String Topology and the Based Loop Space

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Structured Ring Spectra: TNG
University of Hamburg

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Make $H_{*+d}(LM)$ a Batalin-Vilkovisky (BV) algebra:

- $\circ$ and Δ combine to produce a degree-1 Lie bracket $\{\,,\,\}$ on $H_{*+d}(LM)$ (the *loop bracket*)

Hochschild Homology and Cohomology

The Hochschild homology and cohomology of an algebra A exhibit similar operations:

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Why $C_*\Omega M$? Goodwillie, '85: $H_*(LM) \cong HH_*(C_*\Omega M)$, M connected

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Theorem (M.)

Let M be a connected, k-oriented Poincaré duality space of formal dimension d. Then Poincaré duality induces an isomorphism

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- Cap product with $[M]$ still induces an isomorphism

$$H^*(M; E) \to H_{*+d}(M; E).$$

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Compatibility of Hochschild operations under D:

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Generalizes results of Abbaspour-Cohen-Gruher ('05) and Vaintrob ('06) when $M \simeq K(G,1)$, so $C_*\Omega M \simeq kG$.

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- View $[M] \in H_d M$ as a class in $\mathrm{Tor}_d^{C_*\Omega M}(k,k)$. PD says

$$\mathrm{ev}_{[M]} : R\,\mathrm{Hom}_{C_*\Omega M}(k,E) \to E \otimes^L_{C_*\Omega M} \Sigma^{-d} k$$

a weak equivalence for E a $k[\pi_1 M]$-module

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- Algebraic Postnikov tower, compactness of k as a $C_*\Omega M$-module show a weak equivalence for *all* $C_*\Omega M$-modules E.

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Essentially homological; twist allows umkehr map $f^!$ for $f : N \hookrightarrow M$

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- For $f : N \to M$, pullback f^*

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When $\mathcal{E} = S_M$, recovers classical Atiyah duality $M^{-TM} \simeq F(M_+, S)$

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- $(\Sigma^\infty_+ \Omega M^c)^{h\Omega M}$ a ring spectrum via convolution product

Topological Hochschild Constructions

Topological Hochschild Cohomology

$(\Sigma^\infty_+ G^c)^{hG}$ and $THH_S(\Sigma^\infty_+ G)$ both Tots of cosimplicial spectra

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Recover chain-level results by applying $- \wedge Hk$, Thom isomorphism

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- Recover Hochschild Lie bracket $[\,,\,]$ as "free" BV Lie bracket

D and Goodwillie isom take $\cup$ to loop product and $-D^{-1}BD$ to Δ

Thanks for your attention!

Slides online soon at
`http://www.ericmalm.net/work/`