

# Exercises in Algebraic Topology (master)

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## Exercise sheet no 1

due: 15th of April 2013

As we have no-one who will correct the exercises, 'due' means that you should work on the exercises and should be able to present solutions during the exercise class on the 15th of April.

### 1 (Disks and spheres)

Let  $\mathbb{D}^n$  be the chain complex whose only non-trivial entries are in degrees  $n$  and  $n-1$  with  $\mathbb{D}_n^n = \mathbb{D}_{n-1}^n = \mathbb{Z}$ . Its only non-trivial boundary operator is the identity.

Similarly, let  $\mathbb{S}^n$  be the chain complex whose only non-trivial entry is in degree  $n$  with  $\mathbb{S}_n^n = \mathbb{Z}$ .

a) Assume that  $(C_*d)$  is an arbitrary chain complex. Describe chain maps from  $\mathbb{D}^n$  to  $C_*$  and from  $\mathbb{S}^n$  to  $C_*$  in terms of subobjects of  $C_*$ .

b) Are there chain maps between  $\mathbb{D}^n$  and  $\mathbb{S}^m$ ? What is the homology of  $\mathbb{D}^n$  and  $\mathbb{S}^m$ ?

c) Let  $f_* : C_* \rightarrow C'_*$  be a chain map and assume that  $f_n$  is a monomorphism for all  $n$ . Do we then know that  $H_n(f_*)$  is also a monomorphism?

### 2 (Too much to ask for?)

a) What are the homology groups of the chain complex

$$C_* = (\dots \rightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \rightarrow \dots)?$$

b) Is there a chain homotopy from the identity of  $C_*$  to the zero map, *i.e.*, are there maps  $s_n : C_n \rightarrow C_{n+1}$  with  $d \circ s + s \circ d = \text{id}_{C_n}$  for all  $n \in \mathbb{Z}$ ?

### 3 (Induced maps)

a) Let  $X$  and  $Y$  be topological spaces. Is every chain map  $f_* : S_*(X) \rightarrow S_*(Y)$  induced by a map of topological spaces?

b) Let  $p : \tilde{X} \rightarrow X$  be a covering map. We know that the induced map on fundamental groups is a monomorphism. Is that also true for  $H_1(p)$ ?

### 4 (Lego)

Let  $(A_n)_{n \in \mathbb{Z}}$  be an arbitrary family of finitely generated abelian groups. Is there a chain complex  $F_*$  with  $F_n$  free abelian for all  $n \in \mathbb{Z}$  and with  $H_n(F_*) \cong A_n$ ? (Recall the structure theorem for finitely generated abelian groups for this.)