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Characterizing Moišezon Spaces by Almost Positive Coherent Analytic Sheaves

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Introduction

An irreducible normal compact complex space X is a Moišezon space if the transcendence degree of the field of meromorphic functions on X is equal to the complex dimension of X. In a joint paper [7] Grauert and the author introduced the notion of "quasi-positive" coherent analytic sheaves on complex spaces and proved a "vanishing theorem" for such sheaves on Moišezon spaces which is analogous to the vanishing theorem of Nakano [18] for positive vector bundles on compact complex manifolds. It is well known that a normal compact complex space is projective algebraic if and only if it carries a *weakly positive* vector bundle (in the sense of Grauert [6]). Therefore, it is a natural conjecture that *an irreducible normal compact complex space X is a Moišezon space if and only if there exists a (torsion free) coherent analytic sheaf $\mathscr{S}$ on X which is weakly positive almost everywhere.*

This paper is a first but very modest step forward to the proof of this conjecture. First of all we give the sentence "a vector bundle E over a complex space X with singularities is weakly positive almost everywhere" an exact meaning and call such vector bundles *almost positive*. It has been shown by Griffiths [8] and Kobayashi and Ochiai [13] that there are also precise vanishing theorems for almost positive vector bundles on compact Kähler manifolds (cf. § 1). We then investigate in § 2 the observation of Rossi [19] that coherent analytic sheaves can be made free (modulo torsion) by proper modifications, i.e.: If $\mathscr{S}$ is a coherent analytic sheaf on an irreducible compact complex space X then there exists a (minimal) proper modification $\varphi: \hat{X} \to X$ where $\hat{X}$ is an irreducible compact complex space such that the torsion free preimage $\hat{\mathscr{S}} = \mathscr{S} \circ \varphi = \varphi^* \mathscr{S}/\text{torsion}\,(\varphi^* \mathscr{S})$ is locally free. We shall show that Rossi's construction is the (unique) solution of a universal mapping problem and that it coincides with monoidal transformations if $\mathscr{S}$ is an ideal sheaf on X. In § 3 we define a *sheaf $\mathscr{S}$* on X to be *almost positive* if the locally free sheaf $\hat{\mathscr{S}}$ is almost positive in the sense of § 1, and prove a *vanishing theorem for almost positive coherent analytic sheaves on Moišezon spaces.* The methods are the same as in [7].

The last paragraph is devoted to the connection between Moišezon spaces and almost positive sheaves. We show: *Every normal Moišezon space X carries*

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an almost positive (torsion free) coherent analytic sheaf $\mathcal{S}$ *of rank* 1. The set A where $\mathcal{S}$ is not free or not positive is analytic and of codimension at least 2 in X. We are able to prove *the converse only in the case where the set* A *is discrete.* This gives at least *in the case of* 2*-dimensional normal spaces a characterization of Moišezon spaces.* Finally, we show, generalizing an example of Grauert, that there exist Moišezon spaces X in all dimensions $n \geqq 2$ which are not projective algebraic such that X carries an almost positive coherent analytic sheaf of rank 1 with discrete A.

In general the "bad" set A of an almost positive coherent analytic sheaf $\mathcal{S}$ is not assumed to be analytic. In fact the vanishing theorems can be proved without such an additional assumption. But it seems to be very difficult to prove the conjecture in this full generality. However, there is a hope to prove it under the assumption that A itself is a lower dimensional Moišezon subspace of X (this makes sense since every subspace of a Moišezon space is again a Moišezon space). This would give an *inductive characterization* of Moišezon spaces by almost positive coherent analytic sheaves.

In an appendix we shall show that the vanishing theorems can be proved even under a more general definition of almost positivity. Whereas in § 3 the bundles are positive (in some sense) in a dense open subset it is enough to require that they are positive only in a non-empty open subset $U \subset X$ and semi-positive outside U. For this we have to prove an identity theorem for harmonic forms with values in a vector bundle which can be derived from an identity theorem of Aronszajn [2].

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§ 1. Almost Positive Vector Bundles

1. Let X be an irreducible (reduced) complex space and $E \to X$ a complex analytic vector bundle of rank r over X. If E^* denotes the dual bundle of E then $P = P(E)$ is defined as $E^* - X/\mathbb{C}^*$. $\pi\colon P(E) \to X$ is a fibre bundle whose fibre $P(E)_x$ is the projective space $P(E_x^*) \cong \mathbb{P}^{r-1}$ of lines in E_x^*, $x \in X$. $E^* - X \to P(E)$ is a principal bundle; we denote the associated line bundle by $G = G(E)$.

2. There always exists a biholomorphic mapping

$$G^* - P \xrightarrow{\sim} E^* - X, \tag{1}$$

and therefore E is positive (in the sense of Grauert) iff G is positive. This can be seen immediately by Grauert's definition [6]: E is positive if some tubular neighborhood of the zero section in the dual bundle E^* is strongly pseudoconvex.

If P is a complex manifold (i.e., if X is a manifold) then G is positive if and only if there exists an hermitian metric h on G such that the hermitian quadratic form

$$\Theta_G(\eta) = \sum \Theta_{i\bar{j}}\, \eta^i \bar{\eta}^j, \qquad \eta = (\eta^1, \dots, \eta^n), \qquad n = \dim X, \tag{2}$$

is positive definite where $\{\Theta_{i\bar{j}}\}$ denotes the curvature tensor of G with respect to h (Griffiths [8]).

Remark. If the line bundle G is given by transition functions $g_{\iota\kappa}$ with respect to a covering $\{U_\iota\}$ of P (that means we have $w_\iota = g_{\iota\kappa} w_\kappa$ on $U_\iota \cap U_\kappa$ where w_ι is a coordinate in the trivialization $G|U_\iota \xrightarrow{\sim} U_\iota \times \mathbb{C}$) then an hermitian metric h on G is given by positive C^∞ functions h_ι on U_ι such that

$$h_\iota = |g_{\kappa\iota}|^2 h_\kappa \tag{3}$$

on $U_\iota \cap U_\kappa$ (usually such a system $\{h_\iota\}$ is called an hermitian metric on the dual bundle G^*!). Then in order that (2) be positive definite it is necessary and sufficient that the form

$$\sum \frac{\partial^2(-\log h_\iota)}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j \tag{4}$$

is positive definite on U_ι for all ι. Hence

G is positive if and only if there exists a system of positive C^∞ functions h_ι such that (3) *holds and the functions $-\log h_\iota$ are strongly plurisubharmonic on U_ι.*

This is the original definition given by Kodaira [14].

Definition. A vector bundle E over a (reduced) complex space X is called *almost positive* if there exist a (positive definite) C^∞ hermitian metric h on the associated line bundle $G = G(E)$ over $P = P(E)$ and a dense open set $\mathring{R}$ contained in the set $R = R(P)$ of regular points of P such that the hermitian quadratic form (2) is positive definite for all points in $\mathring{R}$.

Since X is reduced, $P = P(E)$ is also reduced and hence $R = R(P)$ is an open and dense subset of P.

Remark. Every positive vector bundle in the sense of Grauert is almost positive.

3. We now introduce the notion of semi-positive vector bundles.

Definition. A vector bundle E over a complex space X is called *semi-positive* if the following two conditions are satisfied:

i) there exists an hermitian metric h on $G = G(E)$ such that the hermitian quadratic form $\Theta_G(\eta)$ is positive semi-definite on $R = R(P)$,

ii) $G^k \otimes \pi^* F$ is almost positive for every almost positive line bundle F over an open set $U \subset X$ and every $k \geqq 1$, where $\pi: P \to X$ is the canonical projection (with respect to the canonical metric on $G^k \otimes \pi^* F$).

Lemma 1. *In the case of a line bundle G, condition* ii) *follows from* i).

Proof. For $E = G$, $P(E) = X$ and $G(E) = G$. If F is an almost positive line bundle on $U \subset X$ such that $\Theta_F(\eta)$ is positive definite on $\mathring{R} \subset U$ then

$$\Theta_{G^k \otimes F}(\eta) = k\,\Theta_G(\eta) + \Theta_F(\eta)$$

is positive definite on $\mathring{R}$, q.e.d.

Lemma 2. *Every positive vector bundle E over X is semi-positive.*

Proof. By definition, $G=G(E)$ carries an hermitian metric such that $\Theta_G(\eta)$ is positive definite on a dense open subset $\mathring{R}\subset R(P)$. Then $\Theta_G(\eta)$ must be positive semi-definite on $R(P)$ because of continuity.

If F is an almost positive line bundle on $U\subset X$ then $\pi^* F$ is obviously semi-positive. Hence $G^k\otimes\pi^* F$ is almost positive because G is almost positive by definition, q.e.d.

