Seminar (Master): Homotopy theory and Hopf algebras **Winter term 2017/18**

Prof. Dr. B. Richter

This seminar is based on the content of the lecture courses *Topology (bachelor) and Topology (master)*. The first talks only need basic knowledge about elementary homotopy theory and fundamental groups, but the later talks also need background in singular homology and cohomology.

Talks

(1) **Higher homotopy groups I** Define the groups $\pi_n(X, x_0)$. Why are the groups $\pi_n(X, x_0)$ abelian for $n \geq 2$? Show that every covering $p\colon \tilde{X} \to X$ induces an isomorphism on higher homotopy groups. Define the action of the fundamental group on the higher homotopy groups. [H, §4.1 including Proposition 4.2]

(2) **Higher homotopy groups II** Define the relative homotopy groups, $\pi_n(X, A, x_0)$ for $x_0 \in A \subset X$. Prove that there is a long exact sequence

$$\ldots \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \to \ldots$$

for every pair of spaces (X, A) and every $x_0 \in A$. [H, pp.343–346]

(3) **Fibrations** You already know the definition of a fibration. Show that one can pull back a fibration and tell us about the difference between Serre fibrations and Hurewicz fibrations. Prove that the homotopy lifting property (HLP) for disks $\mathbb{D}^n$ is equivalent to the HLP for $(\mathbb{D}^n, \partial\mathbb{D}^n)$. Define the based loop space, ΩX, of a based space X.

What is the path-loop fibration? What can you say about the homotopy groups of a loop space? Recall the exponential law for mapping spaces and apply this to $C((\Sigma X, x_0), (Y, y_0)) = C((\mathbb{S}^1 \wedge X, x_0), (Y, y_0))$. [H, pp. 375,395,], [M, Chapter 7 §1, §2]

(4) **Long exact sequence for a fibration** Show that every fibration $p\colon E \to B$ (for a path-connected B) gives rise to a long exact sequence on homotopy groups:

$$\ldots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots$$

What does this give in the examples of the Hopf fibrations? [H, 4.41]

(5) **Associated fibration** You can turn every continuous map into a fibration. Tell us about this construction and show that it doesn't change much if you already start with a fibration. [H, 4.64–4.66 plus some background and examples], [M, Chapter 7, §3]

(6) **Whitehead's theorem** This important theorem says that continuous maps between connected CW complexes that induces isomorphisms on all homotopy groups are actually homotopy equivalences. Prove that! [H, pp. 346–348]

(7) **H-spaces and the James construction** Define the James construction [H, section 3.2] and describe the divided power structure on $H^*(J(\mathbb{S}^{2n}))$. Define what an H-space is and present lots of examples ($\mathbb{S}^i$ for $i = 0, 1, 3, 7$, matrix groups, $\mathbb{C}P^\infty$, $J(X)$, $SP(X)$) [H, 3.C]. Tell us what a Hopf algebra is and describe this structure for $H^*(X; R)$.

(8) **Examples and structure theorem for Hopf algebras** What is a primitive element in a Hopf algebra? What are typical examples of Hopf algebras? When is $F[\alpha]/\alpha^n$ a Hopf algebra? Use the latter example to show that the finite $\mathbb{C}P^n$'s cannot be H-spaces. State and prove the structure theorem for Hopf algebras [H, pp. 284–287].

(9) **Pontryagin products** The homology groups of H-spaces carry an extra structure, the Pontryagin product. Give examples of these products and identify the homology of the James construction in good cases. What is the dual Hopf algebra of a Hopf algebra? [H, 287–291]

(10) **Stable splittings I** Show that for two CW complexes the suspension of their product splits and use this to split the suspension of the James construction on a CW complex. [H, pp. 466–468 and background]

(11) **Stable splittings II** One can show that there is a weak homotopy equivalence $J(X) \to \Omega\Sigma X$ for every connected CW complex X. Give us an overview over the proof and the key steps. In particular, this identifies $H_*(\Omega\Sigma X; R)$ and $H^*(\Omega\Sigma X; R)$ in good cases. [H, 4.J]

References

[H] Allen Hatcher, *Algebraic topology*, Cambridge University Press (2002), xii, 544 p.
Erhältlich auf `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

[M] J. Peter May, *A concise course in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL (1999), x+243 pp.