Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2013

Exercise sheet no 2 due: 19th of April 2013

5 (Snake Lemma)

Prove the famous Snake Lemma:

If

$$\begin{array}{ccccccc} & & A' & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & A'' & \longrightarrow & 0 \\ & & \downarrow{f'} & & \downarrow{f} & & \downarrow{f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\alpha'} & B & \xrightarrow{\beta'} & B'' & & \end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \mathrm{coker}(f') \to \mathrm{coker}(f) \to \mathrm{coker}(f'').$$

Here, δ has to be suitably defined. All other maps are induced by the maps in the diagram.

(If you are lazy: `http://www.youtube.com/watch?v=etbcKWEKnvg`)

6 (Cones)

Let $f\colon A_* \to B_*$ be a chain map. The *mapping cone* of f, $C(f)$, is a chain complex with $C(f)_n = A_{n-1} \oplus B_n$ and whose differential is $D(a,b) = (-da, db - f(a))$.

a) Let A_* be a chain complex and $C = C(\mathrm{id}_{A_*})$. What can you say about the identity map of C?

b) Let $f_*\colon A_* \to B_*$ be a chain map. Develop a criterion for f_* being null-homotopic in terms of C.

7 (Surfaces)

Let F_g denote the closed orientable surface of genus g. Use the Seifert van Kampen theorem to determine the fundamental group of F_g and then apply the Hurewicz theorem to calculate $H_1(F_g)$.

8 (Exactness)

Let C_* be an arbitrary chain complex and let p be a prime. Is it always true that the sequence of chain complexes

$$0 \to C_* \xrightarrow{p\cdot} C_* \to C_*/pC_* \to 0$$

is exact?