Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2013

Exercise sheet no 9 due: 14th of June 2013

33 ((co)homology of real projective spaces)

a) Calculate $H^m(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ and $H^m(\mathbb{R}P^n; \mathbb{Z})$ for all m and n.
b) Let $\alpha \in H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ and $a \in H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ be generators. What is $\alpha \cap a$?

34 (suspensions) Why are cup products of elements in positive degree trivial in the cohomology of suspensions?

35 (half-exactness of Hom) Let M be an abelian group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \qquad (\bullet)$$

be a short exact sequence of abelian groups.

a) What can you say about the exactness of the sequences

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{\beta^*} \mathrm{Hom}(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}(A, M) \longrightarrow 0.$$

b) What happens if the sequence $(\bullet)$ splits?

36 (relative variant of the cap-product) Let A and B be subspaces of a topological space X such that the inclusion $S_*^{\mathfrak{U}}(A \cup B) \hookrightarrow S_*(A \cup B)$ induces an isomorphism in homology (with $\mathfrak{U} = \{A, B\}$). Show that there is a variant of the cap-product

$$\cap \colon H^q(X, A) \otimes H_n(X, A \cup B) \to H_{n-q}(X, B).$$