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Projective Resolutions of Cohen-Macaulay Algebras

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Introduction

The problem of explicitly finding a free resolution, minimal in some suitable sense, of a module over a polynomial ring is solved in principle by the algorithm of Hilbert [H]. However, this algorithm is of enormous computational difficulty. If the module happens to be finite dimensional over the ground field, and if the module structure is given by specifying the commuting linear transformations induced by the indeterminates, then a little-known result of Scheja and Storch, resumed in Sect. 1, allows one to write down an explicit free resolution of the right length without computation. The same idea can be used for many Cohen-Macaulay modules (Example 1.1). But although the Scheja-Storch resolution is minimal in some cases, it is not minimal in the main case of interest where the module is a factor-ring of the polynomial ring and not the ground field itself.

It is the goal of this paper to write down a universal resolution for certain factor-rings which is minimal in some cases of interest. We obtain in particular a free resolution, over a polynomial ring, for any affine Cohen-Macaulay ring (Example 3.3), and the resolution is minimal in a suitable sense if the ring has (locally at some point) minimal multiplicity for its embedding dimension; the cases of main interest are perhaps the 2-dimensional rings with rational singularities (Sect. 4), and the ("relatively Cohen-Macaulay") total space of the versal deformation of a ring of the form $k[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^2$ (Example 3.1).

We now describe the situation in which we work, beginning with the leading special case (all rings in this paper are commutative and associative, algebras have a unit element 1):

Let R be a ring, and let A be an R-algebra which is finitely generated and projective as an R-module, with $R \cong R \cdot 1 \subset A$. Since 1 is locally part of a minimal system of generators of A as an R-module, we may write $A = R \oplus E$ (as R-modules) for some finitely generated projective module E. Given the decomposition above, there is a natural epimorphism from the symmetric algebra $S := S(E) = \sum_{0 \leq k} S_k E$ to A, which is an isomorphism on $R \oplus E = S_0(E) \oplus S_1(E) \subset S$.

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More generally, suppose that E is a finitely generated projective R-module and A is any factor-ring of $S(E)$ such that for some $l \geqq 2$ the induced map

$$\sum_{0 \leqq k \leqq l-1} S_k(E) \to A$$

is an isomorphism. Our main result (Theorem 3.2) gives a projective resolution of A as an $S(E)$-module in this case, and describes when this resolution is minimal.

This is applied to affine Cohen-Macaulay rings, as indicated above, by involving the Noether normalization theorem.

Of course the simplest case of our result is that in which $A \cong R[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^l$ as an algebra. The well-known minimal free resolution of A as an $R[x_1, \ldots, x_r]$-module may be described as an Eagon-Northcott complex associated to the $l \times l$ minors of the $r+l-1$ by l matrix

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_{r-1} & x_r & 0 & \ldots & 0 \\ 0 & x_1 & x_2 & \ldots & x_{r-1} & x_r & \ldots & 0 \\ & & \ddots & \ddots & & \ddots & \ddots & \\ 0 & \ldots & & x_1 & x_2 & \ldots & x_{r-1} & x_r \end{pmatrix},$$

or more intrinsically, by the complexes in [B-E]. The resolutions of our paper can be seen as deformations of this last.

The resolution described below was originally obtained by the third author, in the setting of an explicit versal deformation of $k[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^2$; he used the techniques of the generalized preparation theorem to give a direct proof of exactness. The current version of the results was subsequently obtained jointly.

1. The "Universal" Resolution of Scheja and Storch

As in the introduction, let R be a (commutative) ring, and let E be a finitely generated projective R-module. Let $S = S(E) = \sum_{0 \leqq k} S_k E$ be the symmetric algebra of E over R; for any R-module M, we write $\tilde{M}$ for the S-module $S \otimes_R M$.

For each k, there is a "diagonal" map $\Delta : \Lambda^k E \to \Lambda^{k-1} E \otimes_R E$, defined by

$$\Delta(e_1 \wedge \ldots \wedge e_k) = \sum_{i=1}^{k} (-1)^{i-1} e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_k \otimes e_i,$$

where the presence of $\widehat{e_i}$ means that e_i has been left out.

Given any S-module N, there is a "multiplication" map $m : E \otimes_R N \to N$, and we may define for each k a map

$$d_N : \Lambda^k E \otimes_R N \to \Lambda^{k-1} E \otimes_R N$$

as the composite

$$\Lambda^k E \otimes_R N \xrightarrow{\Delta \otimes 1} \Lambda^{k-1} E \otimes_R E \otimes_R N \xrightarrow{1 \otimes m} \Lambda^{k-1} E \otimes_R N.$$

From the fact that $e_i(e_j n) = e_j(e_i n)$ for all i, j and all $n \in N$, it follows at once that $d_N^2 = 0$.

