 1\,,$$

and therefore the existence of the iterated integrals

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}_+^*} \varphi(t)\,e^{-sy}\,F(y)\,dy\right)dt\,, \quad \sigma > 0\,.$$

But this implies by the Theorem of TONELLI, that I_σ exists for all $\sigma > 0$ such that we get by FUBINI's Theorem (for fixed $\sigma > 0$ and x):

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\sigma y}\,F(y)\,e^{-iyt}\,dy\right)e^{ixt}\,\varphi(t)\,dt = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-i(y-x)t}\,\varphi(t)\,dt\right)e^{-\sigma y}\,F(y)\,dy\,,$$

hence

$$\frac{1}{\sqrt{2\pi}}\int_T e^{ixt}\,\frac{g(1+\sigma+it)}{1+\sigma+it}\,\varphi(t)\,dt = \int_0^\infty e^{-\sigma y}\,F(y)\,\Phi(x-y)\,dy\,.$$

Since also

$$s\int_0^\infty e^{-(s-1)x}\,dx = \frac{s}{s-1}\,,$$

we arrive at WITT's main equality (slightly generalized):

$$(***) \qquad \frac{1}{\sqrt{2\pi}}\int_T e^{ixt}\left\{\frac{g(1+\sigma+it)}{1+\sigma+it} - \frac{\alpha}{\sigma+it}\right\}\varphi(t)\,dt = \int_0^\infty e^{-\sigma y}\,(F(y)-\alpha)\,\Phi(x-y)\,dy\,.$$

We now let σ go to 0. Since the function $(z+1)^{-1}(g(z+1) - \alpha(z+1)/z)$ is uniformly continuous on the rectangle $[0,1]\times T$ we can apply LEBESGUE's Theorem in order to see that the left hand side converges to

$$L(x) := \frac{1}{\sqrt{2\pi}}\int_T e^{ixt}\,\frac{g(1+it) - \alpha/it - \alpha}{1+it}\,\varphi(t)\,dt\,.$$

On the other side, the family $e^{-\sigma y}\,\Phi(x-y)$ converges monotone for fixed x to the integrable function $\Phi(x-y)$. Thus, this part of the right hand side converges due to LEVI's Theorem to

$$\alpha\int_{-\infty}^{x}\Phi(y)\,dy\,.$$

Therefore, the limit

$$\lim_{\sigma \searrow 0} \int_{\mathbb{R}} e^{-\sigma y} F(y)\, \Phi(x - y)\, dy$$

exists, too, and the same reasoning yields as limit the value

$$\int_{\mathbb{R}} F(y)\, \Phi(x - y)\, dy = (F * \Phi)(x) .$$

So, we see that the convolution $F * \Phi$ exists, as desired, and we have at each x the identity

$$(F * \Phi)(x) = L(x) + \alpha \int_{-\infty}^{x} \Phi(y)\, dy .$$

Finally, $L(x)$ being the FOURIER transform of a continuous function on $\mathbb{R}$ with compact support, $\lim_{x \to \infty} L(x) = 0$ by the RIEMANN - LEBESGUE Lemma (see the proof of Lemma 2.8). Thus,

$$\lim_{x \to \infty} (F * \Phi)(x) = \alpha \int_{\mathbb{R}} \Phi(y)\, dy . \qquad \square$$

Proof of Theorem 4.1. We just have to make sure that pairs (φ, Φ) as in Theorem 4.4 and Lemma 4.3 exist. The task can easily be achieved by functions of type $\varphi(u) := \sup(0, b^2(2a - |u|))$. Obviously, φ is equal to the convolution $\lambda * \lambda$ of the function $\lambda := b\, \chi_{[-a,a]}$, χ_I denoting the characteristic function of the interval I. It is well known or easy to compute that

$$\widehat{\lambda}(t) = \sqrt{\frac{2}{\pi}}\, b\, \frac{\sin at}{t} ,$$

and from this it follows that

$$\Phi(t) := \widehat{\varphi}(t) = \widehat{\lambda * \lambda}(t) = \sqrt{2\pi}\, \widehat{\lambda}^{\,2}(t) = c \left(\frac{\sin at}{t} \right)^2 \quad \text{with} \quad c := 2b^2 \sqrt{2/\pi} . \qquad \square$$

