# Seminar: Hochschild homology and the homology of free loop spaces

**Winter term 2015/16**

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This seminar aims at a thorough introduction to Hochschild homology, its connection to the homology theory for groups and its relationship to the (singular) homology of free loop spaces.

If you have any questions concerning your talk, don't hesitate to ask. Please hand in an outline of your talk 14 days before you actually give the talk.

## List of talks

(1) **Hochschild homology I: basics** Define what Hochschild homology of an associative algebra is. There are several variants of the definition and you should present the one by the Hochschild complex and the one via the two-sided bar construction leading to a Tor-interpretation. Please give examples, describe the module structure over the center and present some low-dimensional calculations. [L, 1.1], [W, Chapter 9]

(2) **Morita invariance of $HH_*$, Kähler differentials** Morita equivalence is a concept that allows to compare module categories. Show that Hochschild homology satisfies Morita invariance. Start with the case of matrix algebras and then proceed to the general case. We saw the module of Kähler differentials in the first talk. Generalize this to a map from algebraic differential forms to Hochschild homology. [L, 1.2, 1.3], [W, Chapter 9]

(3) **Hochschild cohomology** Dual to Hochschild homology there is a corresponding cohomology theory. Describe it and and its low dimensional cohomology groups. Morita equivalence carries over, and there is a Lie-algebra structure on Hochschild cohomology. [L, 1.5], [W, Chapter 9]

(4) **Cyclic (co)homology** If one considers the Hochschild complex of an algebra $A$ with coefficients in $A$, then this complex carries a cyclic structure. Define what cyclic homology and cohomology of an algebra is. There are several equivalent definitions: in terms of the cyclic bicomplex, via a slightly smaller bicomplex and in characteristic zero, there is an even leaner description. Prove that we have a periodicity sequence relating Hochschild and cyclic homology. Relate cyclic homology to algebraic differential forms. Mention the cohomological variant. [L, parts of 2.1–2.4], [W, 9.6]

(5) **Simplicial and cyclic modules** Define what a simplicial object is and treat the case of simplicial modules in detail. Describe how simplicial modules are related to chain complexes. A cyclic module is a simplicial module with an extra compatible cyclic symmetry. Describe what they are and show that one can define Hochschild and cyclic homology for such modules. [L, 1.6, 2.5], [W, 8.1, 8.4]

(6) **Group (co)homology** Define what the (co)homology groups of a discrete group are and compare them to the corresponding Hochschild (co)homology groups of the group algebra. Do as many examples as possible; we need the example of the homology of cyclic groups. [L, 7.4], [W, Chapter 6 and 9.7]

(7) **Hochschild homology and $\mathbb{S}^1$-equivariant homology** Simplicial and cyclic sets have geometric realizations as CW complexes. The geometric realization of a cyclic set carries and action of the group $\mathbb{S}^1$. Prove these facts and describe the connection between cyclic homology and $\mathbb{S}^1$-equivariant homology. [L, 7.1, 7.2]

(8) **Spectral sequences – basics** What are spectral sequences? Define them and describe the spectral sequence associated to a filtered chain complex. Every bicomplex gives rise to two spectral sequence. Mention the example of the spectral sequence converging to cyclic homology with group homology of the cyclic groups as input. [BT, III §14], [L, Remark p. 55], [McC, W]

(9) **The Leray-Serre spectral sequence** Associated to a fibration there is the Leray-Serre spectral sequence. It computes the (co)homology of the total space from the (co)homology of the base and the fiber. [McC, 5.1,5.2]

(10) **Free loop spaces** Free loop spaces are spaces of (unbased) maps from $\mathbb{S}^1$ to a topological space $X$: $LX = \mathrm{Maps}(\mathbb{S}^1, X)$. Even if $X$ is a finite CW complex, $LX$ will be large. Describe the fibration from $LX$ to $X$ with based loops as the fiber. Topological spaces sometimes carry compositions that are associative up to homotopy: define what an H-space is and describe the splitting of a free loop space of an H-space.

(11) **Hochschild homology and free loop spaces** Give an overview of the results by Goodwillie and Jones that compare the (co)homology of free loop spaces to Hochschild (co)homology and which relate the $\mathbb{S}^1$-equivariant (co)homology of free loop spaces to cyclic (co)homology. What is the strategy of proof? [L, 7.3], [G], [J].

(12) **The Chas-Sullivan product I** Chas and Sullivan defined a product on the homology of free loop spaces. Describe this product. [CS, F].

(13) **The Chas-Sullivan product II** Hochschild cohomology carries a cup product. Under the identification of Hochschild cohomology with the cohomology of the free loop spaces this product corresponds to the Chas Sullivan product.

However, both cohomology theories carry extra structure, a so-called Gerstenhaber algebra structure (a compatible combination of a graded commutative product and a (graded) Lie structure). Report on work of Cohen und Jones who establish that one can actually identify the Gerstenhaber algebra structures. [CJ, F]

You will have access to [F].

## References


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