Theorem 1.1 (Scheja-Storch). *Let N be an S-module which is finitely generated and projective as an R-module. For each k, define*

$$\delta : \Lambda^k \tilde{E} \otimes_R N \to \Lambda^{k-1} \tilde{E} \otimes_R N$$

as

$$\delta = d_S \otimes 1 - 1 \otimes d_N : S \otimes_R \Lambda^k E \otimes_R N \to S \otimes_R \Lambda^{k-1} E \otimes_R N .$$

We have $\delta^2 = 0$, and (assuming that $\Lambda^{r+1} E = 0$) the complex

$$\mathbf{K}_N : 0 \longrightarrow \Lambda^r \tilde{E} \otimes_R N \xrightarrow{\delta} \dots \xrightarrow{\delta} \tilde{E} \otimes_R N \xrightarrow{\delta} S \otimes_R N \longrightarrow 0$$

is a projective resolution of the S-module N.

If R is local with maximal ideal $\mathfrak{m}$ and $EN \subset \mathfrak{m}N$, then $\mathbf{K}_N$ is minimal in the sense that it gives a minimal resolution of N as an $S_{(\mathfrak{m}, S_1 E)}$-module.

Proof [Sch-St, pp. 87–88]. Consider the "enveloping algebra" $\tilde{S} = S \otimes_R S = S(E \oplus E)$, and write $\tilde{\tilde{E}}$ for $\tilde{S} \otimes_S \tilde{E}$. The "diagonal" map $E \to E \oplus E$, given by $e \mapsto (e, -e)$, induces a map $\tilde{\tilde{E}} \to \tilde{S}$, from which we may form the Koszul complex

$$\tilde{\mathbf{K}}_S : 0 \to \Lambda^r \tilde{\tilde{E}} \to \dots \to \Lambda^2 \tilde{\tilde{E}} \to \tilde{\tilde{E}} \to \tilde{S}$$

which is an $\tilde{S}$-projective resolution of S as an $\tilde{S}$-module under the natural augmentation $\tilde{S} \to S$.

Regarding $\tilde{S}$ as an S-module by multiplication in the second component, one sees that

$$\mathbf{K}_N = \tilde{\mathbf{K}}_S \otimes_S N .$$

Since $\tilde{\mathbf{K}}_S$ is split exact as a complex of S-modules, this is exact, and gives a resolution of N as an S-module, as required.

The minimality statement follows at once because, under the given hypothesis, $\delta \otimes S/(\mathfrak{m}, S_1 E) = 0$. □

Example 1.1. Let $S = k[y_1, \dots, y_{r+d}]$ be a polynomial ring over a field k, and let F be any finitely generated S-module of dimension d. By the Noether normalization theorem, we may choose new variables $x_1, \dots, x_r, t_1, \dots, t_d$ so that $S = k[x_1, \dots, x_r, t_1, \dots, t_d]$ and F is a finitely generated $R = k[t_1, \dots, t_d]$-module.

If now F is a Cohen-Macaulay module – that is with $\operatorname{depth}_{(y_1, \dots, y_{r+d})} F = d$ – then by the Auslander-Buchsbaum-Serre theorems F will be free over R, and Theorem 1.1 will apply. The resolution will be minimal [after localizing at $(y_1, \dots, y_{r+d}) = \mathfrak{m}$, say] if and only if the minimal number of generators of $F_\mathfrak{m}$ as an $S_\mathfrak{m}$-module is equal to the rank of F over R.

Example 1.2 (the generic case). Let R be a polynomial ring over $\mathbb{Z}$ in nr^2 variables, which we think of as forming n r by r matrices X_i, modulo the $r^2\binom{n}{2}$ quadratic relations making these matrices commute. If F is a free R-module of rank r, we may make F into an $S = R[x_1, \dots, x_n]$-module by letting x_i act as X_i. The theory above applies to give a resolution of this module.

2. Some Projective Modules and Complexes

In this section we recall from [B-E] the construction of the modules which play a role in the resolutions written down in the next section. For a treatment in the natural full generality of Schur functors, see [Las], [A-B-W], and the works cited there, and for a slightly different construction, see [Tow].

Let E be a module over a ring R. As before, we write $\Delta : \Lambda^{k+1}E \to \Lambda^k E \otimes E$ for the diagonal map of the exterior algebra, $m : E \otimes S_{l-1}E \to S_l E$ for the multiplication in the symmetric algebra, and we define d_E as the composite

$$\Lambda^{k+1}E \otimes S_{l-2}E \xrightarrow{\Delta \otimes 1} \Lambda^k E \otimes E \otimes S_{l-2}E \xrightarrow{1 \otimes m} \Lambda^k E \otimes S_{l-1}E .$$

(All tensor products in this section are taken over R.)

Definition. $L_l^k E = \operatorname{coker}(\Lambda^{k+1}E \otimes S_{l-2}E \xrightarrow{d_E} \Lambda^k E \otimes S_{l-1}E)$.
It is obvious from the definition that $L_1^k E = \Lambda^k E$ and $L_l^k E = 0$ whenever $\Lambda^k E = 0$.

