CROSSED MODULES OVER OPERADS AND OPERADIC COHOMOLOGY

HANS-JOACHIM BAUES, ELIAS GABRIEL MINIAN, AND BIRGIT RICHTER

Abstract. It is known that elements in the cohomology of groups and in the Hochschild cohomology of algebras are represented by crossed extensions. We introduce the notion of crossed modules and crossed extensions for algebras over operads and obtain in this way an operadic version of Hochschild cohomology. Applications are given for the operads Com, Ass and for E_∞ operads.

1. Introduction

An interpolation between the category of graded modules and the category of chain complexes is the category $\mathcal{P}air(\mathcal{M}od)$ of pairs of graded modules. An object in this category consists of two graded modules A and B and a map $\partial : A \longrightarrow B$ of degree zero between them.

In fact, consider the homology functor H_* from the category of chain complexes, $\mathcal{C}hain$, to the category of graded modules, $\mathcal{M}od$. One has the secondary homology functor D as in the diagram

$$\begin{array}{ccc} \mathcal{C}hain & \xrightarrow{\quad H_* \quad} & \mathcal{M}od \\ & {\scriptstyle D}\searrow \quad \nearrow{\scriptstyle \pi_0} & \\ & \mathcal{P}air(\mathcal{M}od) & \end{array}$$

which is a factorization of H_* carrying a chain complex (C_*, d) to the pair

$$D(C_*) = (\partial : s^{-1}(\mathrm{coker}(d)) \longrightarrow \ker(d)).$$

Here ∂ is induced by the differential d and s^{-1} denotes the suspension. Over a field the functor H_* is compatible with the monoidal structure given by the tensor product. We will equip the category of pairs of modules with a canonical symmetric monoidal structure, such that all functors in the diagram preserve the monoidal structure up to a natural transformation. In particular, every operad $\mathcal{O}$ in the category $\mathcal{C}hain$ yields an operad $D(\mathcal{O})$ in the category $\mathcal{P}air(\mathcal{M}od)$.

In this paper we investigate the basic facts concerning secondary operads. These are operads in the symmetric monoidal category $\mathcal{P}air(\mathcal{M}od)$; algebras over such operads are called crossed modules and prolongations of crossed modules are crossed extensions.

For any secondary operad $\mathcal{D} = (\partial : \mathcal{D}_1 \to \mathcal{D}_0)$ we define operadic cohomology $H^*_{\mathcal{D}}$ associated to $\mathcal{D}$ as the set of equivalence classes of crossed extensions. Weak equivalence classes of crossed modules over $\mathcal{D}$ yield the elements in the third operadic cohomology $H^3_{\mathcal{D}}$. This corresponds to the classical fact that crossed modules in the category of groups

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represent elements in the third cohomology of groups (see [Br, IV.5] or [MLW, ML3]), and Huebschmann's result [Hue] ensures that in fact every cohomology class in group cohomology can be gained as an equivalence class of a crossed extension of groups.

In general, if $\mathcal{O}$ is an operad of graded modules, then operadic cohomology $H^*_{(0\to\mathcal{O})}(A, M)$ vanishes whenever A is a free $\mathcal{O}$-algebra. So for operads in graded modules we obtain a cohomology theory which behaves similar to the one, which arises from the homology theory for quadratic operads defined in [GK].

If Ass is the operad for associative algebras and if we work relative to a field, then $H^*_{0\to\mathsf{Ass}}$ is Hochschild cohomology. This case is studied in [BM].

Let $\mathcal{E}(\infty)^{\mathsf{Ass}}$ be an E_∞ operad with a map from Ass. Then the cohomology theory $H^*_{(D(\mathcal{E}(\infty)^{\mathsf{Ass}}))}$ is a cohomology theory for commutative algebras, such as André-Quillen cohomology and Gamma cohomology defined by A. Robinson and S. Whitehouse [RW].