Remark. Using the full strength of L^1–theory, we obviously need only the assumption that for $x \searrow 1$ the function $g_x(iy) := g(x + iy)$ converges to $g(1 + y)$ in L^1 on every finite interval $|y| \le C$. Under the stronger assumptions of Theorem 4.4, the use of LEBESGUE's Convergence Theorem can be replaced by arguments with the RIEMANN integral and *uniform* convergence; FUBINI - TONELLI is also true under these assumptions. Finally, it is not difficult to replace B. LEVI's Theorem by an adequate RIEMANN equivalent (see [1], Appendix 5, THEOREM).

Appendix: LEPTIN's interpretation

In the classical case, we have for $\sigma > 0$:

$$e^{-\sigma t} F(t) = e^{-(\sigma+1)t}\, \psi(e^t) \le C\, t\, e^{-\sigma t} .$$

Therefore, the function F_σ defined by

$$F_\sigma(t) := e^{-\sigma t} (F(t) - 1)$$

is contained in $L^1 \cap L^2$. More generally, this is true when F is bounded from above by a polynomial in t, in particular if F is bounded. Then, also the FOURIER transform $\widehat{F_\sigma} \in L^2$. But, by Theorem 3.2,

$$\widehat{F_\sigma}(t) = \frac{g(1 + \sigma + it)}{1 + \sigma + it} - \frac{1}{\sigma + it} .$$

Henceforth, the formula $(***)$ is nothing else but the PLANCHEREL formula applied to the function Φ and the translate $\tau_x F_\sigma$.

5 WITT's proof of the generalized PNT and application to the IKEHARA - WIENER Theorem

Of course, it is not quite reasonable to use the relation $(***)$ just for proving $F \le M$. In fact, it leads to a simple analytic proof of the generalized PNT which depends on a little deeper but still elementary knowledge of FOURIER analysis. The arguments even apply to prove the IKEHARA - WIENER Theorem 2.1 directly.

WITT begins his page with the following remark:

Lemma 5.1 *For every* $\varepsilon > 0$ *there exists an even non negative continuous function* φ *with compact support such that its* FOURIER *transform* Φ *has the following properties* :

$$\int_{\mathbb{R}} \Phi(t)\, dt = 1 \,, \quad e^{-\varepsilon} < \int_{-\varepsilon}^{\varepsilon} \Phi(t)\, dt < 1 \,.$$

Proof. Choosing the constants a and b for φ in the proof of Theorem 4.4 appropriately gives the result: By the inversion formula for FOURIER transforms, we get

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(t)\, e^{itu}\, dt$$

and therefore, with $u = 0$,

$$2\, a\, b^2 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(t)\, dt = \frac{2b^2}{\pi} \int_{\mathbb{R}} \left(\frac{\sin\, at}{t} \right)^2 dt \,,$$

and in particular, as one can also show by complex analytic methods,

$$\int_{\mathbb{R}} \left(\frac{\sin\, t}{t} \right)^2 dt = \pi \,.$$

For fixed $a > 0$, we choose $b > 0$ such that $2\, a\, b^2 \sqrt{2\pi} = 1$. Then we have for all $\varepsilon > 0$:

$$\int_{-\varepsilon}^{\varepsilon} \Phi(t)\, dt < \int_{-\infty}^{\infty} \Phi(t)\, dt = 1 \,.$$

On the other hand, we get by the substitution $\tau := a\, t$ because of $c = 1/(\pi\, a)$ that

$$\int_{-\varepsilon}^{\varepsilon} \Phi(t)\, dt = \frac{1}{\pi a} \int_{-\varepsilon}^{\varepsilon} \left(\frac{\sin\, at}{t} \right)^2 dt = \frac{1}{\pi} \int_{-\varepsilon a}^{\varepsilon a} \left(\frac{\sin\, \tau}{\tau} \right)^2 d\tau \,.$$