Proposition 2.1. *Suppose that E is projective, and $k + l > 0$.*
1) *The complex*

$$(*)_{k+1} \qquad \dots \xrightarrow{d_E} \Lambda^{k+1}E \otimes S_{l-1}E \xrightarrow{d_E} \Lambda^k E \otimes S_l E \xrightarrow{d_E} \dots$$

is split exact. In particular, $L_l^k E$ is a projective R-module, $L_l^1 E \cong S_l E$, $l \geqq 1$, and there is a natural exact sequence

$$0 \to L_l^k E \to \Lambda^{k-1}E \otimes S_l E \to L_{l+1}^{k-1} E \to 0 .$$

2) *Let $F = E \oplus Rf$, f being linearly independent over R, and let $\pi : F \to E$ be the projection. Defining the dotted arrows in the following diagram to be the indicated composites, the sequence of dotted arrows is exact:*

$$\begin{tikzcd}
 & & \Lambda^k F \otimes S_{l-1}F \arrow[d, "\Lambda^k\pi\otimes 1"] & & \\
0 \arrow[r, dotted] & L_{l-1}^{k+1}F \arrow[ur] \arrow[r, dotted, "in_{l-1}^{k+1}"'] & \Lambda^k E \otimes S_{l-1}F \arrow[r, dotted] \arrow[d, "1\otimes S_{l-1}\pi"] & L_l^k E \arrow[r, dotted] & 0 \\
 & & \Lambda^k E \otimes S_{l-1}E \arrow[ur] & &
\end{tikzcd}$$

Remark. If, in part 1), E is actually free, then so is $L_l^k E$. As in [B-E] (or, in a more general case [A-B-W]) one can show that if $e_1, \dots, e_r$ is a basis for E, then $L_l^k E$ admits as basis the images of the elements

$$e_{i_1} \wedge \dots \wedge e_{i_k} \otimes e_{j_1} \dots e_{j_1} \dots e_{j_{l-1}} \in \Lambda^k E \otimes S_{l-1}E$$

with $i_1 < \dots < i_k$ and $i_1 \leqq j_1 \leqq \dots \leqq j_{l-1}$.

Proof [B-E]. 1) The identity map $E \to E$ induces a map

$$\tilde{E} = S(E) \otimes E \to S(E),$$

and we may form the Koszul complex

$$(**) \qquad \dots \to \Lambda^2 \tilde{E} \to \tilde{E} \to S(E),$$

which is exact except at $S(E)$, where it has homology R. It is easy to see that $(**)$ is the direct sum, over all $k+l \geqq 0$, of the sequences $(*)_{k+l}$. Further the sequence $(*)_0$ is $0 \to R \to 0$, accounting for the homology of $(**)$, and thus $(*)_{k+l}$ for $k+l>0$ is exact as claimed.

Now for any given k, l, we have an exact sequence

$$0 \to L_l^k E \to \Lambda^{k-1} E \otimes S_l E \to \ldots \to S_{l+k-1} E \to 0 .$$

If E is projective, then all $\Lambda^i E \otimes S_j E$ are projective, so the sequence splits, and $L_l^k E$ is projective too.

The exactness of $(*)_l$ for $l \geqq 1$ includes in particular the exactness of

$$\Lambda^2 E \otimes S_{l-2} E \to E \otimes S_{l-1} E \to S_l E \to 0 ,$$

whence $L_l^1 E \cong S_l E$. $(*)_{k+l-1}$ further yields short exact sequences

$$0 \to L_{l-1}^{k+1} E \to \Lambda^k E \otimes S_{l-1} E \to L_l^k E \to 0 ,$$

concluding the proof of part 1).

2) Write C for the cokernel of the map

$$\Lambda^{k+1} F \otimes S_{l-2} F \xrightarrow{(\Lambda^k \pi) \circ d_F} \Lambda^k E \otimes S_{l-1} F .$$

It suffices to show that in the commutative diagram

$$\begin{tikzcd}
\Lambda^{k+1} F \otimes S_{l-2} F \arrow[r, two heads] \arrow[rr, bend left, "d_F"] \arrow[d, "1"] & L_{l-1}^{k+1} F \arrow[r] \arrow[dr, dashed] & \Lambda^k F \otimes S_{l-1} F \arrow[d, "(\Lambda^k \pi) \otimes 1"] & \\
\Lambda^{k+1} F \otimes S_{l-2} F \arrow[rr, "(\Lambda^k \pi) \circ d_F"] \arrow[d, "(\Lambda^{k+1} \pi) \otimes (S_{l-2} \pi)"] & & \Lambda^k E \otimes S_{l-1} F \arrow[r, two heads] \arrow[d, "1 \otimes (S_{l-1} \pi)"] & C \arrow[d, dashed] \\
\Lambda^{k+1} E \otimes S_{l-2} E \arrow[rr, "d_E"] & & \Lambda^k E \otimes S_{l-1} E \arrow[r, two heads] & L_l^k E
\end{tikzcd}$$

the lower square induces an isomorphism $C \dashrightarrow L_l^k E$ and the upper square induces a monomorphism $L_{l-1}^{k+1} F \dashrightarrow \Lambda^k E \otimes S_{l-1} F$.

