Exercises in Algebraic Topology (master)

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Exercise sheet no 13 due: 12th of July 2013

49 (lim-one)

a) Show that $\varprojlim^1 M_i = 0$ if $(M_i)_{i\in\mathbb{N}}$ is an inverse system of finite abelian groups.

b) Assume that

$$M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \dots$$

is an inverse system of abelian groups and assume that each M_i has an underlying countable set. Prove that the system either satisfies the Mittag-Leffler condition or $\varprojlim^1 M_i$ is uncountable.

c) If every M_i is a finitely generated abelian group, then $\varprojlim^1 M_i$ is divisible.

50 (maps between real projective spaces)

a) Prove that for $n > m$ any map $f\colon \mathbb{R}P^n \to \mathbb{R}P^m$ induces the trivial map on π_1.

b) Show that if $g\colon \mathbb{S}^n \to \mathbb{S}^m$ is an odd map, *i.e.*, $g(-x) = -g(x)$ for all x, then $n \leqslant m$.

51 (invariant for lens spaces) We'll investigate three-dimensional lens spaces, *i.e.*, $L(p,q) = L(p;1,q)$ for a prime p.

Let $\alpha \in H^1(L(p,q);\mathbb{Z}/p\mathbb{Z})$ and $\beta = \beta(\alpha) \in H^2(L(p,q);\mathbb{Z}/p\mathbb{Z})$ be generators. Then $\alpha\cup\beta \in H^3(L(p,q);\mathbb{Z}/p\mathbb{Z})$ is a generator as well. We have choices for these generators.

a) Show that a different generator α' gives

$$\langle \alpha' \cup \beta', [L(p,q)]\rangle = n^2 \langle \alpha \cup \beta, [L(p,q)]\rangle,$$

if $\alpha' = n\alpha$ for $n \in (\mathbb{Z}/p\mathbb{Z})^\times$.

b) For $a, b \in \mathbb{Z}/p\mathbb{Z}$ we say that a is equivalent to b, $a \sim b$, if $a \equiv \pm bn^2$ for some n prime to p. The number

$$t_q \in \mathbb{Z}/p\mathbb{Z}, \qquad t_q \sim \langle \alpha \cup \beta, [L(p,q)]\rangle$$

is therefore an invariant of $L(p,q)$.

Show that $t_q \sim q t_1$.

52 ($L(5,2) \not\simeq L(5,1)$)

a) Show that if two lens spaces $L(p,q)$ and $L(p,q')$ are homotopy equivalent, then

$$qq' \equiv \pm n^2 \bmod p.$$

b) Apply this to distinguish the lens spaces $L(5,2)$ and $L(5,1)$.