Exercises in Algebraic Topology (master)

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Exercise sheet no 12 due: 5th of July 2013

45 (cup pairing)

a) What are the cup pairings on $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$?

b) What can you say about the symmetry of the cup pairing if the dimension of the manifold is $4n$ or $4n+2$?

46 (inverse limits)

a) Consider the short exact sequence of inverse systems

$$0 \to \{p^i\mathbb{Z}\} \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0.$$

Determine the inverse limits and the $\lim^1$-terms.

b) Let $\{A_i\}_{i\in\mathbb{N}_0}$ be an inverse system of abelian groups such that the structure maps $A_{i+1} \to A_i$ are monomorphisms. Define a topology on $A = A_0$ by declaring the sets $\{a + A_i\}$ to be open for $a \in A$ and $i \geqslant 0$. (Of course, here the A_i are viewed as subsets of A via the monomorphisms.) Show that the inverse limit of the A_i is trivial if A is hausdorff. When does the $\lim^1$-term vanish?

c) Show that the inverse limit of the inverse system $\{k[x]/x^n\}_{n\geqslant 1}$ is isomorphic to the formal power series ring $k[[x]]$. Here, k is a commutative ring with unit.

47 (complements in spheres) Consider the m-sphere $\mathbb{S}^m$ for $m \geqslant 2$ and a subset $K \subset \mathbb{S}^m$. Prove the following facts:

a) If $K \cong \mathbb{D}^k$, then $\tilde{H}_k(\mathbb{S}^m \backslash K) \cong 0$ for all $k \geqslant 0$.

b) In particular, for $K \cong \mathbb{D}^k$ the complement $\mathbb{S}^m \backslash K$ is path-connected for all $k \geqslant 0$.

c) If $K \cong \mathbb{S}^k$, then $k \leqslant m$ and

$$\tilde{H}_p(\mathbb{S}^m \backslash K) \cong \tilde{H}_p(\mathbb{S}^m \backslash \mathbb{S}^k) \cong \tilde{H}_p(\mathbb{S}^{m-k-1})$$

and you know these groups.

d) In particular, $\mathbb{S}^m \backslash \mathbb{S}^k$ is pathconnected if and only if $k \neq m-1$. How many pathcomponents does $\mathbb{S}^m \backslash \mathbb{S}^{m-1}$ always have?

48 (Jordan Separation Theorem) Use 47 to prove the *Jordan Separation Theorem*: If $K \subset \mathbb{S}^m$ $(m \geqslant 2)$ with $K \cong \mathbb{S}^{m-1}$, then $\mathbb{S}^m \backslash K$ has two components and both have K as boundary.