Homotopical algebra and homotopy colimits

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New interactions between homotopical algebra and quantum field theory
Oberwolfach, 19th of December 2016

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Flexible framework, can be used for chain complexes, topological spaces, algebras over operads, and many more – allows us to do homotopy theory.

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The objects are families of R-modules $C_n, n \geq 0$, together with R-linear maps, the differentials, $d = d_n \colon C_n \to C_{n-1}$ for all $n \geq 1$ such that $d_{n-1} \circ d_n = 0$ for all n.

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Morphisms are chain maps $f_* \colon C_* \to D_*$. These are families of R-linear maps $f_n \colon C_n \to D_n$ such that $d_n \circ f_n = f_{n-1} \circ d_n$ for all n.

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The nth homology group of a chain complex C_* is

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Here, we use the convention that $Z_0 C_* = C_0$. Chain maps f_* induce well-defined maps on homology groups $H_n(f)$:
$H_n(f) \colon H_n(C_*) \to H_n(D_*)$, $H_n(f)[c] := [f_n(c)]$.

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Model categories give such a construction.

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- the weak equivalences, (*we*)
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We indicate weak equivalences by $\xrightarrow{\sim}$, cofibrations by $\rightarrowtail$ and fibrations by $\twoheadrightarrow$.
These classes of maps have to satisfy a lot of compatibility conditions...

Model categories, II

M1 The category $\mathcal{C}$ has all limits and colimits.

M2 (2-out-of-3): If f, g are morphisms in $\mathcal{C}$ such that $g \circ f$ is defined, then if two of the maps f, g, $g \circ f$ are weak equivalences, then so is the third.

M3 If f is a retract of g and g is in *we*, *cof* or *fib*, then so is f.

M4 For every commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r, "\beta"] & Y \end{tikzcd}$$

in $\mathcal{C}$ where i is a cofibration and q is an acyclic fibration or where i is an acyclic cofibration and q is a fibration, a lift ξ exists with $q \circ \xi = \beta$ and $\xi \circ i = \alpha$.

M5 Every morphism f in $\mathcal{C}$ can be factored as $f = p \circ j$ and $q \circ i$, where j is an acyclic cofibration and p is a fibration, q is an acyclic fibration and i is a cofibration.

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$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "Id_U"] & X \arrow[r] \arrow[d, "g"] & U \arrow[d, "f"] \\
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M4: The lift ξ in $\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r, "\beta"] \arrow[ur, dotted, "\xi"] & Y \end{tikzcd}$ is not required to be unique!

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M5: Can be used for constructing projective/injective resolutions, CW-approximations etc.

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The category Ch_R has several model category structures. The one we will use is: A chain map $f \colon C_* \to D_*$ is a

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What are projective modules?

Projective modules

Let R be a ring. A left R-module P is projective if for every epimorphism $\pi\colon M \to Q$ of R-modules and every morphism $f\colon P \to Q$ of R-modules there is an R-linear morphism $\xi\colon P \to M$ that lifts f to M:

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If R is a field, then every module is projective.

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Here $d\colon (\mathbb{D}^n)_n = R \to R = (\mathbb{D}^n)_{n-1}$ is the identity map.
Exercise: Calculate the homology groups of spheres and disks. Show that every chain map from $\mathbb{S}^n$ to a chain complex C_* picks out an n-cycle $c \in Z_n(C_*)$ and that every chain map from $\mathbb{D}^n$ to a chain complex C_* picks out an element $x \in C_n$. Therefore there is a canonical map $i_n\colon \mathbb{S}^{n-1} \to \mathbb{D}^n$.

Lemma

1) A morphism in Ch_R is a fibration if and only if it has the lifting property with respect to all maps $0 \to \mathbb{D}^n$ with $n \geq 1$.
2) A morphism in Ch_R is an acyclic fibration if and only if it has the lifting property with respect to all maps $i_n \colon \mathbb{S}^{n-1} \to \mathbb{D}^n$ for $n \geq 0$.