Example 1. Let E be a vector bundle over X and h an hermitian metric on E such that the hermitian biquadratic form ($\{\Theta^{\rho}_{\sigma i\bar{j}}\}$ denotes the curvature tensor of E with respect to h)

$$\Theta_E(\xi,\eta)=\sum \Theta^{\rho}_{\sigma i\bar{j}}\,\xi^{\sigma}\,\bar{\xi}^{\rho}\,\eta^i\,\bar{\eta}^j \tag{5}$$

is positive definite in the two variables $\xi=(\xi^1,\ldots,\xi^r)$, $r=\operatorname{rank} E$, $\eta=(\eta^1,\ldots,\eta^n)$, $n=\dim X$, for all points in a dense open subset $\mathring{R}\subset R(X)$ (resp. positive semi-definite in $R(X)$), that means $E|\mathring{R}$ is *positive in the sense of Griffiths* (resp. $E|R(X)$ is semi-positive in the sense of Griffiths). Then E is almost positive (resp. semi-positive).

The proof follows easily from formula (2.36) in [8]. This formula says that if E carries an hermitian metric with positive definite (positive semi-definite) $\Theta_E(\xi,\eta)$ then $G=G(E)$ has an hermitian metric such that $\Theta_G(\eta)$ is positive definite (resp. positive semi-definite and positive definite in the direction of the fibres of $\pi\colon P\to X$). If F is an almost positive line bundle on X then $\pi^* F$ is semi-positive and positive in the directions which are orthogonal to the fibres of π. This implies the statement of Example 1.—A special case of this example is

Example 2. Every trivial vector bundle $X\times\mathbb{C}^r$ is semi-positive.

Example 3. Let E be a vector bundle generated by its sections. Then E is semi-positive.

For every vector bundle which is generated by its sections is semi-positive in the sense of Griffiths (cf. [8], formula (2.24) and the proof of Theorem G on p. 212).

4. We need some functorial properties of almost positive and semi-positive vector bundles. We say that a *holomorphic mapping* $\varphi\colon\hat{X}\to X$ *is of class* $(*)$, if X and $\hat{X}$ are irreducible complex spaces of equal dimension and φ is discrete at at least one point $\hat{x}_0\in\hat{X}$.

Proposition 1. *Let $\varphi\colon\hat{X}\to X$ be a holomorphic mapping of class $(*)$, and let E be an almost positive (resp. semi-positive) vector bundle on X. Then $\hat{E}=\varphi^* E$ is almost positive (resp. semi-positive).*

Proof. There exists a canonical diagram

$$\begin{array}{ccc} P(\varphi^* E) & \xrightarrow{\hat{\varphi}} & P(E) \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \hat{X} & \xrightarrow{\varphi} & X, \end{array}$$

such that $\hat{\varphi}$ is of class $(*)$ and $\hat{G}=G(\varphi^* E)=\hat{\varphi}^* G$, $G=G(E)$. Therefore the metric h on G defines a metric $\hat{h}$ on $\hat{G}$ in an obvious manner. Since $\hat{\varphi}$ is also of class $(*)$ the set of points where $\hat{\varphi}$ is locally biholomorphic is open and dense in $P(\varphi^* E)$. From that it follows easily that $\hat{h}$ possesses the desired properties, q.e.d.

5. Let X, E, P and G be given as in Section 1, let F be a line bundle on X and let π be the canonical projection $\pi\colon P \to X$. We denote by $\mathscr{E}, \mathscr{G}, \mathscr{F}$, etc., the sheaf of germs of holomorphic sections in E, G, F, etc. Take an open subset U of X such that $E \mid U$ and $F \mid U$ are trivial. Then

$$\pi_{(l)}(\mathscr{G}^k \otimes \pi^* \mathscr{F}) = \pi_{(l)}(\mathscr{G}^k) \otimes \mathscr{F},$$

$$\pi^{-1}(U) = U \times \mathbb{P}^{r-1} \quad \text{and} \quad \mathscr{G}^k \mid \pi^{-1}(U) = \mathcal{O}_U \otimes_{\mathcal{O}_{\pi^{-1}(U)}} \mathscr{H}^k,$$

where $\mathscr{H}$ denotes the sheaf of germs of holomorphic sections in a positive hyperplane bundle H of $\mathbb{P}^{r-1}$ and $\pi_{(l)}(\mathscr{S})$ is the l-th direct image sheaf of a sheaf $\mathscr{S}$. Now

$$H^l(\mathbb{P}^{r-1}, \mathscr{H}^k) = 0, \qquad l > 0, \ k \geqq 1,$$

and

$$H^0(\mathbb{P}^{r-1}, \mathscr{H}^k) \cong E^{(k)},$$

where $E^{(k)}$ denotes the k-th symmetric tensor power of E. This implies

$$\pi_{(l)}(\mathscr{G}^k \otimes \pi^* \mathscr{F}) = 0, \qquad l > 0,$$

and hence

$$\begin{aligned} H^l(P, \mathscr{G}^k \otimes \pi^* \mathscr{F}) &= H^l\big(X, \pi_{(0)}(\mathscr{G}^k \otimes \pi^* \mathscr{F})\big) \\ &= H^l(X, \mathscr{E}^{(k)} \otimes \mathscr{F}), \qquad l \geqq 0, \ k \geqq 0. \end{aligned} \tag{6}$$

6. It is now very easy to give a proof of the vanishing theorem of Griffiths [8], Theorem G, and Kobayashi and Ochiai [13], Corollary 2.4.

Theorem 1. *Let X be a projective algebraic manifold with canonical bundle K_X, and let E resp. F be a vector bundle resp. a line bundle on X such that*

i) *E is almost positive and $F \otimes \det E^*$ is semi-positive, or*

ii) *E is semi-positive and $F \otimes \det E^*$ is almost positive, then*

$$H^l(X, \mathscr{E}^{(k)} \otimes \mathscr{F} \otimes \mathscr{K}_X) = 0, \qquad l \geqq 1, \ k \geqq 1.$$

Proof. Since (4) holds we have only to show that

$$H^l(P, \mathscr{G}^k \otimes \pi^*(\mathscr{F} \otimes \mathscr{K}_X)) = 0, \qquad l \geqq 1, \ k \geqq 1, \tag{7}$$

where $P = P(E)$, $G = G(E)$ and $\pi\colon P \to X$. In both cases i) and ii)

$$G^{k+r} \otimes \pi^*(F \otimes \det E^*), \qquad k \geqq 1, \ r = \operatorname{rank} E,$$

is almost positive on P by definition. Since P is again projective algebraic ([15], Theorem 8) one gets by a slight generalization of the proof of Kodaira's vanishing theorem [14]:

$$H^l(P, \mathscr{G}^{k+r} \otimes \pi^* \mathscr{F} \otimes \pi^* \det \mathscr{E}^* \otimes \mathscr{K}_P) = 0, \qquad l \geqq 1, \; k \geqq 1$$

(cf. for instance [7], Satz 2.2, and also Theorem 6 in the Appendix of this paper). By [8], Formula (2.38)

$$K_P = G^{-r} \otimes \pi^* \big((\det E) \otimes K_X\big),$$

and this implies (7), q.e.d.

§ 2. Monoidal Transformations with Respect to Coherent Analytic Sheaves

1. It was shown by Rossi [19] that coherent analytic sheaves can be "made free" by means of modifications. We want to give here a more systematic treatment of this subject similar to the representation of monoidal transformations in [10].

Definition. Let φ: $Y \to X$ be a holomorphic mapping of complex spaces and $\mathscr{S}$ a coherent analytic sheaf on X. Then we define the *torsion free preimage $\mathscr{S} \circ \varphi$ of $\mathscr{S}$ under φ* as the analytic preimage of $\mathscr{S}$ under φ modulo torsion:

$$\mathscr{S} \circ \varphi = \varphi^* \mathscr{S} / T(\varphi^* \mathscr{S}).$$

We note some properties of the torsion free preimage (proofs can be found in [7], § 1.3):

Proposition 2. *Let φ: $Y \to X$, ψ: $Z \to Y$ be holomorphic mappings, and let $\mathscr{S}$ be an arbitrary and $\mathscr{F}$ a locally free sheaf on X. Then*

i) $(\mathscr{S} \otimes \mathscr{F}) \circ \varphi \cong (\mathscr{S} \circ \varphi) \otimes \varphi^* \mathscr{F}$.

ii) *If φ is of class* $(*)$ (cf. § 1.4) *then*

$$\mathscr{S} \circ \varphi \cong \big(\mathscr{S} / T(\mathscr{S})\big) \circ \varphi.$$

iii) *If ψ is of class* $(*)$ *then*

$$(\mathscr{S} \circ \varphi) \circ \psi = \mathscr{S} \circ (\varphi \circ \psi).$$

2. We define monoidal transformations axiomatically. All complex spaces are assumed to be reduced.

Definition. Let X be a complex space and $\mathscr{S}$ a coherent analytic sheaf on X. Then a pair $(X_{\mathscr{S}}, \varphi_{\mathscr{S}})$ of a complex space $X_{\mathscr{S}}$ and a proper modification mapping $\varphi_{\mathscr{S}}$: $X_{\mathscr{S}} \to X$ will be called the *monoidal transformation of X with respect to $\mathscr{S}$* if the following two conditions are satisfied:

i) the torsion free preimage $\mathscr{S} \circ \varphi_{\mathscr{S}}$ is locally free on $X_{\mathscr{S}}$,

ii) if φ: $Y \to X$ is any proper modification mapping such that i) holds then there exists a unique holomorphic mapping ψ: $Y \to X_{\mathscr{S}}$ with $\varphi = \varphi_{\mathscr{S}} \circ \psi$.