The first of these facts follows from the exactness of the rows and columns in the commutative diagram

$$\begin{tikzcd}
& \Lambda^k E \otimes S_{l-2} F \arrow[dl, "(f \wedge -) \otimes 1"'] \arrow[d, "1 \otimes (f \cdot -)"] & & \\
\Lambda^{k+1} F \otimes S_{l-2} F \arrow[r] \arrow[d] & \Lambda^k E \otimes S_{l-1} F \arrow[r, two heads] \arrow[d] & C \arrow[r] \arrow[d, dashed] & 0 \\
\Lambda^{k+1} E \otimes S_{l-2} E \arrow[r] \arrow[d] & \Lambda^k E \otimes S_{l-1} E \arrow[r] \arrow[d] & L_l^k E \arrow[r] & 0 \\
0 & 0 , & &
\end{tikzcd}$$

where we have written $f\cdot -$ and $f\wedge -$ for multiplication by f in the symmetric and exterior algebras, respectively.

The second fact follows from the exactness of the row and column in the following commutative diagram, together with the fact that $f\cdot -$ is a monomorphism:

$$\begin{tikzcd}
 & & \Lambda^{k-1}E\otimes S_{l-1}F \arrow[r, "1\otimes(f\cdot -)"] \arrow[d, "(f\wedge -)\otimes 1"] & \Lambda^{k-1}E\otimes S_l F \\
0 \arrow[r] & L^{k+1}_{l-1}F \arrow[r] \arrow[dr, dashed] & \Lambda^k F\otimes S_{l-1}F \arrow[r, "d_F"] \arrow[d] & \Lambda^{k-1}F\otimes S_l F \arrow[u, "(\Lambda^{k-1}\pi)\otimes 1"'] \\
 & & \Lambda^k E\otimes S_{l-1}F \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

This concludes the proof of Proposition 2.1. $\square$

3. Resolving an Algebra

For the moment, let S be a ring and G a projective S-module, of rank $r+1$, say. Given a map $\phi: G\to S$, there is for each $l\geqq 1$ an induced map

$$S_l\phi: S_l G\to S,$$

given by

$$(S_l\phi)(g_1 g_2\dots g_l)=\phi(g_1)\,\phi(g_2)\dots\phi(g_l).$$

One finds in [B-E] a complex of projective S-modules

$$\mathbf{P}_l(\phi): 0\to L_l^{r+1}G\xrightarrow{d_l\phi}\cdots\to L_l^2 G\xrightarrow{d_l\phi} S_l G\xrightarrow{S_l\phi} S,$$

whose homology is annihilated by the image of $S_l\phi$, defined as follows:

For each $k\geqq 2$, we define $d_l\phi: L_l^k G\to L_l^{k-1}G$ as the map induced by the commutative diagram

$$\begin{tikzcd}
\Lambda^{k+1}G\otimes_S S_{l-2}G \arrow[r, "d_\phi\otimes 1"] \arrow[d] & \Lambda^k G\otimes_S S_{l-2}G \arrow[d] \\
\Lambda^k G\otimes_S S_{l-1}G \arrow[r, "d_\phi\otimes 1"] \arrow[d] & \Lambda^{k-1}G\otimes_S S_{l-1}G \arrow[d] \\
L_l^k G \arrow[r, dashed, "d_l\phi"] \arrow[d] & L_l^{k-1}G \arrow[d] \\
0 & 0
\end{tikzcd}$$

where $d_\phi : \Lambda^{s+1}G \to \Lambda^s G$ is the differential of the Koszul complex associated to the map $G \to S$. [Note that $\mathbf{P}_1(\phi)$ is just the Koszul complex.] Here we identify $L_l^1 G$ with $S_l G$.

If E is a projective R-module, $S = S(E)$, $G = S \otimes_R E$, and if $\phi : G \to S$ denotes the multiplication map, then $\mathbf{P}_l(\phi)$ becomes a projective resolution of the ideal $\sum_{l \leq k} S_k E \subset S$; more generally, if the image of G in S contains a regular sequence of length $r+1$, then $\mathbf{P}_l(\phi)$ yields a resolution for the l^{th} power of the ideal generated by this sequence.

The proof that the homology of $\mathbf{P}_l(\phi)$ is annihilated by the image of $S_l \phi$ is a rather easy induction on l from the well-known case $l = 1$ (see [B-E], where $\mathbf{P}_l(\phi)$ is called $\mathbf{L}_l^1(\phi)$, for more details).