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Towards the homotopy category

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\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[dl, "H_{n+1}"'] \arrow[d, bend right, "f_{n+1}"'] \arrow[d, bend left, "g_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[dl, "H_n"'] \arrow[d, bend right, "f_n"'] \arrow[d, bend left, "g_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[dl, "H_{n-1}"'] \arrow[d, bend right, "f_{n-1}"'] \arrow[d, bend left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

If f_* is chain homotopic to g_*, then $H_* f = H_* g$.

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The cylinder on C_* is the chain complex $cyl(C)_*$ with $cyl(C)_n = C_n \oplus C_{n-1} \oplus C_n$ and with $d \colon cyl(C)_n \to cyl(C)_{n-1}$ given by the matrix

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The "top" and the "bottom" of the cylinder embed as

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There is also a map $q\colon cyl(C)_* \to C_*$ sending (c_1, c_2, c_3) to $c_1 + c_3$. These maps are chain maps.
Exercise: Two chain maps $f_*, g_*\colon C_* \to D_*$ are chain homotopic if and only if they extend to a chain map

$$f_* + H_* + g_*\colon cyl(C)_* \to D_*.$$

Cylinder objects in a model category

Let C be an object in a model category $\mathcal{C}$. We call an object cyl_C a cylinder object for C, if there are morphisms

$$C \sqcup C \xrightarrow{i} cyl_C \xrightarrow[\sim]{q} C$$

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$cyl(C)_*$ as above is a cylinder object: we can take $i(c_1, c_2) = (c_1, 0, c_2)$ and $q\colon cyl(C)_* \to C_*$ as above.

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Good and very good cylinder objects exist thanks to M5.
The map $i\colon C \sqcup C \to cyl_C$ has components $i_0\colon C \to cyl_C$ and $i_1\colon C \to cyl_C$ given by the two maps $C \to C \sqcup C$.

Left homotopies

Two morphisms in a model category $f, g\colon C \to D$ are called left homotopic, if there is a cylinder object cyl_C of C and a morphism $H\colon cyl_C \to D$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

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We need to restrict to nice objects!

Digression: initial and terminal objects

Every chain complex C_* receives a unique chain map f from the trivial chain complex 0 with $0_n = 0$ for all $n \geq 0$, the trivial abelian group,

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So, 0 is initial and terminal in the category Ch_R and $\varnothing$ is inital in Top whereas any one-point space is terminal in Top.

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For every object X in a model category, we can factor the unique map $i \to X$ as

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with $f \in cof$ and $q \in fib \cap we$. We call this a cofibrant replacement of X. (This can be made functorial in X.)

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In Ch_R this gives projective resolutions of any R-module M viewed as $\mathbb{S}^0(M)$.

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For a cofibrant object QX we can factor the unique map $QX \to t$ as

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Definition: The homotopy category, $Ho(\mathcal{C})$, of a model category $\mathcal{C}$ has as objects the objects of $\mathcal{C}$ and $Ho(\mathcal{C})(X, Y)$ is the set of (left) homotopy classes of maps from RQX to RQY.

This is the right thing: There is a functor $\gamma\colon \mathcal{C} \to Ho(\mathcal{C})$ with $\gamma(X) = X$ and $\gamma(f\colon X \to Y) = [RQf\colon RQX \to RQY]$.

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So $Ho(\mathcal{C})$ is a model for $\mathcal{C}[we^{-1}]$!

What are diagrams?

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- $\mathcal{D} = (2 \leftarrow 0 \rightarrow 1)$ and $\mathcal{C} = Ch_R$ gives a diagram $F(2) \leftarrow F(0) \rightarrow F(1)$ of chain complexes and chain maps.

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- For $\mathcal{D} = (0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots)$ and $\mathcal{C} = Top$ we get a sequence $F(0) \rightarrow F(1) \rightarrow F(2) \rightarrow \ldots$ of topological spaces and continuous maps.
- If S is any set, then we can consider it as a category whose only morphisms are identity maps. A functor $F\colon S \to \mathcal{C}$ for any $\mathcal{C}$ is just an S-indexed family of objects.

What are colimits?

Let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor as above. Then a colimit of F is an object $\mathrm{colim}_{\mathcal{D}}F$ of $\mathcal{C}$ that is "as close to the diagram that F defines as it can be".

