Brauer groups for commutative S-algebras.

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Fact: If R is a field and A is finite dimensional, then A is central simple iff μ_A is an isomorphism.

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Two Azumaya algebras A and B are called Brauer equivalent, if there are faithful, finitely generated projective R-modules P, Q such that

$$A \otimes_R \mathrm{End}_R(P) \cong B \otimes_R \mathrm{End}_R(Q).$$

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We denote by $[A]$ the Brauer equivalence class of A.

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- $[A][A^{\mathrm{op}}] = [A \otimes_R A^{\mathrm{op}}] = [\mathrm{End}_R(A)] = 0$.

Examples

- If k is an algebraically closed field, then $Br(k) \cong 0$.
- $Br(\mathbb{F}_q) \cong 0$, if $\mathbb{F}_q$ is a finite field,
- $Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} = \langle [\mathbb{H}] \rangle$,
- $Br(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Q}/\mathbb{Z}$.

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- $Br(\mathbb{Z}) \cong 0$,
- $Br(\mathbb{Z}[\pi]) \cong 0$ for any finite abelian group π,
- $(R, \mathfrak{m})$ complete and local, then $Br(R) \to Br(R/\mathfrak{m})$ is an isomorphism (Azumaya, Auslander-Goldman), *e.g.*, $Br(\mathbb{Z}_p) \cong 0$.
- $Br(\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}/2\mathbb{Z}$.

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Let A be a cofibrant R-algebra. Then A is a (topological) Azumaya algebra over R if

1. A is a dualizable R-module,
2. $\mu_A \colon A \wedge_R A^{\mathrm{op}} \to F_R(A, A)$ is a weak equivalence, and
3. A is faithful as an R-module.

Base change

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2. Let $A \in \mathscr{A}_R$ and let C be a commutative R-algebra, such that C is dualizable and faithful as an R-module. If $A \wedge_R C$ is Azumaya over C, then A is Azumaya over R.

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3. If A and B are Azumaya over R, then so is $A \wedge_R B$.
4. For any cofibrant faithful dualizable R-module M, $F_R(M, M)$ is Azumaya over R.

Brauer groups

Let A and B be Azumaya algebras over R. Then A is Brauer equivalent to B, if there are cofibrant faithful dualizable R-module spectra M, N such that

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It is a priori *not* clear, that any Azumaya algebra A which is Brauer equivalent to R is of the form $F_R(M, M)$ for suitable R-module spectra M.

Separability and centrality

Proposition Let A be an Azumaya algebra over R.

- A is separable, *i.e.*, the multiplication $m\colon A \wedge_R A^{\mathrm{op}} \to A$ has a section in the derived category of $A \wedge_R A^{\mathrm{op}}$-module spectra.
- A is homotopically central, *i.e.*, the canonical map $R \to THH_R(A)$ is a weak equivalence.

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Here, $THH_R(A)$, the topological Hochschild cohomology spectrum, is $F_{A \wedge_R A^{\mathrm{op}}}(\tilde{A}, A)$ where $\tilde{A}$ is a cofibrant replacement of A in the model category of $A \wedge_R A^{\mathrm{op}}$-module spectra.

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In algebra, $\mathrm{Hom}_{A \otimes_R A^{\mathrm{op}}}(A, A)$ is the center of A.

Galois extensions of commutative S-algebras

Let G be a finite group and B a commutative cofibrant R-algebra, such that G acts on B via commutative R-algebra maps.
Definition[Rognes] The extension $R \to B$ is G-Galois if

- $R \sim B^{hG} = F_G(EG_+, B)$ (htp fixed points)
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$HR \to HT$ is a G-Galois extension iff $R \to T$ is a G-Galois extension of commutative rings (Rognes).
For a Galois extension $R \to B$ we can define the relative Brauer group, $Br(B/R)$, as

$$Br(B/R) = \ker(Br(R) \to Br(B))$$

where the map is given by extension of scalars.

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From now on, we assume that B is faithful as an R-module.