With this out of the way, we now suppose that $A \cong \sum_{k \leq l-1} S_k E$. We put $F = S_0 E \oplus S_1 E = Rf \oplus E$ and $S = S(E)$. Note that as an R-module A may be naturally identified with $S_{l-1} F$. Thus the Scheja-Storch resolution of A as an S-module may be written

$$\mathbf{K}_A : \ldots \to \Lambda^k \tilde{E} \otimes_R S_{l-1} F \xrightarrow{\delta} \Lambda^{k-1} \tilde{E} \otimes_R S_{l-1} F \to \ldots$$

On the other hand, the natural map $F = S_0 E \oplus S_1 E \hookrightarrow S(E) = S$ induces a map

$$\phi : \tilde{F} = S \otimes_R F \to S$$

(whose image is all of S), and we may form the complex

$$\mathbf{P}_{l-1}(\phi) : \ldots \to L_{l-1}^{k+1} \tilde{F} \xrightarrow{d_{l-1}\phi} L_{l-1}^k \tilde{F} \to \ldots$$

which is exact, since its homology is annihilated by all of S.

Lemma 3.1. *For each $l \geq 2$ and $k \geq 1$, the diagram*

$$\begin{array}{ccc} L_{l-1}^{k+1} \tilde{F} & \xrightarrow{d_{l-1}\phi} & L_{l-1}^k \tilde{F} \\ \downarrow{\scriptstyle in_{l-1}^{k+1}} & & \downarrow{\scriptstyle in_{l-1}^k} \\ \Lambda^k \tilde{E} \otimes_R S_{l-1} F & \xrightarrow{\delta} & \Lambda^{k-1} \tilde{E} \otimes_R S_{l-1} F \end{array}$$

is anti-commutative.

Proof. The proof is a computation, which we outline. From the definitions we see that one must check the anti-commutativity of a diagram of the form

$$\begin{array}{ccc} \Lambda^{k+1} F \otimes_R S_{l-2} F & \xrightarrow{d_\phi \otimes 1} & \Lambda^k \tilde{F} \otimes_R S_{l-2} F \\ \downarrow{\scriptstyle (\Lambda^k \pi) \circ d_F} & & \downarrow{\scriptstyle S \otimes_R ((\Lambda^{k-1}\pi) \circ d_F)} \\ \Lambda^k E \otimes_R S_{l-1} F & \xrightarrow{\delta} & \Lambda^{k-1} \tilde{E} \otimes_R S_{l-1} F, \end{array}$$

where we have written π for the projection $F \to E$ with kernel Rf.

Let $\Phi \in S_{l-2}F$ and $e_1, \ldots, e_{k+1} \in E$. An element of $\Lambda^{k+1}F \otimes S_{l-2}F$ is a linear combination of elements of the form

$$e_1 \wedge \ldots \wedge e_{k+1} \otimes \Phi = e_I \otimes \Phi$$

or

$$f \wedge e_1 \wedge \ldots \wedge e_k \otimes \Phi = f \wedge e_J \otimes \Phi .$$

Define $e_{I-i} := (-1)^{i-1} e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{k+1}$, and in a similar way e_{J-i} and e_{I-i-j}. Examining the first case, we see that

$$[S \otimes ((\Lambda^{k-1}\pi) \circ d_F][d_\phi \otimes 1](e_I \otimes \Phi) = \sum_{i \neq j} e_i \otimes e_{I-i-j} \otimes (e_j \Phi),$$

while

$$\delta[(\Lambda^k \pi) \circ d_F](e_I \otimes \Phi) = \sum_{i \neq j} e_i \otimes e_{I-j-i} \otimes (e_j \Phi) + \sum_{i \neq j} e_{I-i-j} \otimes (e_i e_j \Phi).$$

Since $e_i e_j = e_j e_i$ and $e_{I-i-j} = -e_{I-j-i}$, the last term is zero and the first differs by sign from the term above. In the second case, we get the following two results:

$$\sum_i e_{J-i} \otimes (e_i \Phi) - \sum_i e_i \otimes e_{J-i} \otimes (f \Phi)$$

resp.

$$\sum_i e_i \otimes e_{J-i} \otimes (f \Phi) - \sum_i e_{J-i} \otimes (e_i f \Phi).$$

Since f acts as the identity element in A, the proof is finished. □

From Lemma 3.1 and Proposition 2.1, 2), we see that δ induces a differential

$$\partial_A : L_l^k \tilde{E} \to L_l^{k-1} \tilde{E}.$$

We can now state our main result:

Theorem 3.2. *Let A be an algebra over the ring R, and suppose $E \subset A$ is a projective submodule with the property that, for some $l \geqq 2$, the natural map*

$$\sum_{k \leqq l-1} S_k E \to A$$

is an isomorphism. Let

$$\partial_A : S_l E \to S = S(E)$$

be given by

$$\partial_A(e) = e - \bar{e},$$

where $\bar{e}$ is the image of e under the map $S(E) \to A \cong \sum_{k \leqq l-1} S_k E \subset S$, and let ∂_A also denote the extended map

$$\partial_A : S_l \tilde{E} \to S, \quad \text{where} \quad \tilde{E} = S \otimes_R E.$$

