Exercises in Algebraic Topology (master)

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Exercise sheet no 5 For the exercise classes on the 6th and 13th of May 2015

17 (NDR) Let X be a topological space. We call $A \subset X$ a *neighbourhood deformation retract* (NDR), if there is a continuous $u\colon X \to [0,1]$ with $u^{-1}(0) = A$ together with a homotopy $h\colon X \times [0,1] \to X$, such that

(1) $h(-,0) = \mathrm{id}_X$,
(2) $h(a,t) = a$ for all $a \in A, t \in [0,1]$,
(3) $h(x,1) \in A$ for all $x \in X$ with $u(x) < 1$.

If u is strictly smaller than 1 for all x, then A is a deformation retract of X.

Decide whether the following are equivalent, *i.e.*, prove the equivalences or provide counterexamples for those which aren't. Assume that A is closed.

a) $A \subset X$ is an NDR.
b) $X \times \{0\} \cup A \times [0,1] \subset X \times [0,1]$ is a deformation retract.
c) $X \times \{0\} \cup A \times [0,1] \subset X \times [0,1]$ is a retract.
d) $A \subset X$ has the homotopy extension property, *i.e.*, for all $f\colon X \times \{0\} \cup A \times [0,1] \to Y$ there is an $F\colon X \times [0,1] \to Y$ such that $F \circ i = f$. Here i is the inclusion $i\colon X \times \{0\} \cup A \times [0,1] \to X \times [0,1]$.

18 (Brouwer fixed-point theorem) Prove the Brouwer fixed-point theorem: Let X be a closed ball $B_R(x) \subset \mathbb{R}^n$ for $n \geqslant 1$ and let f be a continuous map $f\colon B_R(x) \to B_R(x)$. Prove that f has a fixed point.

Use this to show that every $(a_{ij}) = A \in M(n \times n; \mathbb{R})$ with non-negative a_{ij} must have an eigenvector with non-negative coordinates.

19 (More linear algebra) Let $A \in O(n+1)$. Then multiplication by A induces a continuous self-map on $\mathbb{S}^n$. What is the degree?

20 (A product) Give a CW model of $\mathbb{S}^n \times \mathbb{S}^m$.