For arbitrarily given $\varepsilon > 0$, one can finally choose a so large that

$$\int_{-\varepsilon a}^{\varepsilon a} \left(\frac{\sin\, \tau}{\tau} \right)^2 d\tau > e^{-\varepsilon}\, \pi \,. \qquad \square$$

Remark. The function

$$K(t) = \frac{1 - \cos\, t}{\pi\, t^2} = \frac{1}{2\,\pi} \left(\frac{\sin\, t/2}{t/2} \right)^2$$

is known in the literature as the FEJÉR *kernel*. For $\Phi(t) = r\, K(rt)$ the formula $(***)$ is nothing else but the "basic relation" in the proofs by WIENER and BOCHNER (see [18], p. 483).

In order to prove the generalized PNT and even the IKEHARA - WIENER Theorem 2.1 after WITT, we follow once more the notes of LEPTIN. We put again $F(x) = e^{-x} f(e^x)$ and know by Theorem 4.1

that F is bounded. Similarly as in Section 4, it follows by the isotony of the function $x \longmapsto e^x F(x)$ that

$$e^{x-\varepsilon} F(x-\varepsilon) \leq e^{x-y} F(x-y) \leq e^{x+\varepsilon} F(x+\varepsilon)$$

for all y with $|y| \leq \varepsilon$ independently of x. Multiplication with e^{y-x} gives then

$$e^{-2\varepsilon} F(x-\varepsilon) \leq F(x-y) \leq e^{2\varepsilon} F(x+\varepsilon) .$$

After choosing Φ with respect to ε as in Lemma 5.1, this implies

$$e^{-3\varepsilon} F(x-\varepsilon) \leq e^{-2\varepsilon} F(x-\varepsilon) \int_{-\varepsilon}^{\varepsilon} \Phi(y)\, dy \leq \int_{-\varepsilon}^{\varepsilon} F(x-y)\, \Phi(y)\, dy \leq e^{2\varepsilon} F(x+\varepsilon)$$

for $|y| \leq \varepsilon$ and

$$e^{-3\varepsilon} F(x-\varepsilon) \leq (F * \Phi)(x) .$$

Using the bound $F \leq M$ the choice of Φ gives the inequalities

$$0 \leq (F * \Phi)(x) - \int_{-\varepsilon}^{\varepsilon} F(x-y)\, \Phi(y)\, dy = \int_{|y| \geq \varepsilon} F(x-y)\, \Phi(y)\, dy \leq M(1 - e^{-\varepsilon}) .$$

Therefore,

$$F(x+\varepsilon) \geq e^{-2\varepsilon} \left((F * \Phi)(x) - M(1 - e^{-\varepsilon})\right) \quad \text{and} \quad F(x-\varepsilon) \leq e^{3\varepsilon} \left((F * \Phi)(x) + M(1 - e^{-\varepsilon})\right) .$$

Because of $\lim_{x \to \infty} (F * \Phi)(x) = \alpha$ (Theorem 4.4), the claim follows: $\lim_{x \to \infty} F(x) = \alpha$. □

Remark. According to our remarks at the end of Section 4, the proof of the IKEHARA - WIENER Theorem 2.1 presented here works under weaker conditions on the extension of g thus leading to the conclusion of Theorem 4.2 of [19] by almost the same path of reasoning. Moreover, Theorem 2.1 itself can be proved without referring to the LEBESGUE integral.