Proposition Let $R \to B$ be a G-Galois extension, $M \in B\langle G \rangle$-mod of the form $M = B \wedge_R N$ such that $N \in R$-mod and such that the $B\langle G \rangle$-action on M is only given on the B-factor. Then $N \simeq M^{hG}$.

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Theorem (Galois descent) If C is an Azumaya algebra over B and an $B\langle G\rangle$-module with $R \to B$ G-Galois and if $B \wedge_R C^{hG} \to C$ is a weak equivalence of $B\langle G\rangle$-modules, then C^{hG} is Azumaya over R.

Setting

Let $R \to B$ be a C_n-Galois extension and let $C_n = \langle \sigma \rangle$. In addition we assume that we have a $\mathbb{Z}$-action on R via R-module maps and $\mathbb{Z} = \langle u \rangle$.

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Let $M_n(B) = \bigvee_{i,j=1}^{n} B_{i,j}$ be the B-algebra of $n \times n$-matrices with $B_{i,j} = B$ for all i, j. Then $M_n(B)$ has a $G = C_n$-action which is given on the (i,j)-summand via

$$B_{i,j} \xrightarrow{\mathrm{id}} B_{i+1,j+1} \xrightarrow{u^{\delta_{i,n}-\delta_{j,n}}} B_{i+1,j+1} \xrightarrow{\sigma} B_{i+1,j+1}.$$

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We define the cyclic algebra with respect to B, σ and u as

$$A(B, \sigma, u) := M_n(B)^{hG}.$$

Proposition The R-algebra $A(B, \sigma, u)$ is Azumaya over R and is split by B, *i.e.*,

$$A(B, \sigma, u) \in Br(B/R).$$

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$$\pi_*(E_n) = W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}].$$

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Similarly, E_n^{nr} is a commutative S-algebra with

$$\pi_*(E_n^{nr}) = W\bar{\mathbb{F}}_p[[u_1, \ldots, u_{n-1}]][u^{\pm 1}].$$

Azumaya algebras via homotopy centers

Theorem Let p be a prime that does not divide the order of the non-abelian group G ($G \neq 1$) and let R be an Eilenberg-MacLane spectrum Hk for an algebraically closed field k of characteristic p or let R be E_n^{nr}. Then

$$\mathrm{THH}_R(R[G]) \to R[G]$$

is an Azumaya extension.

Idea of proof

Algebraic case: Artin-Wedderburn gives $k[G] \cong \prod_{i=1}^{r} M_{m_i}(k)$, *i.e.*, the center is étale and hence realizable via Robinson and Goerss-Hopkins obstruction theory as a commutative Hk-algebra C.

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Lubin-Tate case: Reduction modulo the maximal ideal in $\pi_0 E_n^{nr}$ gives a splitting similar to the one above. We can lift the corresponding idempotents and get

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These homotopy fixed points are equivalent to $\mathrm{THH}_{E_n^{nr}}(E_n^{nr}[G], E_n^{nr}[G])$ and $E_n^{nr}[G]$ can actually be realized as an associative $\prod_{i=1}^{r} E_n^{nr}$-algebra, such that the extension is Azumaya.

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This yields a group homomorphism $Br(R) \to Br(HR)$. For the converse, we need (so far) that R satisfies the following: If M is a finitely presented R-module and $\mathrm{Tor}_k^R(M, M) = 0$ for $k > 0$, then M is flat.
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For such R we have, that if HT is Azumaya over HR, then so is T over R. Using results of Toën we can show that $Br(Hk) = 0$ if k is an algebraically closed field.

Importing examples from algebra

Assume that R is a commutative S-algebra and $\pi_0(R) \to A_0$ is an algebraic Azumaya algebra.
Then using Angeltveit's obstruction theory, we can realize $A_* = \pi_*(R) \otimes_{\pi_0(R)} A_0$ as an associative R-algebra spectrum.

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In that way one can realize for instance algebraic quaternionic extensions or cyclic extensions in brave new rings.

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- What is $Br(S)$?