If $X_{\mathscr{S}}$ exists it is uniquely determined by ii) up to biholomorphic mappings.

3. We shall show now that $(X_{\mathscr{S}}, \varphi_{\mathscr{S}})$ locally exists (cf. [19]). Let U be an open subset of X such that there exists an epimorphism $\mathcal{O}^q \to \mathscr{S} \to 0$ over U and let D be the set of points in U where $\mathscr{S}$ is not locally free.

In the following $L = L(\mathscr{S})$ denotes the *linear space* associated to $\mathscr{S}$ in the sense of Grauert [6] (cf. also [5]) and $V(L)$ is the coherent analytic sheaf of germs of holomorphic functions on the linear space L which are linear along the fibres of L. Then

$$V(L(\mathscr{S})) \cong \mathscr{S} \quad \text{and} \quad L(\mathscr{E}) \cong E^*,$$

if E is a vector bundle.

$L(\mathscr{S}) \mid U - D$ is a complex analytic vector bundle over $U - D$. Set $r = \operatorname{rank} L(\mathscr{S}) \mid U - D$ (we may assume that $U - D$ is connected). The sequence $\mathcal{O}^q \to \mathscr{S} \to 0$ gives rise to an embedding $L(\mathscr{S}) \hookrightarrow U \times \mathbb{C}^q$ of linear spaces. Attaching to every point $x \in U - D$ the r-dimensional linear subspace $L(\mathscr{S})_x \subset \mathbb{C}^q$ we get a holomorphic mapping

$$\mu\colon U - D \to G(r, q)$$

of $U - D$ into the Grassmann manifold $G(r, q)$ of r-dimensional subspaces of $\mathbb{C}^q$. It has been shown by Rossi ([19], Proposition 3.4) that μ is a *meromorphic mapping* in the sense of Remmert; that means:

The closure $\hat{U}$ of the graph of μ in $U \times G(r, q)$ is a (reduced) complex space. In the canonical diagram

$$\begin{tikzcd} & \hat{U} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ U & & G(r, q) \end{tikzcd}$$

φ is proper and induces a biholomorphic mapping of $\hat{U} - \varphi^{-1}(D)$ onto $U - D$. Moreover $\hat{U} - \varphi^{-1}(D)$ is dense in $\hat{U}$ (i.e., *φ is a proper modification mapping*).

Proposition 3. *$\mathscr{S} \circ \varphi$ is locally free.*

Proof. Let $G = G(r, q)$, then we have an exact holomorphic vector bundle sequence

$$0 \to B \to G \times \mathbb{C}^q \to Q \to 0$$

on G where B is the universal bundle of G (whose fibre B_g at a subspace $g \in G$ is exactly the vector space g itself). $0 \to B \to G \times \mathbb{C}^q$ is lifted under μ to

$$L(\mathscr{S}) \hookrightarrow U \times \mathbb{C}^q$$

on $U - D$. Now let $\hat{\mathscr{S}} = V(\psi^* B)$ be the locally free sheaf of germs of holomorphic functions on $\psi^* B$ which are linear along the fibres of $\psi^* B \to \hat{U}$. Obviously

$$\psi^* B = \hat{U} \times_G B \hookrightarrow \hat{U} \times \mathbb{C}^q$$

such that we have an epimorphism

$$\mathcal{O}_{\hat{U}}^q \xrightarrow{\alpha} \hat{\mathscr{S}} \to 0.$$

Since $V(L(\mathscr{S}))=\mathscr{S}$, $\hat{\mathscr{S}}$ coincides with $\varphi^*\mathscr{S}$ on $\hat{U}-\varphi^{-1}(D)$ and α is there equal to the canonical epimorphism

$$\mathcal{O}^q_{\hat{U}}=\varphi^*\mathcal{O}^q_U \xrightarrow{\beta} \varphi^*\mathscr{S}\to 0.$$

Therefore α annihilates $\ker\beta$ on $\hat{U}-\varphi^{-1}(D)$. This remains true on $\varphi^{-1}(D)$ since $\hat{\mathscr{S}}$ is locally free. Thus we get

$$\begin{tikzcd} \mathcal{O}^q_{\hat{U}} \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \hat{\mathscr{S}} \arrow[r] & 0 \\ & \varphi^*\mathscr{S}, \arrow[u, "\gamma"'] & \end{tikzcd}$$

γ obviously being surjective. In the exact sequence

$$0\to\mathscr{K}\to\varphi^*\mathscr{S}\to\hat{\mathscr{S}}\to 0$$

we have $T(\mathscr{K})=\mathscr{K}$ since $\operatorname{supp}\mathscr{K}\subset\varphi^{-1}(D)$ is nowhere dense in $\hat{U}$. This implies because of $T(\hat{\mathscr{S}})=0$

$$T(\varphi^*\mathscr{S})=\mathscr{K} \quad\text{and hence}\quad \mathscr{S}\circ\varphi=\hat{\mathscr{S}}, \quad \text{q.e.d.}$$

Proposition 4. *The pair* $(\hat{U},\varphi)$ *has the universal property* ii) *of the definition in Section* 2.

Proof. Let $\psi\colon V\to U$ be an arbitrary proper modification mapping such that $\mathscr{S}\circ\psi$ is locally free. Then $V_0=\psi^{-1}(U-D)$ is open and dense in V. The exact sequence $\mathcal{O}^q\to\mathscr{S}\to 0$ leads to an exact sequence $\mathcal{O}^q_V\to\psi^*\mathscr{S}\to 0$ and thus to an epimorphism $\mathcal{O}^q_V\to\mathscr{S}\circ\psi\to 0$. Since $\mathscr{S}\circ\psi$ is locally free of rank r we construct as at the beginning of this section a holomorphic mapping

$$v\colon V\to G(r,q).$$

Since on $V_0=\psi^{-1}(U-D)$ we have $L(\mathscr{S}\circ\psi)=L(\psi^*\mathscr{S})=V\times_U L(\mathscr{S})$ the following diagram is commutative:

$$\begin{tikzcd} V_0 \arrow[d, "\psi"'] \arrow[dr, "v"] & \\ U-D \arrow[r, "\mu"] & G(r,q). \end{tikzcd}$$

Let χ be the holomorphic mapping of V into $U\times G(r,q)$ defined by (ψ,v). χ is obviously a holomorphic mapping of V into $\hat{U}$ such that $\psi=\varphi\circ\chi$.

The uniqueness of χ is clear since V_0 is open and dense in V, q.e.d.

4. Let X be a complex space, $\mathscr{S}$ a coherent analytic sheaf on X and U_1 and U_2 open subsets of X such that $X=U_1\cup U_2$. Suppose that $\varphi_i\colon\hat{U}_i\to U_i$ are monoidal transformations of U_i with respect to $\mathscr{S}_i=\mathscr{S}\,|\,U_i$, $i=1,2$. Then there exists a unique isomorphism $i\colon\varphi_1^{-1}(U_1\cap U_2)\to\varphi_2^{-1}(U_1\cap U_2)$ such that $\varphi_2\circ i=\varphi_1$ on $\varphi_1^{-1}(U_1\cap U_2)$. The proof of this statement can be reduced to the case in which one of the U_i is equal to X, and this case is an easy consequence of the universal property of monoidal transformations. By piecing together the local solutions of Section 3 we have proved (cf. [19], Theorem 3.5):

Theorem 2. *Let X be an (irreducible) complex space, $\mathcal{S}$ a coherent analytic sheaf on X and $D=\{x\in X$: $\mathcal{S}$ is not locally free at $x\}$. Then there exists the monoidal transformation $(X_{\mathcal{S}}, \varphi_{\mathcal{S}})$ of X with respect to $\mathcal{S}$. $X_{\mathcal{S}}$ is a reduced (irreducible) complex space and $\varphi_{\mathcal{S}}$ is a proper modification such that $\varphi_{\mathcal{S}}$: $X_{\mathcal{S}}-\varphi_{\mathcal{S}}^{-1}(D)\to X-D$ is biholomorphic.*

If $U\subset X$ is an open subspace of X then $(\varphi_{\mathcal{S}}^{-1}(U), \varphi_{\mathcal{S}})$ is the monoidal transformation of U with respect to $\mathcal{S}\,|\,U$.

Remark. Let $(X_{\mathcal{S}}, \varphi_{\mathcal{S}})$ be the monoidal transformation of X with respect to a coherent analytic sheaf $\mathcal{S}$ of rank r. Then one should be able to prove the more general fact

ii′) *If φ: $Y\to X$ is* any *holomorphic (not necessarily proper modification) mapping such that $\mathcal{S}\circ\varphi$ is locally free* of rank r *then there exists a unique* holomorphic mapping ψ: $Y\to X_{\mathcal{S}}$ such that $\varphi=\varphi_{\mathcal{S}}\circ\psi$.

The statement ii′) is obviously false if one only requires that $\mathcal{S}\circ\varphi$ is locally free of arbitrary rank.