Appendix: Simple proof under BEURLING's conditions

Let us finish this survey with some comments on what we can achieve with our approach when we only assume one of the less restrictive conditions (B_γ) instead of (W_β). We first notice (see Section 3) that for $\sigma > 1$ we have

$$(+++) \qquad \zeta_A(s) = \frac{\alpha\, s}{s-1} + s\, \rho(s) \quad \text{with} \quad \rho(s) = \int_0^\infty e^{-sx} \left(A(x) - \alpha\, e^x\right) dx .$$

Certainly, $\rho(s)$ is a holomorphic function with derivatives

$$\rho^{(n)}(s) = \int_0^\infty (-x)^n e^{-sx} \left(A(x) - \alpha\, e^x\right) dx .$$

It is easily seen that this derivative can continuously be extended to $\sigma \geq 1$ if the semigroup A satisfies (B_γ) with $\gamma > n + 1$. So, we may conclude:

Theorem 5.2 *If $\gamma > n+1$, then ζ_A and all its derivatives up to the n–th order have* continuous *extensions to the punctured line* $\{\mathrm{Re}\ s = 1\} \setminus \{1\}$.

In particular, if $\gamma > 1$, the function ζ_A has a continuous extension to the punctured line $\{\sigma = 1\} \setminus \{1\}$, and, moreover (cf. [1] and the concluding remarks at the end of this appendix), this extension has no zeros if $\gamma > 3/2$. Therefore, if we suppose that $\gamma > 2$, we know that the quotient ζ_A'/ζ_A can continuously be extended to the line $\sigma = 1$ except at the point 1. More precisely, one can easily prove the following lemma (see [2], Lemma 3.4).

Lemma 5.3 *Define the function* h *on the open half plane* $\{\sigma = \mathrm{Re}\, s > 1\}$ *by*

$$h(s) := -\frac{\zeta_A'(s)}{\zeta_A(s)} - \frac{1}{s-1} .$$

Then, for $\gamma > 2$, h *can continuously be extended to the line* $\sigma = 1$, *and it exists the limit*

$$\lim_{\substack{s \to 1 \\ \mathrm{Re}\, s \geq 1}} \frac{h(s) - h(1)}{s-1} .$$

Using also our former considerations, especially Corollary 2.9, this immediately yields a (quite) *simple* proof of the following part of Beurling's Theorem, as was first noticed by Bekehermes [2].

Theorem 5.4 *The abstract PNT holds under the condition* (B_γ), $\gamma > 2$.

Remark. Bekehermes also invoked the Chebyshev estimates of [7] for his proof. We can avoid this by referring, once more, to Theorem 4.1.

Let us add a few remarks on the non vanishing of the ζ–function on $\sigma = 1$ for $\gamma > 3/2$ following [1]. Bateman and Diamond first conclude that $\zeta = \zeta_A$ cannot tend to zero too quickly as $s \to 1 + it$. In fact, if $\gamma > 1$, then for any non zero real number t and any positive integer m there exists a positive constant $C = C(m, t)$ such that

$$|\zeta(\sigma + it)| \geq C(\sigma - 1)^{1/2+1/2m}$$

(see [1], Lemma 8C; to come to this result, the authors apply $(+{+}+)$ for arbitrary m). On the other hand (loc. cit., Lemma 8D) one has a Lipschitz condition for $1 < \gamma < 2$ (this is no loss of generality since (B_γ) implies $(B_{\gamma'})$ for all $\gamma' < \gamma$):

$$|\zeta(s_1) - \zeta(s_2)| \leq D\, |s_1 - s_2|^{\gamma-1}$$

for all $s_1, s_2 \in \Omega(\delta, \Delta)$ and $D = D(\delta, \Delta)$, where $0 < \delta < \Delta - 1$ and

$$\Omega(\delta, \Delta) := \{ s \in \mathbb{C} : \mathrm{Re}\, s \geq 1 ,\ |s - 1| \geq \delta ,\ |s| \leq \Delta \} .$$

So, if $\zeta(1 + it_0) = 0$ for some $t_0 \in \mathbb{R}^*$, then for all m and all σ with $1 < \sigma < 2$:

$$C(\sigma - 1)^{1/2+1/2m} \leq |\zeta(\sigma + it_0)| \leq D(\sigma - 1)^{\gamma-1} .$$

This obviously leads to a contradiction when m is so large that $1/(2m) < \gamma - 3/2$.

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