Exercises in Algebraic Topology (master)

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Summer term 2017

Exercise sheet no 8 For the exercise class on the 26th of June 2017

1 (Suspensions) Let X be a space and let ΣX be its suspensions. Take $\alpha, \beta \in H^*(\Sigma X; R)$ in positive degrees. Here R is an arbitrary commutative ring with unit. What is $\alpha \cup \beta$?

2 (Orientation covering) Let M be an n-dimensional topological manifold.

a) Prove that there is an oriented manifold $\hat{M}$ and a 2-fold covering $p\colon \hat{M} \to M$ called the orientation covering.

b) Are the following statements equivalent? 1. M is orientable. 2. The orientation covering is a trivial covering, *i.e.*, $\hat{M} \cong M \times \mathbb{Z}/2\mathbb{Z}$ as spaces over M.

c) Assume that M is finite dimensional, path connected with $\pi_1(M) = 1$. Is M orientable?

d) What is the orientation covering of $\mathbb{R}P^n$ for even n? What about the Klein bottle and the open Möbius strip?

3 (Cut-and-paste) Assume $g \geqslant 2$ and let E_{2g} be a regular $2g$-gon with vertices $z_1, \ldots, z_{2g}$. You already encountered the quotient space N_g of E_{2g} by the relation

$$(1-t)z_{2j-1} + tz_{2j} \sim (1-t)z_{2j} + tz_{2j+1}.$$

Here, you should interpret the indices mod $2g$. We called N_g the non-orientable surface of genus g. Justify that name.

4 (R-orientations) Let R be a commutative ring with unit and let M be a connected m-dimensional manifold together with an R-orientation. Show that the group of units of R, $R^\times$, acts free and transitively on the set of all R-orientations of M. For $R = \mathbb{Z}$ this should look familiar.