5. We want to show that in the special case of ideal sheaves our construction coincides with the usual notion of *monoidal transformations.*

Let the ideal $\mathcal{I}$ be locally generated by holomorphic functions $f_1, \ldots, f_q$; thus we have an epimorphism

$$\mathcal{O}_U^q \xrightarrow{p} \mathcal{I}\to 0, \qquad p(h_{1,x}, \ldots, h_{q,x})=\sum_{\nu=1}^{q} h_{\nu,x} f_{\nu,x},$$

and therefore

$$L(\mathcal{I})=\left\{(x, w_1, \ldots, w_q)\in U\times\mathbb{C}^q\colon \sum_{\nu=1}^{q} h_\nu(x)\, w_\nu=0 \text{ for all } (h_1, \ldots, h_q)\in\ker p_*\right\},$$

where p_*: $\Gamma(U, \mathcal{O}_U^q)\to\Gamma(U, \mathcal{I})$. Since obviously $(x, f_1(x), \ldots, f_q(x))\in L(\mathcal{I})_x$, $L(\mathcal{I})_x$ is, at a point $x\in X$ where $\mathcal{I}_x$ is free, equal to the 1-dimensional linear space through 0 and $(f_1(x), \ldots, f_q(x))$ in $\mathbb{C}^q$, i.e., μ: $U-D\to G(1,q)=\mathbb{P}^{q-1}$ is given by $\mu(x)=(f_1(x):\ldots:f_q(x))$ and this is exactly the construction of the monoidal transformation with center $Z(\mathcal{I})=\{x\in X\colon \mathcal{I}_x\neq\mathcal{O}_{X,x}\}$ (cf. [10], Remark 2).

6. For the investigations in § 4 we need in particular the case where $\mathcal{S}$ is a rank one sheaf. We look at the local situation

$$\begin{array}{ccc} & \hat{U} & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ U & & G(1,q). \end{array}$$

$\hat{\mathcal{S}}=\mathcal{S}\circ\varphi$ is the sheaf of germs of holomorphic functions on $L(\psi^* B)$ which are linear along the fibres of $\psi^* B\to\hat{U}$ where B is the universal bundle of $G(1,q)$. But $G(1,q)=\mathbb{P}^{q-1}$ and B is the dual of the positive hyperplane bundle on $\mathbb{P}^{q-1}$. From that we derive

Proposition 5. *Let $(X_{\mathscr{S}}, \varphi_{\mathscr{S}})$ be the monoidal transformation of X with respect to the rank one sheaf $\mathscr{S}$, and let $x \in D$. Then $\varphi_{\mathscr{S}}^{-1}(x) = X_{\mathscr{S},x}$ is a projective algebraic space and $\mathscr{S} \circ \varphi_{\mathscr{S}} | X_{\mathscr{S},x}$ is the sheaf of germs of holomorphic sections in the positive hyperplane bundle of $X_{\mathscr{S},x}$.*

We have only to apply the formula $V(E) = \mathscr{E}^*$ where E denotes a vector bundle and $\mathscr{E}^*$ the sheaf of germs of holomorphic sections in the dual bundle E^*.

§ 3. Almost Positive Coherent Analytic Sheaves

1. Let $\mathscr{S}$ be a coherent analytic sheaf over a (reduced) irreducible complex space X. Then there exists, as was shown in § 2, a (minimal) irreducible complex space $X_{\mathscr{S}}$ and a proper modification mapping $\varphi_{\mathscr{S}}: X_{\mathscr{S}} \to X$ such that $\mathscr{S} \circ \varphi_{\mathscr{S}}$ is locally free. $X_{\mathscr{S}}$ is reduced, and it is compact if X is compact. Denote the set of nonsingular points of X by $R(X)$ and the set of points where $\mathscr{S}$ is not locally free by $D(\mathscr{S})$.

Definition. $\mathscr{S}$ is called *almost positive* if the vector bundle $E(\mathscr{S} \circ \varphi_{\mathscr{S}})$ is almost positive over $X_{\mathscr{S}}$ in the sense of § 1.2 ($E = E(\mathscr{S} \circ \varphi_{\mathscr{S}})$ denotes the uniquely determined vector bundle with $\mathscr{E} \cong \mathscr{S} \circ \varphi_{\mathscr{S}}$).

Moreover, if $A \subset X$ is a nowhere dense closed subset such that

$$X - A \subset R(X) \cap D(\mathscr{S})$$

and $G = G(E)$ is positive on $P(E) - (\varphi_{\mathscr{S}} \circ \pi)^{-1}(A)$, where $\pi: P(E) \to X_{\mathscr{S}}$, we say that $\mathscr{S}$ is positive on $X - A$.

Semi-positivity is defined in exactly the same fashion.

2. Let $\mathscr{S}$ be a coherent analytic sheaf over X and $L = L(\mathscr{S})$ the complex linear space associated to $\mathscr{S}$. Assume that there exists an hermitian form h_x on each fibre L_x. $h = \{h_x\}$ is called an *hermitian form on* L in [7], if there exist a neighborhood U of each point $x_0 \in X$, an embedding $L|U \hookrightarrow U \times \mathbb{C}^q$ and a positive hermitian form

$$h = \sum \hat{h}_{i\bar{j}} w_i \bar{w}_j$$

on $U \times \mathbb{C}^q$ with C^∞ functions $\hat{h}_{i\bar{j}}$ such that $h_x = \hat{h} | L_x$ for all $x \in U$.

If $\mathring{R}$ is open and dense in $R(X) \cap D(\mathscr{S})$ then $L|\mathring{R}$ is a vector bundle on the manifold $\mathring{R}$ with an hermitian metric h.

$\mathscr{S}$ is called *Nakano quasi-positive* in [7], if there exists an hermitian form h on $L = L(\mathscr{S})$ and a dense open subset $\mathring{R}$ of $R(X) \cap D(\mathscr{S})$ such that $L|\mathring{R}$ is negative with respect to $h|\mathring{R}$ in the sense of Nakano [18]. (Unfortunately there is a mistake in the definition of [7], p. 265. $\mathscr{S}$ is called positive, if $L(\mathscr{S})$ is *positive*. But this is wrong since $L(\mathscr{E}) = E^*$ for a vector bundle E and E should be positive if and only if $\mathscr{E}$ is positive!)

In the same way one can define *Griffiths quasi-positive* sheaves $\mathscr{S}$. It is well known that

Nakano quasi-positive $\Rightarrow$ *Griffiths quasi-positive*.

We want to show now that

$$\textit{Griffiths quasi-positive} \Rightarrow \textit{almost positive}.$$

Let h be the hermitian form on $L(\mathscr{S})$ and $\varphi_{\mathscr{S}}\colon X_{\mathscr{S}} \to X$ the monoidal transformation with respect to $\mathscr{S}$. As was shown in [7], Satz 1.4, $L(\mathscr{S} \circ \varphi_{\mathscr{S}})$ carries an hermitian metric which is obviously Griffiths negative in a dense open subset of $X_{\mathscr{S}}$. Since $\mathscr{S} \circ \varphi_{\mathscr{S}}$ is locally free, $L(\mathscr{S} \circ \varphi_{\mathscr{S}}) = E(\mathscr{S} \circ \varphi_{\mathscr{S}})^*$ and therefore $E(\mathscr{S} \circ \varphi_{\mathscr{S}})$ is almost positive in the sense of Griffiths. This implies that $E(\mathscr{S} \circ \varphi_{\mathscr{S}})$ is almost positive (Example 1, § 1.3), q.e.d.

3. A (reduced) compact complex space X is called a *Moišezon space* if every irreducible component X_j of X admits $n_j = \dim_{\mathbb{C}} X_j$ (algebraically and analytically) independent meromorphic functions. Let $\mathscr{S}$ be a coherent analytic sheaf on an irreducible Moišezon space then the monoidal transformation $\varphi_{\mathscr{S}}\colon X_{\mathscr{S}} \to X$ with respect to $\mathscr{S}$ is a modification and hence $X_{\mathscr{S}}$ is again a Moišezon space. If $\mathscr{S}_1, \ldots, \mathscr{S}_t$ are finitely many coherent analytic sheaves,

$$\varphi = \varphi_{\mathscr{S}_1, \ldots, \mathscr{S}_t} = \varphi_{\mathscr{S}_1} \times \cdots \times \varphi_{\mathscr{S}_t}\colon X_{\mathscr{S}_1, \ldots, \mathscr{S}_t} = X_{\mathscr{S}_1} \times_X \cdots \times_X X_{\mathscr{S}_t} \to X$$

is a modification of X such that $\mathscr{S}_\tau \circ \varphi$ is locally free for all $\tau = 1, \ldots, t$ (here one needs Proposition 2, iii)). $X_{\mathscr{S}_1, \ldots, \mathscr{S}_t}$ is a Moišezon space since each finite product of Moišezon spaces and each subspace of a Moišezon space is again a Moišezon space (cf. for example [16], Chapt. I, Theorem 3). Moreover $\mathscr{S}_\tau \circ \varphi$ is almost positive if $\mathscr{S}_\tau$ is almost positive because of Proposition 1.

