Exercises in Algebraic Topology (master)

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Exercise sheet no 5 For the exercise class on the 29th of May

1 (Moore spaces) Let G be an arbitrary finitely generated abelian group.

a) Construct a CW space $M(G,n)$ whose reduced homology is concentrated in degree n with $\tilde{H}_n(M(G,n)) \cong G$. Such a space is called a *Moore space of type* (G,n).

b) How does $M(G,n)$ look like if G is a finite cyclic group?

c) Do you recognize $M(\mathbb{Z}/2\mathbb{Z}, 1)$?

d) Let $A_1, A_2, \ldots$ be a sequence of abelian groups. Construct a space X with $\tilde{H}_i(X) \cong A_i$ for all $i \geqslant 1$.

2 (Euler characteristics of covering spaces) Assume that X is a finite CW complex and that $p\colon \tilde{X} \to X$ is an n-sheeted covering.

(1) Show that $\tilde{X}$ is also a finite CW complex and that $\chi(\tilde{X}) = n\chi(X)$.

(2) What can you say about finite coverings of the form $p\colon \mathbb{S}^{2n} \to X$ with X a finite CW complex. How many sheets can they have?

3 (Non-orientable surfaces) For $g \geqslant 2$ consider a regular $2g$-gon $P_{2g} \subset \mathbb{R}^2$ with vertices $z_1, \ldots, z_{2g}$. We identify edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim tz_{2j+1} + (1-t)z_{2j}$$

(here the indices are to be read mod $2g$) and call the quotient $N_g = P_{2g}/\sim$ the *closed non-orientable surface of genus* g. What is N_2? Calculate the homology of N_{2g} using the cellular chain complex.

4 (Right-exactness) Show that for every short exact sequence

$$0 \to A \to B \to C \to 0$$

of abelian groups and any abelian group D, the sequence

$$A \otimes D \to B \otimes D \to C \otimes D \to 0$$

is exact.

Prove that for a split-exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$0 \to A \otimes D \to B \otimes D \to C \otimes D \to 0$$

is exact.