If $\partial_A : L_l^k\tilde{E} \to L_l^{k-1}\tilde{E}$ is defined as above for $k \geqq 2$, and if $\Lambda^{r+1}E = 0$, then the complex

$$\mathbf{L}_{A,E} : 0 \to L_l^r\tilde{E} \xrightarrow{\partial_A} L_l^{r-1}\tilde{E} \to \ldots \to L_l^2\tilde{E} \xrightarrow{\partial_A} S_l\tilde{E} \xrightarrow{\partial_A} S$$

is a projective resolution of A as an S-module.

Further, if R is local with maximal ideal $\mathfrak{m}$, then the above resolution is minimal (after localizing S at $(\mathfrak{m}, E)$, say) if the product $E \cdot S_{l-1}E \subset A$ is contained in $\mathfrak{m}A$.

Proof. Consider the short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & L_{l-1}^{r+1}\tilde{F} \arrow[r, "d_{l-1}\phi"] \arrow[d] & \ldots \arrow[r] & L_{l-1}^2\tilde{F} \arrow[r] \arrow[d] & S_{l-1}\tilde{F} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda^r\tilde{E} \otimes S_{l-1}F \arrow[r, "\delta"] \arrow[d] & \ldots \arrow[r] & \tilde{E} \otimes S_{l-1}F \arrow[r] \arrow[d] & S_{l-1}\tilde{F} \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
0 \arrow[r] & L_l^r\tilde{E} \arrow[r, "\partial_A"] & \ldots \arrow[r] & L_l^1\tilde{E} \arrow[r] & 0 & &
\end{tikzcd}$$

Considering the long exact sequence in homology associated with the parts of these complexes to the left of the vertical line, we see that there is an exact sequence

$$0 \to L_l^r\tilde{E} \xrightarrow{\partial_A} \ldots \xrightarrow{\partial_A} L_l^1\tilde{E} \xrightarrow{\text{“}\partial_A\text{”}} S \to A \to 0,$$

where we have written “∂_A” for the connecting homomorphism. Identifying $L_l^1\tilde{E}$ with $S_l\tilde{E}$, and tracing through the diagram to identify the connecting homomorphism, we get the desired result.

The minimality statement is verified by noting that under the map

$$\Lambda^k E \otimes S_{l-1}F \to L_l^k E,$$

the submodule $\Lambda^k E \otimes S_{l-1}E$ is mapped onto $L_l^k E$, and under the given hypothesis for minimality, $\delta \otimes S/(\mathfrak{m}, E)$ is zero on $\Lambda^k E \otimes S_{l-1}E$. □

Example 3.1 (the generic algebra). Let

$$R = \mathbb{Z}[a_{ij}^\varrho]/I,$$

where I is the ideal of relations that make the product law

$$e_i e_j = \sum_{\varrho=0}^{r} a_{ij}^\varrho e_\varrho, \qquad 0 \leqq i, j \leqq r,$$

into a commutative and associative algebra structure on $F = R^{r+1}$ with unit element e_0; i.e.: I is generated by the elements

$$a_{ij}^k - a_{ji}^k,\ a_{i0}^k - \delta_i^k,\ \sum_{\varrho=0}^{r} (a_{ij}^\varrho a_{k\varrho}^l - a_{kj}^\varrho a_{i\varrho}^l).$$

Let $E \subset F$ be the free submodule spanned by $e_1, \ldots, e_r$. The above construction provides a minimal resolution of F as an $S(E)$-module.

If we reduce modulo the ideal generated by all a_{ij}^k for which $i \neq 0$ and $j \neq 0$, we obtain the algebra

$$\bar{F} = S(\bar{E}) \Big/ \sum_{2 \leqq k} S_k\bar{E} = \mathbb{Z}[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^2,$$

and $\mathbf{L}_{F,E}$ reduces to $\mathbf{P}_2(\bar{\phi})$ where $\bar{\phi}$ denotes the multiplication map $\bar{E}\otimes_{\mathbb{Z}}\bar{S}\to\bar{S}$, $\bar{S}=S(\bar{E})=\mathbb{Z}[x_1,\ldots,x_r]$.

