Exercises in Algebraic Topology (master)

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Exercise sheet no 10 For the exercise class 10th of July 2017

This is the last sheet. You might not have everything at hand on the 3rd of July for doing all of the exercises; try to do as much as possible.

1 (Cup pairing)

a) What are the cup pairings on $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$?

b) What can you say about the symmetry of the cup pairing if the dimension of the manifold is $4n$ or $4n+2$?

2 (Complements in spheres) Consider the m-sphere $\mathbb{S}^m$ for $m \geqslant 2$ and a subset $K \subset \mathbb{S}^m$. Prove the following facts:

a) If $K \cong \mathbb{D}^k$, then $\tilde{H}_k(\mathbb{S}^m \setminus K) \cong 0$ for all $k \geqslant 0$.

b) In particular, for $K \cong \mathbb{D}^k$ the complement $\mathbb{S}^m \setminus K$ is path-connected for all $k \geqslant 0$.

c) If $K \cong \mathbb{S}^k$, then $k \leqslant m$ and

$$\tilde{H}_p(\mathbb{S}^m \setminus K) \cong \tilde{H}_p(\mathbb{S}^m \setminus \mathbb{S}^k) \cong \tilde{H}_p(\mathbb{S}^{m-k-1})$$

and you know these groups.

d) In particular, $\mathbb{S}^m \setminus \mathbb{S}^k$ is pathconnected if and only if $k \neq m-1$. How many pathcomponents does $\mathbb{S}^m \setminus \mathbb{S}^{m-1}$ always have?

3 (Jordan Separation Theorem) Prove the *Jordan Separation Theorem*: If $K \subset \mathbb{S}^m$ ($m \geqslant 2$) with $K \cong \mathbb{S}^{m-1}$, then $\mathbb{S}^m \setminus K$ has two components and both have K as boundary.

4 (Inverse limits)

a) Consider the short exact sequence of inverse systems

$$0 \to \{p^i\mathbb{Z}\} \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0.$$

Determine the inverse limits and the $\lim^1$-terms.

b) Let $\{A_i\}_{i \in \mathbb{N}_0}$ be an inverse system of abelian groups such that the structure maps $A_{i+1} \to A_i$ are monomorphisms. Define a topology on $A = A_0$ by declaring the sets $\{a + A_i\}$ to be open for $a \in A$ and $i \geqslant 0$. (The A_i are viewed as subsets of A via the monomorphisms.) Show that the inverse limit of the A_i is trivial if A is Hausdorff. When does the $\lim^1$-term vanish?

c) Let k be a commutative ring with unit. Show that the inverse limit of the inverse system $\{k[x]/x^n\}_{n \geqslant 1}$ is isomorphic to the formal power series ring $k[[x]]$.