Due to Moišezon [16] there exists a desingularization $\pi\colon \hat{X} \to X_{\mathscr{S}_1, \ldots, \mathscr{S}_t}$ where $\hat{X}$ is a projective algebraic manifold (cf. also § 4.1). In summary we get

Let $\mathscr{S}_1, \ldots, \mathscr{S}_t$ be finitely many coherent analytic sheaves on a Moišezon space X. Then there exists a projective algebraic manifold $\hat{X}$ and a modification $\varphi\colon \hat{X} \to X$, such that all $\mathscr{S}_\tau \circ \varphi$ are locally free. If one $\mathscr{S}_\tau$ is almost positive then $\mathscr{S}_\tau \circ \varphi$ is also almost positive.

In the situation above we define the torsion free sheaf

$$\mathscr{S}_1^{(\mu_1)} \cdot \ldots \cdot \mathscr{S}_t^{(\mu_t)} \cdot \mathscr{K}_X$$

as the zeroth direct image of $(\mathscr{S}_1 \circ \varphi)^{(\mu_1)} \otimes \cdots \otimes (\mathscr{S}_t \circ \varphi)^{(\mu_t)} \otimes \mathscr{K}_{\hat{X}}$ under φ, where $K_{\hat{X}}$ denotes the canonical line bundle of $\hat{X}$. This definition is independent of the choice of $\hat{X}$ (cf. [7], § 2.1 and3).

If $\mathscr{S}_1, \ldots, \mathscr{S}_t$ are locally free we have

$$\mathscr{S}_1^{(\mu_1)} \cdot \ldots \cdot \mathscr{S}_t^{(\mu_t)} \cdot \mathscr{K}_X = (\mathscr{S}_1^{(\mu_1)} \otimes \cdots \otimes \mathscr{S}_t^{(\mu_t)}) \otimes \mathscr{K}_X,$$

where $\mathscr{K}_X$ is the canonical sheaf of X defined in [7], § 2.1.

Finally let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two sheaves on X, let $Y = X_{\mathscr{S}_1} \times_X X_{\mathscr{S}_2}$ and $\varphi = \varphi_1 \times \varphi_2\colon Y \to X$ be the canonical modification. Then we say that $\mathscr{S}_1 \cdot (\det \mathscr{S}_2)^*$ is almost positive (resp. semi-positive) if the vector bundle $(\mathscr{S}_1 \circ \varphi) \otimes \det(\mathscr{S}_2 \circ \varphi)^*$ is almost positive (resp. semi-positive).

4. We are now in the position to state the vanishing theorem of Griffiths, Kobayashi and Ochiai for almost positive coherent analytic sheaves on Moišezon spaces with singularities.

Theorem 3. *Let X be a Moišezon space, $\mathscr{S}$ and $\mathscr{T}$ coherent analytic sheaves with* rank $\mathscr{T}=1$ *such that*

i) *$\mathscr{S}$ is almost positive and $\mathscr{T}\cdot(\det\mathscr{S})^*$ is semi-positive, or*

ii) *$\mathscr{S}$ is semi-positive and $\mathscr{T}\cdot(\det\mathscr{S})^*$ is almost positive, then*

$$H^l(X,\mathscr{S}^{(k)}\cdot\mathscr{T}\cdot\mathscr{K}_X)=0,\qquad l\geqq 1,\ k\geqq 1.$$

We proceed as in the proof of [7], Satz 2.1. Hence we have first to show the analogue to Satz 2.3.

Proposition 6. *Let X be a projective algebraic space and $\varphi\colon\hat{X}\to X$ a desingularization of X. Further let $\hat{E}$ resp. $\hat{F}$ be a vector bundle resp. a line bundle over $\hat{X}$ such that*

i) *$\hat{E}$ is almost positive and $\hat{F}\otimes\det\hat{E}^*$ is semi-positive, or*

ii) *$\hat{E}$ is semi-positive and $\hat{F}\otimes\det\hat{E}^*$ is almost positive, then*

$$\varphi_{(\nu)}(\hat{\mathscr{E}}^{(k)}\otimes\hat{\mathscr{F}}\otimes\mathscr{K}_{\hat{X}})=0,\qquad \nu\geqq 1,\ k\geqq 1.$$

Proof. Assume that we have already proved

$$\varphi_{(\mu)}(\hat{\mathscr{E}}^{(k)}\otimes\hat{\mathscr{F}}\otimes\mathscr{K}_{\hat{X}})=0$$

for all k and $1\leqq\mu<\nu$, where ν is fixed. Then we take Stein coverings $\mathscr{U}=\{U_\rho\}$ of X resp. $\mathscr{V}=\{V_\sigma\}$ of $\hat{X}$ and form the double complex

$$\{C^{r,s}=C^{r,s}(\hat{\mathscr{U}},\mathscr{V};\hat{\mathscr{E}}^{(k)}\otimes\hat{\mathscr{F}}\otimes\mathscr{K}_{\hat{X}});\ \delta',\delta''\}$$

belonging to the two coverings $\hat{\mathscr{U}}=\{\hat{U}_\rho=\varphi^{-1}(U_\rho);\ U_\rho\in\mathscr{U}\}$ and $\mathscr{V}$ of $\hat{X}$:

$$\begin{tikzcd}
C^{0,0} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & C^{1,0} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & C^{2,0} \arrow[r] \arrow[d, "\delta''"'] & \cdots \arrow[r] & C^{\nu,0} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & C^{\nu+1,0} \arrow[r] \arrow[d, "\delta''"'] & \cdots \\
C^{0,1} \arrow[r, "\delta'"] \arrow[d] & C^{1,1} \arrow[r, "\delta'"] \arrow[d] & C^{2,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C^{\nu,1} \arrow[r, "\delta'"] \arrow[d] & C^{\nu+1,1} \arrow[r] \arrow[d] & \cdots \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
C^{0,\nu-1} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & C^{1,\nu-1} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & C^{2,\nu-1} \arrow[r] \arrow[d, "\delta''"'] & \cdots \arrow[r] & C^{\nu,\nu-1} \arrow[r, "\delta'"] & C^{\nu+1,\nu-1} \arrow[r] & \cdots \\
C^{0,\nu} \arrow[r, "\delta'"] & C^{1,\nu} \arrow[r, "\delta'"] & C^{2,\nu} \arrow[r] & \cdots . & & &
\end{tikzcd}$$

All horizontal sequences in this diagram are exact. Since we have by the induction hypothesis

$$H^\mu(\varphi^{-1}(U), \hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) = \Gamma(U, \varphi_{(\mu)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}})) = 0$$

for all Stein open sets $U \subset X$ and all $1 \leqq \mu < \nu$, $k \geqq 1$, the vertical sequences

$$C^{r,0} \xrightarrow{\delta''} C^{r,1} \xrightarrow{\delta''} \cdots \xrightarrow{\delta''} C^{r,\nu-1}$$

are exact up to $\nu - 1$ for all r. From this one derives by "ascending" a canonical isomorphism

$$H^{2,\nu-1} \cong H^{\nu+1,0}.$$

Since X is projective algebraic there exists an almost positive line bundle H on X; $\hat{H} = \varphi^* H$ is almost positive on $\hat{X}$ because of Proposition 1. Obviously for all μ and $l \geqq 0$

$$\varphi_{(\mu)}(\hat{\mathscr{E}}^{(k)} \otimes (\hat{\mathscr{F}} \otimes \hat{\mathscr{H}}^l) \otimes \mathscr{K}_{\hat{X}}) = \varphi_{(\mu)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) \otimes \mathscr{H}^l.$$

Since therefore $\varphi_{(\mu)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) = 0$ if and only if $\varphi_{(\mu)}(\hat{\mathscr{E}}^{(k)} \otimes (\hat{\mathscr{F}} \otimes \hat{\mathscr{H}}^l) \otimes \mathscr{K}_{\hat{X}})$ vanishes and $\hat{F} \otimes \det \hat{E}^*$ is almost positive (resp. semi-positive) if and only if $(\hat{F} \otimes \hat{H}^l) \otimes \det \hat{E}^*$ is so for one $l \geqq 0$ we may replace $\hat{F}$ by a suitable product $\hat{F} \otimes \hat{H}^l$ without loss of generality.

Due to Grauert [6] there exists an integer l_0 such that

$$H^{\nu+1}(X, \varphi_{(0)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) \otimes \mathscr{H}^l) = 0$$

for all $l \geqq l_0$, k fixed. Hence

$$H^{\nu+1}(X, \varphi_{(0)}(\hat{\mathscr{E}}^{(k)} \otimes (\hat{\mathscr{F}} \otimes \hat{\mathscr{H}}^l) \otimes \mathscr{K}_{\hat{X}})) = 0$$

and we may assume that

$$\begin{aligned} H^{\nu+1,0} &= H^{\nu+1}(\hat{\mathscr{U}}, \hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) \\ &= H^{\nu+1}(X, \varphi_{(0)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}})) = 0. \end{aligned}$$

Suppose now that $\varphi_{(\nu)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) \neq 0$. Since we are again allowed to tensorize by a suitable power of H we may assume that there exists a global section $\sigma \neq 0$ in $\Gamma(X, \varphi_{(\nu)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}))$.