Example 3.2 (the case of an augmented algebra). Suppose that the R-algebra A is projective and finitely generated as an R-module and that A contains an ideal E with $A/E\cong R$ as algebras. Then, of course, $A\cong R\oplus E$ as R-modules. But $E^2\subset E$, so E has an $S(E)$-module structure as well as A. In this case, the resolution of Theorem 3.2 is somewhat simpler, and may be defined by giving the short exact sequence of complexes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
 & 0 \arrow[r] \arrow[d] & \Lambda^r\tilde{E} \arrow[r] \arrow[d] & \ldots \arrow[r] & \Lambda^2\tilde{E} \arrow[d] \\
0 \arrow[r] & \Lambda^r\tilde{E}\otimes_R E \arrow[r] \arrow[d] & \Lambda^{r-1}\tilde{E}\otimes_R E \arrow[r] \arrow[d] & \ldots \arrow[r] & \tilde{E}\otimes_R E, \arrow[d] \\
0 \arrow[r] & L_2^r\tilde{E} \arrow[r] \arrow[d] & L_2^{r-1}\tilde{E} \arrow[r] \arrow[d] & \ldots \arrow[r] & L_2^1\tilde{E} \arrow[d] \\
 & 0 & 0 & & 0
\end{tikzcd}$$

augmented as above.

Example 3.3. Suppose A is an affine ring; that is, A is of the form $k[x_1,\ldots,x_n]/I$, with k a field. By the Noether normalization theorem, A will be, after a change of variables, a finitely generated module over its subring $R=k[x_1,\ldots,x_d]$, where $d=\dim A$. The ring A is Cohen-Macaulay if and only if it is free as an R-module.

Supposing that this is the case, we may write $A=R\oplus E$ (as R-modules) and apply Theorem 3.2 to get a resolution of A as a $k[x_1,\ldots,x_d,\, e_1,\ldots,e_r]$-module.

One may further obtain a resolution of A over $S'=k[x_1,\ldots,x_n,\, e_1,\ldots,e_r]$ by tensoring $\mathbf{L}_{A,E}$ with the Koszul complex over S' of a regular sequence of the form $x_i-p_i(x_1,\ldots,x_d,\, e_1,\ldots,e_r)$, $i=d+1,\ldots,n$, where p_i is a polynomial in the indicated variables with the same image in A as x_i. Since one can also go from a resolution for A over $k[x_1,\ldots,x_n]$ to a resolution for A over S' by tensoring with a suitable Koszul complex, this yields some informations about resolutions over $k[x_1,\ldots,x_n]$.

4. Applications to Singularities

Let k be an infinite field, $k\{y_1,\ldots,y_n\}$ the formal power series ring on k, and let A be a factor-ring $k\{y_1,\ldots,y_n\}/I$ of dimension d and multiplicity m. Suppose that A is Cohen-Macaulay and that there exists a number $l\geqq 2$ such that $I\subset(y_1,\ldots,y_n)^l$. Then

$$m\geqq\binom{(n-d)+l-1}{n-d}.$$

If m is equal to the smallest possible value, one can find new coordinates $u_1, \ldots, u_d$, $x_1, \ldots, x_r$, $r=n-d$, such that A is free over $R=k\{u_1, \ldots, u_d\}$ and

$$A/(u_1, \ldots, u_d)A = k\{x_1, \ldots, x_r\}/(x_1, \ldots, x_r)^l.$$

(For more details, see e.g. [Sal, Chap. 2].)

Under this assumption, let E be the free R-module on a basis $e_1, \ldots, e_r$, and let

$$S(E) \to A$$

be given by sending e_i to the residue class of x_i in A. Then

$$\sum_{k \leqq l-1} S_k E \to A$$

is bijective, since it is an isomorphism modulo $u_1, \ldots, u_d$. Hence, Theorem 3.2 applies to give a finite free resolution for A as an $R[e_1, \ldots, e_r]$-module, where e_i acts as x_i on A. Localizing at $(u_1, \ldots, u_d, x_1, \ldots, x_r)$ and completing with respect to the maximal ideal yields a minimal finite free resolution for A as $k\{u_1, \ldots, u_d, x_1, \ldots, x_r\}$-module.

Example 4.1. If A is the local ring of a two-dimensional rational singularity (over an algebraically closed field k) of embedding dimension n and multiplicity m, then the equation

$$m = n - 1$$

is satisfied (see [Art]). Since, by assumption, A is normal and $l=2$, our theory applies. Moreover, if the map $S_2 E \to A$ is explicitly given, i.e. if the equations of A are written in the form

$$x_i x_j = \sum_{\varrho=1}^{r} a_{ij}^{\varrho} x_\varrho + a_{ij}^0, \qquad 1 \leqq i, j \leqq r,$$

with $a_{ij}^{\varrho} \in k\{u_1, u_2\}$ satisfying the relations of Example 3.1, and if one uses special bases for the free $k\{u_1, u_2\}$-modules $L_2^k E$ (see the remark after Proposition 2.1), it is possible to write down the minimal free resolution for A by concrete matrices. (For a homological construction of such resolutions, see [Wahl]).