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Definition: A colimit of F over $\mathcal{D}$ is an object $\mathrm{colim}_D F$ of $\mathcal{C}$ together with morphisms $\tau_D\colon F(D) \to \mathrm{colim}_D F$ in $\mathcal{C}$ such that for all $f \in \mathcal{D}(D_1, D_2)$

$$\begin{tikzcd}
F(D_1) \arrow[r, "\tau_{D_1}"] \arrow[d] & \mathrm{colim}_D F \\
F(D_2) \arrow[ur, "\tau_{D_2}"'] &
\end{tikzcd}$$

commutes. Furthermore, if C is any other object of $\mathcal{C}$ with morphisms $\eta_D\colon F(D) \to C$ such that

$$\eta_{D_2} \circ F(f) = \eta_{D_1} \quad \forall\, f \in \mathcal{D}(D_1, D_2)$$

then there is a unique morphism $\xi\colon \mathrm{colim}_{\mathcal{D}} F \to C$ with $\xi \circ \tau_D = \eta_D$ for all objects D of $\mathcal{D}$.

Examples of colimits

- Colimits for $\mathcal{D} = (2 \leftarrow 0 \rightarrow 1)$ are called pushouts. In Ch_R the pushout of $F(2) \leftarrow F(0) \rightarrow F(1)$ is the chain complex

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For $\tau_0 \colon F(0) \rightarrow (F(2) \oplus F(1))/\sim$ you take the map from $F(0)$ to $(F(2) \oplus F(1))/\sim$ in the diagram (they are both the same).

Examples of colimits – continued

- For a diagram of the form $F(0) \to F(1) \to F(2) \to \dots$ in *Top* the colimit is given by $\coprod_{n \geq 0} F(n)/\sim$ where $\sim$ identifies $x \in F(m)$ with the image of x in $F(n)$ under the maps in the sequence for $m \leq n$. Such colimits are called sequential colimits.

Examples of colimits – continued

- For a diagram of the form $F(0) \to F(1) \to F(2) \to \dots$ in *Top* the colimit is given by $\bigsqcup_{n \geq 0} F(n)/\sim$ where $\sim$ identifies $x \in F(m)$ with the image of x in $F(n)$ under the maps in the sequence for $m \leq n$. Such colimits are called sequential colimits.
- A colimit over a diagram indexed on a set S viewed as a category is the coproduct of the objects $F(s)$, $s \in S$ and is denoted by $\bigsqcup_S F(s)$. For sets or topological spaces you get the disjoint union of the $F(s)$, for chain complexes you get $\bigoplus_S F(s)$.

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Take the pushout of

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Here $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}, |x| = 1\}$ is the unit sphere in $\mathbb{R}^{n+1}$.

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An explicit formula for the pushout is $* \sqcup */ \sim$ where the two points are glued together, so

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

is a pushout diagram.

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Thus replacing $*$ by the homotopy equivalent $\mathbb{D}^{n+1}$ changed the homotopy type of the pushout.
That's bad, if you want to work up to homotopy...

What should a homotopy colimit do for us?

In a model category all colimits exist by assumption. We can actually view the colimit as a functor

$$\mathrm{colim}_{\mathcal{D}} \colon \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$$

where $\mathcal{C}^{\mathcal{D}}$ denotes the category of functors from $\mathcal{D}$ to $\mathcal{C}$. It is left adjoint to the constant functor

$$\Delta \colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}, \quad \Delta(C)(D) = C \quad \forall\, D$$

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...at least, if $\mathcal{C}^{\mathcal{D}}$ possesses a model category structure and thus a homotopy category, $Ho(\mathcal{C}^{\mathcal{D}})$. (Warning: $Ho(\mathcal{C}^{\mathcal{D}}) \neq Ho(\mathcal{C})^{\mathcal{D}}$!)