But then one can find a non-vanishing cohomology class in $H^\nu(\hat{X}, \hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}})$ because of $H^{2,\nu-1} \cong H^{\nu+1,0} = 0$ (cf. [7], p. 276). That means

$$H^\nu(\hat{X}, \hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{X}}) \neq 0,$$

which is a contradiction to Theorem 1 ($\hat{X}$ is necessarily a projective algebraic manifold and hence Kähler).

The proof works also in the case $\nu = 1$, q.e.d.

Proof of Theorem 3. By assumption there exists a projective algebraic manifold $\hat{X}$ and a modification mapping $\varphi\colon \hat{X} \to X$ such that for $\hat{\mathscr{S}} = \mathscr{S} \circ \varphi$ and $\hat{\mathscr{T}} = \mathscr{T} \circ \varphi$ one of the two cases i) and ii) holds:

i) $\hat{\mathscr{S}}$ is almost positive and $\hat{\mathscr{T}} \otimes \det \hat{\mathscr{S}}^*$ is semi-positive,

ii) $\hat{\mathscr{S}}$ is semi-positive and $\hat{\mathscr{T}} \otimes \det \hat{\mathscr{S}}^*$ is almost positive.

Moreover due to Artin [3] there exists for every point $x_0 \in X$ a projective algebraic space Y, a holomorphic map $\sigma\colon Y \to X$ and a (lower dimensional) algebraic subset $A \subset Y$ such that $\sigma | Y - A$ is locally biholomorphic and $x_0 \in \sigma(Y - A)$. In particular σ is of class (*).

The fibre product $\tilde{Y} = Y \times_X \hat{X}$ is a projective algebraic space which possesses a projective algebraic desingularization $\hat{Y}$ (Hironaka [9]). We have thus a diagram

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\tau} & \hat{X} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{\sigma} & X, \end{array}$$

in which all mappings are of class (*). Therefore we can see that $\hat{E} = E(\tau^* \hat{\mathscr{S}})$ and $\hat{F} = E(\tau^* \hat{\mathscr{T}})$ satisfy one of the two conditions i) and ii) of Proposition 2 on $\hat{Y}$, and hence

$$\psi_{(\nu)}(\hat{\mathscr{E}}^{(k)} \otimes \hat{\mathscr{F}} \otimes \mathscr{K}_{\hat{Y}}) = 0, \qquad \nu \geqq 1.$$

Outside $\hat{A} = \psi^{-1}(A)$ the mapping τ is locally biholomorphic, and one has $x_0 \in \varphi(\tau(\hat{Y} - \hat{A}))$. This implies the vanishing of $\varphi_{(\nu)}(\hat{\mathscr{S}}^{(k)} \otimes \hat{\mathscr{T}} \otimes \mathscr{K}_{\hat{X}})$ in a neighborhood of x_0 for all $\nu \geqq 1$ and all $k \geqq 1$. Hence

$$\varphi_{(\nu)}(\hat{\mathscr{S}}^{(k)} \otimes \hat{\mathscr{T}} \otimes \mathscr{K}_{\hat{X}}) = 0, \qquad \nu \geqq 1,$$

and because of Theorem 1

$$\begin{aligned} H^l(X, \mathscr{S}^{(k)} \cdot \mathscr{T} \cdot \mathscr{K}_X) &= H^l(X, \varphi_{(0)}(\hat{\mathscr{S}}^{(k)} \otimes \hat{\mathscr{T}} \otimes \mathscr{K}_{\hat{X}})) \\ &= H^l(\hat{X}, \hat{\mathscr{S}}^{(k)} \otimes \hat{\mathscr{T}} \otimes \mathscr{K}_{\hat{X}}) \\ &= 0, \end{aligned}$$

$l \geqq 1$, $k \geqq 1$, q.e.d.

§ 4. Some Properties of Moišezon Spaces

1. Let X be a (reduced) Moišezon space. Then X is properly bimeromorphically equivalent to a projective algebraic variety Y ([16], Chapt I, Theorem 1); that means:

There exist a projective algebraic variety Y, a compact complex space X' and proper modifications

$$\begin{array}{ccc} & X' & \\ \swarrow & & \searrow \\ X & & Y. \end{array}$$

Since Y is projective algebraic X' may be supposed to be projective algebraic, too. Therefore we can find a desingularization $\hat{X} \to X'$ due to Hironaka, and we have proved the following statement ([16], Chapt. I, Theorem 7):

Every Moišezon space X has a projective algebraic desingularization $\pi: \hat{X} \to X$.

This proposition can be sharpened as follows (Moišezon [17]):

Let X be an (irreducible) Moišezon space. Then there exists a finite sequence of monoidal transformations of the space X, say $\varphi_i: X_{i+1} \to X_i$, $0 \leqq i < r$, $X_0 = X$, such that

i) *the center B_i of the monoidal transformation φ_i is a nonsingular projective algebraic variety,*

ii) *B_i is nowhere dense in X_i, and if X_i is singular, then $B_i \subset \operatorname{sing}(X_i)$, where $\operatorname{sing}(X_i)$ is the set of singular points of X_i,*

iii) *X_r is a projective algebraic manifold.*

2. We now prove

Theorem 4. *Every normal Moišezon space X carries a torsion free almost positive coherent analytic sheaf $\mathcal{S}$ of rank 1. $\mathcal{S}$ is positive outside an analytic subset A of X of codimension at least 2.*

Proof. As was pointed out in Section 1 there exists a projective algebraic desingularization $\pi: \hat{X} \to X$ of X. Since X is normal and π is a proper modification the image $A = \pi(\hat{A})$ of points $\hat{x} \in \hat{X}$ at which π is degenerated is analytic of codimension 2.

Now let E be an ample line bundle on $\hat{X}$ in the sense of Griffiths [8]; that means

i) the global sections of E over $\hat{X}$ generate each fibre $E_{\hat{x}}$,

ii) if T denotes the tangent bundle of $\hat{X}$ and $F_{\hat{x}}$ the vector space of global sections of E vanishing at $\hat{x} \in \hat{X}$ then the natural mapping

$$F_{\hat{x}} \to E_{\hat{x}} \otimes T_{\hat{x}}$$

is surjective for all $\hat{x} \in \hat{X}$.

Each sufficiently high power of a positive line bundle satisfies these conditions ([8], Theorem C).

Since E is generated by its sections we have an epimorphism

$$\mathcal{O}_{\hat{X}}^q \to \mathcal{E} \to 0, \qquad q = \dim_{\mathbb{C}} H^0(\hat{X}, \mathcal{E}),$$

and hence an embedding

$$L(\mathcal{E}) \hookrightarrow \hat{X} \times \mathbb{C}^q.$$

Remembering that $L(\mathcal{E}) = E^*$ we derive from Griffiths proof of his Theorem A ([8], p. 201) the following statement:

The flat metric on $\hat{X} \times \mathbb{C}^q$ induces an hermitian metric on $L(\mathcal{E})$ whose curvature tensor $\Theta = \{\Theta_{i\bar{j}}\}$ has the property that the hermitian quadratic form

$$\Theta(\eta) = \sum \Theta_{i\bar{j}} \, \eta^i \bar{\eta}^j$$

is negative definite.

Define now $\mathcal{S} := \operatorname{im}(\pi_{(0)} \mathcal{O}_{\tilde{X}}^q \to \pi_{(0)} \mathcal{E})$. $\mathcal{S}$ is coherent and as subsheaf of the torsion free sheaf $\pi_{(0)} \mathcal{E}$ again torsion free. Since $\mathcal{S}$ coincides with $\mathcal{E}$ outside A, $\mathcal{S}$ is of rank 1.

Obviously $\pi_{(0)} \mathcal{O}_{\tilde{X}}^q = \mathcal{O}_X^q$ because X is normal. This implies an epimorphism $\mathcal{O}_X^q \to \mathcal{S} \to 0$ which leads to an embedding

$$L(\mathcal{S}) \hookrightarrow X \times \mathbb{C}^q.$$

Restrict the flat metric of $X \times \mathbb{C}^q$ to $L(\mathcal{S})$. This gives an hermitian metric on $L(\mathcal{S})$, and since this metric coincides with the metric on $L(\mathcal{E})$ outside A the sheaf $\mathcal{S}$ is Griffiths quasi-positive and thus almost positive (cf. § 3.2). The positivity of $\mathcal{S}$ on $X - A$ is clear, q.e.d.

3. In a very special case we can prove the converse of Theorem 4:

Theorem 5. *Let X be an irreducible normal compact complex space and $\mathcal{S}$ an almost positive coherent analytic sheaf of rank* 1 *on X which is positive outside an analytic subset $A \subset X$ of dimension* 0 *(in particular X has only isolated singularities since* $\operatorname{sing} X \subset A$*). Then X is a Moišezon space.*

The idea of the *proof* is the following: There exists a proper modification $\pi: Y \to X$ such that Y is a manifold, π induces an isomorphism $Y - \pi^{-1}(A) \xrightarrow{\sim} X - A$ and $\mathcal{T} = \mathcal{S} \circ \varphi$ is a locally free rank one sheaf. $E(\mathcal{T})$ is positive on $Y - \pi^{-1}(A)$. Moreover $\pi^{-1}(A)$ will be locally defined by one function such that $\pi^{-1}(A)$ defines a line bundle F. We will show that F carries an hermitian metric which is positive in a suitable neighborhood W_1 of $\pi^{-1}(A)$ and is trivial on the trivial bundle $F \mid Y - W_2$, $W_1 \subset\subset W_2 \subset\subset Y$. Then a certain product $F \otimes E(\mathcal{T})^l$ is positive everywhere and thus Y is projective algebraic due to Kodaira. But then X must be a Moišezon space.