For example, for each space X, the cochains $C^*(X)$ are an algebra over Mandell's E_∞ operad [M]. Mandell proved that the E_∞ structure on cochains suffices to determine the homotopy type of certain spaces. In order to get insight into the structure of Mandell's operad $\mathcal{M}(\infty)$ we propose to consider the secondary operad $D(\mathcal{M}(\infty))$ and algebras over it. The action of the secondary operad $D(\mathcal{M}(\infty))$ on $D(C^*(X))$ gives rise to a crossed module which is an intermediate step between the bare cohomology-algebra of a space and its homotopy type.

2. The Secondary Homology of a Chain Complex

Let R be a commutative ring with unit. We denote by $\mathcal{M}od$ the category of $\mathbb{Z}$-graded R-modules and by $\mathcal{C}hain$ the category of chain complexes over R.

2.1. *Remark.* Note that $\mathcal{M}od$ and $\mathcal{C}hain$ are symmetric monoidal categories under the graded tensor product $\otimes$ over R. In both cases the unit object is the ring R concentrated in degree 0. These categories are symmetric monoidal categories enriched over the category of R-modules. We have the interchange map $T : X \otimes Y \to Y \otimes X$ in $\mathcal{M}od$ which carries $x \otimes y$ to the element $(-1)^{|x||y|} y \otimes x$ where $|x| = n$ if $x \in X_n$. Such an interchange sign has to be used whenever graded elements are interchanged. For example for a differential d of degree -1 we have $(\mathrm{id} \otimes d)(x \otimes y) = (-1)^{|x|} x \otimes dy$. Hence the differential in a tensor product $X \otimes Y$ of chain complexes is given by $d_\otimes = \mathrm{id} \otimes d + d \otimes \mathrm{id}$.

2.2. **Definition.** For an arbitrary category $\mathcal{M}$, *the category of pairs in* $\mathcal{M}$, $\mathcal{P}air(\mathcal{M})$, is defined as follows. Objects are maps $f : A \to B$ in $\mathcal{M}$ and a morphism $(\alpha, \beta) : f \to g$ is a commutative diagram in $\mathcal{M}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{g} & B' \end{array}$$

We will define a $\bar{\otimes}$-product in the category of pairs of graded R-modules and show that $(\mathcal{P}air(\mathcal{M}od), \bar{\otimes})$ is a symmetric monoidal category.

2.3. **Definition.** Given $V = (\partial : V_1 \to V_0)$ and $W = (\partial' : W_1 \to W_0)$ in $\mathcal{P}air(\mathcal{M}od)$ consider the following diagram of graded modules.

$$V_1 \otimes W_1 \xrightarrow{d_2} V_1 \otimes W_0 \oplus V_0 \otimes W_1 \xrightarrow{d_1} V_0 \otimes W_0$$

where d_1 and d_2 are defined as follows.

$$d_2(v_1 \otimes w_1) = (\partial v_1) \otimes w_1 - v_1 \otimes (\partial' w_1)$$
$$d_1(v_1 \otimes w_0) = (\partial v_1) \otimes w_0$$
$$d_1(v_0 \otimes w_1) = v_0 \otimes (\partial' w_1)$$

Since $d_1 d_2 = 0$ we obtain a map $\tilde{\partial}$ induced by d_1:

$$\tilde{\partial} : (V \bar{\otimes} W)_1 := \frac{V_1 \otimes W_0 \oplus V_0 \otimes W_1}{\operatorname{Im}(d_2)} \to V_0 \otimes W_0 =: (V \bar{\otimes} W)_0.$$

The map $\tilde{\partial} : (V \bar{\otimes} W)_1 \to (V \bar{\otimes} W)_0 \in \mathcal{P}air(\mathcal{M}od)$ is termed the *$\bar{\otimes}$-product of V and W* and is denoted by $V \bar{\otimes} W$.

The definition of the $\bar{\otimes}$-product occurs for instance also in [P, p.232].

