MR0355101 (50 #7578) 32C40 14B05 14D15
Riemenschneider, Oswald
Deformations of rational singularities and their resolutions.
Complex analysis, 1972 (Proc. Conf., Rice Univ., Houston, Tex., 1972). Vol. I: Geometry of singularities.
Rice Univ. Studies **59** (1973), *no. 1,* 119–130.

Let $\pi: X \to S$ be a family of complex manifolds, and suppose that a certain fiber $X_0 = \pi^{-1}(0)$ $(0 \in S)$ contains a subvariety E_0 that can be blown down to a point. Then locally near 0 and E_0, $X \to S$ can be contracted to a family $Y \to S$. The proof consists in showing that π is "1-convex" [see, e.g., K. Knorr and M. Schneider, Math. Ann. **193** (1971), 238–254; MR0293129] by extending an exhaustion function from X_0 to X.

In general, Y_0 will not be isomorphic to the contraction of X_0, because $\dim H^1(X_t, \mathcal{O}_{X_t})$ $(t \in S)$ may jump (up) at $t = 0$. This is no problem when X_0 is the resolution of a rational singularity ($\dim H^1(X_0, \mathcal{O}_{X_0}) = 0$ [see E. V. Brieskorn, Invent. Math. **4** (1967/68), 336–358; MR0222084]); further, nearby fibers X_t will be the resolution of rational singularities Y_t. If Y_0 has multiplicity not exceeding 2, then Y_t will also, and the author lists (without proof) a table of specializations among the singularities A_n, D_n.

Finally, if X_0 is the rational ruled surface of index m, containing a $\mathbf{P}^1$ with self-intersection $-m$, then its versal deformation $X \to \mathbf{C}^{m-1}$ contracts to give a deformation $Y \to \mathbf{C}^{m-1}$ of the ordinary m-fold singularity Y_0; here Y_0 is the locus $z_0/z_1 = z_1/z_2 = \cdots = z_{m-1}/z_m$ in C^{m+1}, Y_t is $z_0/z_1 = (z_1 + t_1)/z_2 = \cdots = (z_{m-1} + t_{m-1})/z_m$ is nonsingular $(0 \neq t \in \mathbf{C}^{m-1})$. M. Artin later showed these to be exactly those deformations of Y_0 that can be simultaneously resolved [see #7143 above].

{For the entire collection, see 48 #552.} *Michael Schlessinger*