Example 4.2. Suppose $k=\mathbf{C}$, and consider the two-dimensional normal singularity given by the equations

$$x_1^2 = u_1 u_2 \qquad x_2^2 = u_1(u_1^2 + u_2^2) \qquad x_3^2 = u_2(u_1^2 + u_2^2)$$

$$x_1 x_2 = u_1 x_3 \qquad x_2 x_3 = (u_1^2 + u_2^2) x_1$$

$$x_1 x_3 = u_2 x_2,$$

which is a fourfold branched covering of $\operatorname{Spec} k\{u_1, u_2\}$ with branch locus

$$u_1 u_2 (u_1^2 + u_2^2) = 0,$$

i.e. four lines intersecting transversely in one point. Blowing up the origin in $\operatorname{Spec} k\{u_1, u_2\}$, we get a Cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

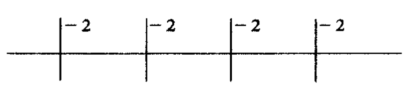
An easy computation shows that two of the four singularities of Y lying over the intersection of the strict transforms of those four lines with the blown up origin $\mathbf{P}_1 \subset X$ are isomorphic to

$$x_1^2=\sigma^2\tau \qquad x_2^2=\sigma^3 \qquad x_3^2=\sigma^3\tau$$
$$x_1x_2=\sigma x_3 \qquad x_2x_3=\sigma^2 x_1$$
$$x_1x_3=\sigma\tau x_2,$$

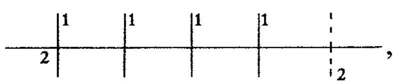
and two are isomorphic to

$$x_1^2=\sigma^2 \qquad x_2^2=\sigma^3\tau \qquad x_3^2=\sigma^3\tau$$
$$x_1x_2=\sigma x_3 \qquad x_2x_3=\sigma^2\tau x_1$$
$$x_1x_3=\sigma x_2.$$

These singularities are of the type described at the beginning of this section, but none is normal. It is easily seen that the first one is irreducible with regular normalization, and the second one is reducible with normalization consisting of two singularities of type $z^2-\sigma\tau=0$. Hence, by desingularizing the normalization $\hat{Y}$ of Y, we get the following configuration of curves



where the vertical lines represent nonsingular rational curves with selfintersection number -2. The horizontal curve C is a twofold cover of $\mathbf{P}_1$, branched at two points, and therefore it is isomorphic to $\mathbf{P}_1$. By standard methods its selfintersection number can be computed with the aid of the divisor of a meromorphic function. If one takes for instance the pullback of the function u_1, the divisor looks as indicated in the following diagram (numbers denoting multiplicities):



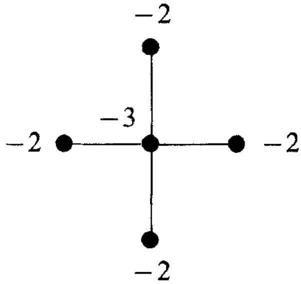
and we get:

$$0=(u_1)\cdot C=2(C\cdot C)+1+1+1+1+2,$$

i.e.

$$C^2=-3.$$

Hence the original singularity is rational with dual graph



(For another treatment of this singularity, see [Wahl, Proposition 4.14].)

A minimal free resolution of A over $\hat{S}=k\{u_1,u_2,x_1,x_2,x_3\}$ may be described in the following form:

$$0 \to \hat{S}^3 \xrightarrow{\partial_3} \hat{S}^8 \xrightarrow{\partial_2} \hat{S}^6 \xrightarrow{\partial_1} \hat{S} \to A=\hat{S}/I \to 0,$$

where the homomorphisms $\partial_1, \partial_2, \partial_3$ are given, respectively, by the following matrices:

$$(x_1^2-u_1u_2, x_1x_2-u_1x_3, x_1x_3-u_2x_2, x_2^2-u_1(u_1^2+u_2^2), x_2x_3-(u_1^2+u_2^2)x_1, x_3^2-u_2(u_1^2+u_2^2)),$$

$$\begin{pmatrix} -x_2 & -x_3 & 0 & 0 & 0 & -(u_1^2+u_2^2) & 0 & 0 \\ x_1 & u_2 & -x_3 & u_1^2+u_2^2 & x_2 & 0 & 0 & 0 \\ u_1 & x_1 & x_2 & 0 & 0 & x_2 & x_3 & u_1^2+u_2^2 \\ 0 & 0 & u_2 & -x_3 & -x_1 & u_2 & 0 & 0 \\ 0 & 0 & 0 & x_2 & u_1 & -x_1 & u_2 & x_3 \\ 0 & 0 & -u_1 & 0 & 0 & 0 & -x_1 & -x_2 \end{pmatrix},$$

$$\begin{pmatrix} x_3 & u_1^2+u_2^2 & 0 \\ -x_2 & 0 & u_1^2+u_2^2 \\ x_1 & x_2 & 0 \\ 0 & -x_1 & -u_2 \\ u_2 & x_3 & 0 \\ 0 & -x_2 & -x_3 \\ -u_1 & 0 & x_2 \\ 0 & -u_1 & -x_1 \end{pmatrix}.$$

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