Exercises in Algebraic Topology (master)

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Summer term 2015

Exercise sheet no 4 For the exercise classes on the 29th of April and the 6th of May 2015

13 (5-Lemma revisited) Consider the following commutative diagram of exact sequences

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5
\end{tikzcd}$$

Check under which assumptions on f_1, f_2, f_4, f_5 we can deduce that the map f_3 is a monomorphism or an epimorphism.

14 (The $9 = 3 \times 3$-lemma) We consider the following commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(a) Prove that the top row is exact if the two bottom rows are exact.

(b) Similarly, prove that the bottom row is exact if the two top rows are exact.

(c) What happens if the top and bottom rows are exact? Can you deduce that the middle row is exact or do you need an extra condition?

15 (Orientation) Take a closed orientable surface of genus g, F_g, and use excision to prove that $H_2(F_g, F_g \setminus \{x\}) \cong \mathbb{Z}$ for $x \in F_g$.

Do the same with the Möbius strip, M. Pick a generator $\mu_x \in H_2(M, M \setminus \{x\})$. What happens with the generator μ_x if you walk along the meridian of the Möbius strip?

16 (Mapping torus) Let $f, g\colon X \to Y$ be two continuous maps. The *mapping torus of f and g* is the space $T(f,g)$ defined as the quotient of $X \times [0,1] \sqcup Y$ by $(x,0) \sim f(x)$ and $(x,1) \sim g(x)$.

Prove that there is a long exact sequence

$$\ldots \longrightarrow H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{i_*} H_n(T(f,g)) \xrightarrow{\delta} H_{n-1}(X) \xrightarrow{f_* - g_*} \ldots$$

and use that to calculate the homology groups of the Klein bottle.