Let us carry out this program. We start with the monoidal transformation (Z, φ) of X with respect to $\mathcal{S}$. Then $X - A$ is nonsingular, $Z - \varphi^{-1}(A)$ is dense in Z and φ induces an isomorphism $Z - \varphi^{-1}(A) \xrightarrow{\sim} X - A$ since $D \subset A$ (if there are points in $A - D$ we make in addition monoidal transformations with respect to the maximal ideals of these points such that φ is degenerated exactly over A). Since the triple (Z, φ, X) satisfies the assumptions of Theorem 1 in [10] we find a complex manifold Y and proper modifications $\pi: Y \to X$, $\psi: Y \to Z$ such that $\pi = \varphi \circ \psi$. Moreover there exists an ideal sheaf $\mathcal{I}$ on X such that $Z(\mathcal{I}) = \{x \in X : \mathcal{O}_{X,x}/\mathcal{I}_x \neq 0\} = A = \{x_1, \ldots, x_t\}$ and (Y, π) is the monoidal transformation of X with center A (with respect to the ideal sheaf $\mathcal{I}$). π is degenerated exactly over A. $E = E(\mathcal{S} \circ \pi) = E(\mathcal{S} \circ \varphi) \times_Z Y$ is an almost positive line bundle which is positive outside $B = \pi^{-1}(A)$.

Moreover $\mathcal{J} = \pi^{-1}(\mathcal{I}) = \mathcal{I} \circ \pi$ is locally free of rank 1 such that $F = E(\mathcal{J})$ is a line bundle on Y. $F^* = E(\mathcal{J})^* = L(\mathcal{J})$ is the normal bundle of the embedding $B = Z(\mathcal{J}) \hookrightarrow Y$. As we pointed out in § 2.6 $F \mid B_\tau$ is the positive hyperplane bundle of the projective algebraic space $B_\tau = \pi^{-1}(x_\tau)$. We now claim more generally the following statement:

(*) *There exist neighborhoods* $V_\tau \subset\subset W_\tau$ *of* B_τ *in* Y, $\tau = 1, \ldots, t$, *such that* $\overline{W}_\sigma \cap \overline{W}_\tau = \emptyset$ *for all* $\sigma \neq \tau$ *and* $W_\sigma \neq Y$ *for all* σ, *and an hermitian metric* h *on* F *such that* F *is positive on* $\bigcup_{\tau=1}^{t} V_\tau$ *and* h *is on* $Y - \bigcup_{\tau=1}^{t} \overline{W}_\tau$ *equal to the flat metric (with curvature* 0*) of the trivial line bundle* $F \,|\, Y - \bigcup_{\tau=1}^{t} \overline{W}_\tau$.

Assume that we have already proved (*). Since E is positive outside $B = \bigcup_{\tau=1}^{t} B_\tau$ we can find a neighborhood $U \subset\subset \bigcup_{\tau=1}^{t} V_\tau$ of B and a positive integer l such that $F \otimes E^l$ is positive on $Y - U$. But on $V = \bigcup_{\tau=1}^{t} V_\tau$ the line bundle F is positive and E is at least semi-positive such that $F \otimes E^l$ is positive on $V \cup (Y - \overline{U}) = Y$. Then Y must be projective algebraic and X is a Moišezon space since π is a proper modification and X is normal.

It remains to prove (*). Since the statement is obviously local with respect to X we only must prove the following:

Let x_0 *be a point of a complex space* U *which is embeddable as an analytic subvariety in a domain of a suitable* $\mathbb{C}^N$ *such that* $x_0 = 0 \in \mathbb{C}^N$. *Assume that there are holomorphic functions* $f_0, \ldots, f_r$ *on* U *such that*

$$\{f_0 = \cdots = f_r = 0\} = x_0,$$

and let $(\hat{U}, \pi)$, $\hat{U} \subset U \times \mathbb{P}^r$, *be the monoidal transformation of* U *with respect to* $\mathscr{I} = (f_0, \ldots, f_r)\, \mathcal{O}_U$. *Assume moreover that* $\hat{U}$ *is nonsingular. Then there exist neighborhoods* $V \subset\subset W \subset\subset U$ *of* x_0 *and an hermitian metric* h *on* $E = E(\pi^{-1}(\mathscr{I}))$ *such that* E *is positive on* $\hat{V} = \pi^{-1}(V)$ *and has curvature* 0 *outside* $\hat{W} = \pi^{-1}(W) \subset\subset \hat{U}$ *with respect to* h.

Proof. Denote by $(\zeta_0, \ldots, \zeta_r)$ a system of homogeneous coordinates of $\mathbb{P}^r$. The functions

$$g_\rho = \frac{|\zeta_\rho|^2}{|\zeta_0|^2 + \cdots + |\zeta_r|^2} \quad \text{on } Z_\rho = \{\zeta_\rho \neq 0\} \subset \mathbb{P}^r$$

are C^∞ and positive. Since $-\log g_\rho$ is strongly plurisubharmonic on Z_ρ,

$$g_\rho = \left| \frac{\zeta_\rho}{\zeta_\sigma} \right|^2 g_\sigma \quad \text{on } Z_\rho \cap Z_\sigma$$

and the hyperplane bundle H on $\mathbb{P}^r$ is given by the transition functions $g_{\rho\sigma} = \left| \frac{\zeta_\sigma}{\zeta_\rho} \right|$ the system $\{g_\rho\}$ is a positive metric on H (cf. § 1.1, Remark).

Denote by ψ the canonical projection $\hat{U} \to \mathbb{P}^r$. Then $E = E(\pi^{-1}(\mathscr{I})) = \psi^* H$ and the transition functions of E are $\hat{f}_\sigma / \hat{f}_\rho$ where $\hat{f}_\rho = f_\rho \circ \pi$. The system $\{h'_\rho = g_\rho \circ \psi\}$ is a metric on E since we have $h'_\rho = |\hat{f}_\rho / \hat{f}_\sigma|^2\, h'_\sigma$.

The function $a_\rho = h'_\rho \circ \pi^{-1}$ is defined on $U_\rho = \{f_\rho \neq 0\}$, positive and satisfies $a_\rho = |f_\rho / f_\sigma|^2\, a_\sigma$ on $U_\rho \cap U_\sigma$. Therefore $a = |f_\rho|^2 / a_\rho$ is a well defined positive

C^∞ function on $\bigcup U_\rho = U - x_0$. We can now find open neighborhoods $V \subset\subset W \subset\subset U$ and a positive C^∞ function δ on U such that δ is, on $U - \overline{W}$, equal to a and, on V, equal to the restriction of $(1+|z_1|^2+\cdots+|z_N|^2)^{-1}$ to V where $z_1, \ldots, z_N$ are complex coordinates of $\mathbb{C}^N$. Define now $h_\rho = \delta \cdot h'_\rho$; the system $\{h_\rho\}$ is a metric on E such that $h_\rho | \pi^{-1}(U - \overline{W}) = a\, h_\rho = |f_\rho|^2$. This means that h_ρ is the trivial metric outside $\pi^{-1}(\overline{W})$. On $\pi^{-1}(V \cap U_\rho) \subset \mathbb{C}^N \times Z_\rho$ we have

$$h_\rho = (1+|z_1|^2+\cdots+|z_N|^2)^{-1} \left. \frac{|\zeta_\rho|^2}{|\zeta_0|^2+\cdots+|\zeta_r|^2} \right| \pi^{-1}(V \cap U_\rho).$$

Thus $-\log h_\rho$ is as restriction of a strongly plurisubharmonic function to the subvariety $\pi^{-1}(V \cap U_\rho)$ again strongly plurisubharmonic which implies the positivity of E on $\pi^{-1}(V)$ with respect to h, q.e.d.

Remark. The proof shows that a complex space satisfying the assumptions of Theorem 5 can be desingularized to a projective algebraic manifold by means of a finite number of monoidal transformations with center a point in X.

4. Obviously Theorem 4 and 5 give a characterization of 2-dimensional Moišezon spaces:

Corollary 1. *A 2-dimensional irreducible normal compact complex space X is a Moišezon space if and only if X carries an almost positive coherent analytic sheaf $\mathcal{S}$ of rank 1 which is positive outside an analytic subset $A \subset X$ of dimension 0.*

The first example of a 2-dimensional Moišezon space with one isolated singularity which is not projective algebraic and moreover not algebraic in the sense of *A.* Weil is due to Grauert [6].

Such an example does not exist if X is nonsingular (Chow and Kodaira [4]):

A 2-dimensional nonsingular Moišezon space is projective algebraic.

