Exercises in Algebraic Topology (master)

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Summer term 2013

Exercise sheet no 1 due: 12th of April 2013

As we have no-one who will correct the exercises, 'due' means that you should work on the exercises and should be able to present solutions during the exercise class on the 12th of April.

1 (Disks and spheres)

Let $\mathbb{D}^n$ be the chain complex whose only non-trivial entries are in degrees n and $n-1$ with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$. Its only non-trivial boundary operator is the identity.

Similarly, let $\mathbb{S}^n$ be the chain complex whose only non-trivial entry is in degree n with $\mathbb{S}^n_n = \mathbb{Z}$.

a) Assume that (C_*d) is an arbitrary chain complex. Describe chain maps from $\mathbb{D}^n$ to C_* and from $\mathbb{S}^n$ to C_* in terms of subobjects of C_*.

b) Are there chain maps between $\mathbb{D}^n$ and $\mathbb{S}^m$? What is the homology of $\mathbb{D}^n$ and $\mathbb{S}^m$?

c) Let $f_*\colon C_* \to C'_*$ be a chain map and assume that f_n is a monomorphism for all n. Do we then know that $H_n(f_*)$ is also a monomorphism?

2 (Too much to ask for?)

a) What are the homology groups of the chain complex

$$C_* = (\ldots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \to \ldots)?$$

b) Is there a chain homotopy from the identity of C_* to the zero map, *i.e.*, are there maps $s_n\colon C_n \to C_{n+1}$ with $d \circ s + s \circ d = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$?

3 (Induced maps)

a) Let X and Y be topological spaces. Is every chain map $f_*\colon S_*(X) \to S_*(Y)$ induced by a map of topological spaces?

b) Let $p\colon \tilde{X} \to X$ be a covering map. We know that the induced map on fundamental groups is a monomorphism. Is that also true for $H_1(p)$?

4 (Lego)

Let $(A_n)_{n\in\mathbb{Z}}$ be an arbitrary family of finitely generated abelian groups. Is there a chain complex F_* with F_n free abelian for all $n \in \mathbb{Z}$ and with $H_n(F_*) \cong A_n$? (Recall the structure theorem for finitely generated abelian groups for this.)