Exercises in Algebraic Topology (master)

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Exercise sheet no 4 due: 3rd of May 2013

13 (5-Lemma revisited) Consider the following commutative diagram of exact sequences

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5
\end{tikzcd}$$

Check under which assumptions on f_1, f_2, f_4, f_5 we can deduce that the map f_3 is a monomorphism or an epimorphism.

14 (The $9 = 3 \times 3$-lemma) We consider the following commutative diagram with exact columns:

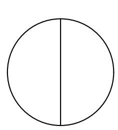
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(a) Prove that the top row is exact if the two bottom rows are exact.

(b) Similarly, prove that the bottom row is exact if the two top rows are exact.

(c) What happens if the top and bottom rows are exact? Can you deduce that the middle row is exact or do you need an extra condition?

15 (MV) Take three copies of $\mathbb{D}^n$ and identify the boundaries of the disks. We call the resulting space X. (For $n = 1$ you get an $\mathbb{S}^1$ with an extra interval glued in.) Use the Mayer-Vietoris sequence in order to calculate the homology of X.



16 (Mapping torus) Let $f, g\colon X \to Y$ be two continuous maps. The *mapping torus of f and g* is the space $T(f, g)$ defined as the quotient of $X \times [0, 1] \sqcup Y$ by $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$.

Prove that there is a long exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{i_*} H_n(T(f, g)) \xrightarrow{\delta} H_{n-1}(X) \xrightarrow{f_* - g_*} \cdots$$

and use that to calculate the homology groups of the Klein bottle.