This can be shown in the following way: Let $\pi: \hat{X} \to X$ be a proper modification such that $\hat{X}$ is projective algebraic. Then due to Hopf [11] π is a finite succession of σ-processes $\pi_\rho: X_\rho \to X_{\rho-1}$, $1 \leq \rho \leq r$, $X_0 = X, X_r = \hat{X}, \pi_1 \circ \cdots \circ \pi_r = \pi$ (a σ-process is the monoidal transformation of a manifold with center a point x_0 where the ideal sheaf is the maximal ideal defined by x_0). Now if $\pi_\rho: X_\rho \to X_{\rho-1}$ is a σ-process and X_ρ is projective algebraic then $X_{\rho-1}$ is projective algebraic (cf. for example [6], § 4).

There are 3-dimensional nonsingular Moišezon spaces which are neither projective algebraic nor an abstract algebraic variety (Moišezon [16], Chapt. III). But *every compact Kähler manifold X with $n = \dim X$ independent meromorphic functions is projective algebraic* ([16], Chapt. I, Theorem 11). Thus we get finally

Corollary 2. *An irreducible compact complex manifold of dimension 2 or an irreducible compact complex Kähler manifold of arbitrary dimension is projective algebraic if and only if there exists an almost positive coherent analytic sheaf of rank 1 on it which is positive outside a discrete set.*

5. At the end of this paper we generalize the example of Grauert [6] in order to show that there exist spaces X in all dimensions $n \geqq 2$ satisfying the assumptions of Theorem 5 which are however not projective algebraic. We prove first:

There exist n-dimensional compact complex manifolds R_n, $n \geqq 1$, with positive canonical bundle K_n such that $H^1(R_n, K_n) \neq 0$.

Let R be a Riemann surface of genus $g \geqq 2$ and K the canonical bundle of R; then

$$H^0(R, K) = g, \qquad H^1(R, K) = 1$$

and $-c(K)$ denotes the Chern number of $K-$:

$$c(K) = 2(g-1) \geqq 2.$$

Therefore $R = R_1$ is an example in the case $n=1$. In the general case take

$$R_n = R \times \cdots \times R \qquad (n \text{ times}).$$

The canonical bundle K_n of R_n is equal to $p_1^* K \otimes \cdots \otimes p_n^* K$ where p_i is the canonical projection of $R_n = R \times \cdots \times R$ onto the i-th factor R. It is an easy exercise to prove the positivity of K_n with the help of the positivity of the K's. Finally, by the Künneth formula (cf. for instance [12]) and induction

$$\dim H^1(R_n, K_n) = n(n-1)\, g \neq 0, \qquad n \geqq 2.$$

The argument now follows the one given by Grauert: Take a covering $\mathscr{U} = \{U_i\}$ of R_n such that K_n is defined by transition functions f_{ij} and such that there exists a cocycle $\xi = \{\xi_{ij}\} \in Z^1(\mathscr{U}, K_n)$ whose cohomology class $\bar{\xi} \in H^1(R_n, K_n)$ is different from zero. Look at the products $U_i \times \mathbb{C}_i$, where $\mathbb{C}_i$ is the complex line $\mathbb{C}$ with the variable z_i, and patch them together by the formula

$$z_i = f_{ij}(x)\, z_j + \xi_{ij}(x), \qquad x \in U_i \cap U_j$$

(remark that ξ_{ij} may be regarded as holomorphic function on $U_i \cap U_j$). Thus we get a complex analytic fibre bundle $Y \to R_n$ with fibre $\mathbb{C}$ and structure group the group of all affine transformations of $\mathbb{C}$. Now by compactification of each fibre $Y_x \cong \mathbb{C}$ one gets a fibre bundle π: $\bar{Y} \to R_n$ with fibre $\mathbb{P}^1 = Y_x \cup Y_{x,\infty}$. $\bar{Y}$ is a projective algebraic manifold.

It is easy to check that the normal bundle of the set $A = \bigcup_{x \in R_n} Y_{x,\infty}$ in $\bar{Y}$ is isomorphic to K_n^*. Since K_n^* is negative A can be blown down to a point x_0 in a normal compact complex space X; the projection φ: $\bar{Y} \to X$ is biholomorphic outside $\varphi^{-1}(x_0) = A$. Exactly as in the proof of Theorem 4 one constructs an almost positive coherent analytic sheaf of rank 1 on X which is positive outside x_0.

That X is not projective algebraic can be seen as in [6], p. 366.

Appendix

We want to show here that one can prove the vanishing theorem (Theorem 3) under much weaker conditions: The almost positive coherent analytic sheaves may be replaced by sheaves which are positive only on a nonempty open subset $U \subset X$ and semi-positive outside U. Since we do not want to repeat all steps we restrict ourselves to the case of complex analytic vector bundles on Kähler manifolds. In order to obtain also a generalization of the vanishing theorem in [7] we deal with Nakano positivity.

Definition. Let E be a complex vector bundle on a complex hermitian manifold X. E is called *(Nakano) quasi-positive* if there exist an hermitian metric h on E and a nonempty open subset U of X such that $E|U$ is positive and $E|X-\bar{U}$ is semi-positive with respect to h in the sense of Nakano [18].

Remark. If E is an almost positive *line bundle* then it is also quasi-positive.

Theorem 6. *Let E be a Nakano quasi-positive vector bundle on a connected compact Kähler manifold X. Then*

$$H^{\nu}(X, \mathscr{E} \otimes \mathscr{K}) = 0, \qquad \nu \geqq 1.$$

Proof. We use the usual notations in the theory of Kähler manifolds (cf. for instance [7], proof of Satz 2.2); $A^{p,q}$ is the complex vector space of C^{∞} forms of type (p, q) on X with values in E and $\square = d'' \delta'' + \delta'' d''$: $A^{p,q} \to A^{p,q}$ denotes the complex Laplace-Beltrami operator. Then we have

$$H^{\nu}(X, \mathscr{E} \otimes \mathscr{K}) \cong \mathscr{H}^{n,\nu}(E) = \{\varphi \in A^{n,\nu} \colon \square\, \varphi = 0\}.$$

Due to Nakano one has for all $\varphi \in \mathscr{H}^{p,q}(E)$ the inequality

$$(\chi \wedge \Lambda\, \varphi, \varphi) = \int_X \chi \wedge \Lambda\, \varphi \wedge \bar{*}\, \varphi \leqq 0$$

On the other hand, since E is Nakano semi-positive, we have for all $\varphi \in A^{n,\nu}$, $\nu \geqq 1$, at each point $x_0 \in X$:

$$\psi = \chi \wedge \Lambda\, \varphi \wedge \bar{*}\, \varphi \geqq 0.$$

(ψ is a (n, n)-form) and ψ is at $x_0 \in U = \{x \in X \colon E$ is Nakano positive at $x\}$ strictly smaller than zero if $\varphi \neq 0$. This implies

$$\varphi | U \equiv 0.$$

The proof is finished if we can show the following *identity theorem for harmonic forms* with values in an hermitian vector bundle.

Theorem 7. *Let E be an hermitian vector bundle on a connected complex hermitian manifold X. Then a harmonic form $\varphi \in \mathscr{H}^{p,q}(E)$ vanishes identically on X if it vanishes on a nonempty open subset $U \subset X$.*

Proof. Let $D = \{x \in X \colon \varphi$ is identically zero in a neighborhood of $x\}$. D is nonempty and open. We have only to show that D is also closed. Since this is a local problem it is enough to prove the following:

Let V be an (arbitrary small) domain in X and let $W \subset V$ be a nonempty open subset of X such that $\varphi | W \equiv 0$. Then $\varphi | V \equiv 0$.

We may choose V so small that $V \subset\subset V' \subset \mathbb{C}^n$ and $E | V'$ is trivial. Represent $\varphi \in \mathscr{H}^{p,q}(E) | V'$ by C^∞ functions $(\varphi_1, \ldots, \varphi_N)$ on V'. Then there exist a strongly elliptic differential operator A of order 2 and differential operators $L_{\nu\mu}$, $1 \leqq \nu$, $\mu \leqq N$, of order $\leqq 1$ on V', such that

$$(\square \varphi)_\nu = A \varphi_\nu + \sum_{\mu=1}^{N} L_{\nu\mu} \varphi_\mu$$

(cf. [1], formula (30)). The coefficients of A and $L_{\nu\mu}$ depend differentiably on the hermitian metrics on X and E and hence they are C^∞. If now $\varphi = (\varphi_1, \ldots, \varphi_N)$ is a solution of $\square \varphi = 0$ on V' there exists a real constant $M > 0$ such that

$$|A \varphi_\nu|^2 \leqq M \sum_{\mu=1}^{N} \left(\sum_{\lambda=1}^{2n} \left| \frac{\partial \varphi_\mu}{\partial x_\lambda} \right|^2 + |\varphi_\mu|^2 \right), \qquad \nu = 1, \ldots, N,$$

for $\varphi | V$ where $(x_1, \ldots, x_{2n})$ is a real coordinate system of $\mathbb{C}^n$. But this is exactly the situation regarded by Aronszajn in [2], (Remark 3, p. 248). From his main theorem it follows immediately that $\varphi | V \equiv 0$ if $\varphi | W \equiv 0$, q.e.d.

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