Exercises in Algebraic Topology (master)

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Exercise sheet no 10 due: 21st of June 2013

37 (orientation covering) Let M be an n-dimensional topological manifold.

a) Prove that there is an oriented manifold $\hat{M}$ and a 2-fold covering $p\colon \hat{M} \to M$ called the orientation covering.

b) Are the following statements equivalent? 1. M is orientable. 2. The orientation covering is a trivial covering, *i.e.*, $\hat{M} \cong M \times \mathbb{Z}/2\mathbb{Z}$ as spaces over M.

c) Assume that M is finite dimensional, path connected with $\pi_1(M) = 1$. Is M orientable?

38 (cut-and-paste) Assume $g \geqslant 2$ and let E_{2g} be a regular $2g$-gon with vertices $z_1, \ldots, z_{2g}$. We glue the edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim (1-t)z_{2j} + tz_{2j+1}.$$

Here, you should interpret the indices mod $2g$. Is the quotient $E_{2g}/\sim$ orientable? What is a suitable definition for $g = 1$?

39 (R-orientations) Let R be a commutative ring with unit and let M be a connected m-dimensional manifold together with an R-orientation. Show that the group of units of R, $R^\times$, acts free and transitively on the set of all R-orientations of M. For $R = \mathbb{Z}$ this should look familiar.

40 (manifolds with boundary) Let

$$\mathbb{R}^m_- := \{(x_1, \ldots, x_m), x_i \in \mathbb{R}, x_1 \leqslant 0\}$$

be an m-dimensional half-space. Its topological boundary is

$$\partial\mathbb{R}^m_- = \{(x_1, \ldots, x_m), x_i \in \mathbb{R}, x_1 = 0\}.$$

An m-dimensional topological manifold with boundary, M with ∂M, is a hausdorff space with a countable basis of its topology together with homeomorphisms $h_i\colon U_i \to V_i$. Here $U_i \subset M$ and $V_i \subset \mathbb{R}^m_-$ are open and the U_i's cover M.

An $x \in M$ is a boundary point of M if there is a homeomorphism $h\colon U \to V$ with U open in M, V open in $\mathbb{R}^m_-$, $x \in U$ and $h(x)$ in $\partial\mathbb{R}^m_-$. The set of boundary points of M is denoted by ∂M.

What is ∂M in the following examples:

a) $\partial(\mathbb{D}^2 \times \mathbb{S}^1)$,

b) $\partial(\mathbb{D}^2 \times \mathbb{D}^2)$,

c) $\partial([0,1])$ and

d) $\partial((\mathbb{S}^1 \times \mathbb{S}^1)\backslash\mathring{\mathbb{D}}^2_\epsilon)$, where $\mathring{\mathbb{D}}^2_\epsilon$ is a an open 2-disk, that is suitably embedded into the torus.

Can you find a general formula for $\partial(M \times N)$ if M and N are manifolds with boundary?