Model category definition of hocolims

Assume that $\mathcal{C}^{\mathcal{D}}$ possesses a model category structure. Then if the colimit functor $\mathrm{colim}_{\mathcal{D}}$ preserves cofibrations and if the functor Δ preserves fibrations, then there is an adjoint pair of functors

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Recipe for $\mathrm{hocolim}_{\mathcal{D}} F$:

1. Take your diagram F and its cofibrant replacement $i \rightarrowtail Q(F) \overset{\sim}{\twoheadrightarrow} F$ in $\mathcal{C}^{\mathcal{D}}$.
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The cofibrant replacement of a diagram in $\mathcal{C}^{\mathcal{D}}$ is *not* just given by the cofibrant replacement of each $F(D)$, but is way more involved.
How do we get explicit models?

Bousfield-Kan, Hirschhorn, Rodríguez-González

- 1972: Bousfield and Kan constructed models for homotopy colimits for diagrams in simplicial sets; those are combinatorial models of topological spaces.
- People observed that the Bousfield-Kan construction transfers to many other settings “with a simplicial structure” (see Hirschhorn’s book [H]).
- Rodríguez-González [RG] gave a systematic account on the question, when there is a Bousfield-Kan model of a homotopy colimit.

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The homotopy colimit of the diagram can be expressed as

$$(X_1 \sqcup X_0 \times [0, 1] \sqcup X_2)/\sim$$

where you glue points $(x_0, 0) \in X_0 \times [0, 1]$ to $g(x_0)$ and $(x_0, 1)$ to $f(x_0)$.

Examples of homotopy colimits, II

For a sequential diagram of topological spaces $X_0 \to X_1 \to X_2 \to \dots$ the telescope is an explicit model of $\mathrm{hocolim}_{\mathbb{N}_0} X$:

1. Replace every X_n by the cylinder $X_n \times [n, n+1]$.
2. Glue the points $(x_n, n+1) \in X_n \times [n, n+1]$ to the points $(f_n(x_n), n+1) \in X_{n+1} \times [n+1, n+2]$.
3. This gives a telescope

$$\left(\bigsqcup_{n \geq 0} X_n \times [n, n+1] \right) / \sim .$$

Example: hocolim in non-negative chain complexes

Let $\mathcal{D}$ be any small category and let $F\colon \mathcal{D} \to Ch_R$ be any functor.
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1. We consider morphisms in the category $\mathcal{D}$. Let $N(\mathcal{D})_n$ be the set of morphisms $D_0 \xrightarrow{f_1} D_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} D_n$. Here, by convention $N(\mathcal{D})_0$ is the set of objects of $\mathcal{D}$.

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2. If we denote an element of $N(\mathcal{D})_n$ as above as $\underline{f} = (f_n, \dots, f_1)$, then we can define

$$d_i(f_n, \dots, f_1) := \begin{cases} (f_n, \dots, f_2), & i = 0, \\ (f_n, \dots, f_{i+2}, f_{i+1} \circ f_i, f_{i-1}, \dots, f_1), & 0 < i < n, \\ (f_{n-1}, \dots, f_1), & i = n. \end{cases}$$

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3. Thus d_i erases the object D_i, so in d_0 f_1 is omitted because its source is gone, in d_n f_n is omitted because it lost its target, and all the inner d_i force a composition because the intermediate object disappeared.
4. We call D_0 the source of $\underline{f} = (f_n, \ldots, f_1)$ and denote it by $s\underline{f}$.

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by using the alternating sum $\sum_{i=0}^{n}(-1)^i d_i$ of the d_i's above.

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by using the alternating sum $\sum_{i=0}^{n} (-1)^i d_i$ of the d_i's above.
The resulting double complex looks as follows:

$$\begin{tikzcd}
\cdots \arrow[d, "\delta"'] & \cdots \arrow[d, "-\delta"'] & \cdots \\
\bigoplus_{(f_1)\in N(\mathcal{D})_1} F(s(f_1))_0 \arrow[d, "\delta"'] & \bigoplus_{(f_1)\in N(\mathcal{D})_1} F(s(f_1))_1 \arrow[l, "d"'] \arrow[d, "-\delta"'] & \cdots \arrow[l, "d"'] \\
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The associated total complex is a model for the homotopy colimit.

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The associated total complex is a model for the homotopy colimit. This is rather involved, but explicit and useful for constructions.

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