Exercises in Algebraic Topology (master)

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Exercise sheet no 6 For the exercise class on the 12th of June 2017

1 (Tor and Hom)

a) Is the abelian group $\mathbb{Q}$ free? Show that the functor $A \mapsto A \otimes \mathbb{Q}$ is exact

b) Let A be a finitely generated abelian torsion group. Can you identify $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ and/or $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ with A?

c) Let M be an abelian group and let

$$(*) \qquad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of abelian groups.

What can you say about the exactness of the sequences

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{\beta^*} \mathrm{Hom}(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}(A, M) \longrightarrow 0.$$

What happens if the sequence $(*)$ splits?

2 (How bad can it be?)

Give an example of a chain complex $(C_*.d)$ with trivial homology, such that the chain complex $C_* \otimes \mathbb{Z}/2\mathbb{Z}$ has non-vanishing homology in every degree.

3 (Same for R-modules?)

(1) Assume that R is a commutative ring with unit. Can you define Tor for R-modules in the same way as we did for $R = \mathbb{Z}$? What is different?
(2) What happens if R is a field? What about $R = \mathbb{Z}/4\mathbb{Z}$?
(3) Let k be a field and let X and Y be arbitrary topological spaces. Show that for all $n \geqslant 0$

$$\bigoplus_{p+q=n} H_p(X; k) \otimes_k H_q(Y; k) \cong H_n(X \times Y; k).$$

Here, $H_p(X; k) \otimes_k H_q(Y; k)$ denotes the tensor product over k of the k-vector spaces $H_p(X; k)$ and $H_q(Y; k)$.

4 (non-natural)

Give an explicit example for the fact that the splitting in the topological Künneth formula is not natural.