Exercises in Algebraic Topology (master)

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Exercise sheet no 7 For the exercise classes on the 20th of May and 3rd of June 2015

25 (Moore spaces) Let G be an arbitrary finitely generated abelian group.

a) Construct a CW space $M(G,n)$ whose reduced homology is concentrated in degree n with $\tilde{H}_n(M(G,n)) \cong G$. Can you extend the construction to the case where G is not necessarily finitely generated?

Such a space is called a *Moore space of type* (G,n).

b) How does $M(G,n)$ look like if G is a finite cyclic group?

c) Do you recognize $M(\mathbb{Z}/2\mathbb{Z},1)$?

d) Let $A_1, A_2, \ldots$ be a sequence of abelian groups. Construct a space X with $\tilde{H}_i(X) \cong A_i$.

26 (Euler characteristic II)

(1) Let X be a finite CW complex and let $c_n(X)$ denote the number of n-cells of X. Prove that

$$\chi(X) = \sum_{n \geqslant 0} (-1)^n c_n(X).$$

(2) For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(3) For a finite CW complex X and $p\colon \tilde{X} \to X$ an n-sheeted covering space, show that $\chi(\tilde{X}) = n\chi(X)$.

27 (Distorted dice!) Let $W = \{(x,y,z) \,|\, |x|,|y|,|z| \leqslant 1\}$ be the standard 3-cube. We identify opposite faces according to $(x,y,1) \sim (-y,x,-1)$, $(x,1,z) \sim (-z,-1,x)$ and $(1,y,z) \sim (-1,-z,y)$. What are the homology groups of $W/\sim$?

28 (Cellular homology – odds and ends) Let X be a CW complex.

- Can $H_n(X^n)$ have torsion or is $H_n(X^n)$ always free abelian?
- Calculate the homology of $\mathbb{S}^n \times \mathbb{S}^m$ for $n,m \geqslant 2$ using cellular chains.
- If X has k n-cells, then $H_n(X)$ is generated by at most k elements.
- For $g \geqslant 2$ consider a regular $2g$-gon $P_{2g} \subset \mathbb{R}^2$ with vertices $z_1, \ldots, z_{2g}$. We identify edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim tz_{2j+1} + (1-t)z_{2j}$$

  (here the indices are to be read mod $2g$ of course) and call the quotient $N_g = P_{2g}/\sim$ the *closed non-orientable surface of genus* g. What is N_2? Calculate the homology of N_{2g} using the cellular chain complex.