The symmetry isomorphism $T : X \otimes Y \to Y \otimes X$ on chain complexes from 2.1 clearly passes to an isomorphism $V \bar{\otimes} W \cong W \bar{\otimes} V$. We denote also by $R \in \mathcal{P}air(\mathcal{M}od)$ the pair $(0 : 0 \to R)$ concentrated in degree zero. Taking all these structures together we obtain the following result:

2.4. **Proposition.** *The category $\mathcal{P}air(\mathcal{M}od)$ with the $\bar{\otimes}$-product is a symmetric monoidal category. The unit of this product is R.*

2.5. **Definition.** Given $V = (\partial : V_1 \to V_0) \in \mathcal{P}air(\mathcal{M}od)$ we denote the cokernel of ∂ in the category $\mathcal{M}od$ by $\pi_0(V)$ and the kernel by $\pi_1(V)$.

2.6. **Proposition.** *Given two pairs V and W then*

$$\pi_0(V \bar{\otimes} W) \cong \pi_0(V) \otimes \pi_0(W)$$

Moreover there exists a map

$$\psi : \pi_0(V) \otimes \pi_1(W) \oplus \pi_1(V) \otimes \pi_0(W) \to \pi_1(V \bar{\otimes} W)$$

which is an isomorphism if R is a field.

Proof. Consider the map

$$\phi : \pi_0(V \bar{\otimes} W) \to \pi_0(V) \otimes \pi_0(W)$$

defined by $\phi(\overline{x \otimes y}) = \overline{x} \otimes \overline{y}$. Here $\overline{x}$ denotes the class of x in the quotient. It is easy to check that this is an isomorphism. The map

$$\psi : \pi_0(V) \otimes \pi_1(W) \oplus \pi_1(V) \otimes \pi_0(W) \to \pi_1(V \bar{\otimes} W)$$

is defined by $\psi(\overline{v_0} \otimes w_1) = \overline{v_0 \otimes w_1}$ and $\psi(v_1 \otimes \overline{w_0}) = \overline{v_1 \otimes w_0}$. It is easy to check that this is a well defined map. If R is a field we can find linear sections $q_1 : \operatorname{Im}(\partial) \to V_1$ and $q_2 : \operatorname{Im}(\partial') \to W_1$ to prove that in this case ψ is an isomorphism. □

We introduce now the *secondary homology functor* $D : \mathcal{C}hain \to \mathcal{P}air(\mathcal{M}od)$.

2.7. **Definition.** Let C be a chain complex with differential $d : C \to C$ of degree -1. Consider the graded modules $D_1(C) = s^{-1} \operatorname{coker}(d)$ and $D_0(C) = \ker(d)$. Here we define for a graded module W the *shifted* graded module $s^{-1}W$ by

$$W_n = (s^{-1}W)_{n-1}, \; w \mapsto s^{-1}(w)$$

We denote by $s^{-1}(\overline{x}) \in s^{-1} \operatorname{coker}(d)$ the element corresponding to $x \in C$ via the projection $C \to \operatorname{coker}(d)$. Then d induces a map of graded modules

$$D(C) = (\partial : D_1(C) = s^{-1} \operatorname{coker}(d) \to D_0(C) = \ker(d))$$

carrying $s^{-1}(\overline{x})$ to dx. A map $f : C \to C'$ of chain complexes induces a map $D(f) : D(C) \to D(C')$ in $\mathcal{P}air(\mathcal{M}od)$ and therefore we obtain a functor

$$D : \mathcal{C}hain \to \mathcal{P}air(\mathcal{M}od)$$

which is termed the *secondary homology functor.*

For any chain complex C the homology of C, $H_*(C)$ is isomorphic to $\pi_0(D(C))$ and $\pi_1(D(C))$ is the shifted homology $s^{-1}H_*(C)$. Thus the composition $\pi_0 \circ D$ is a factorization of the homology functor.

Note that for the ground ring $R \in \mathcal{C}hain$ concentrated in degree 0 we have

$$D(R) = (0 : s^{-1}R \to R)$$

where $s^{-1}R$ denotes the ring R concentrated in degree -1. Therefore $D(R) \in \mathcal{P}air(\mathcal{M}od)$ does *not* coincide with the unit $R = (0 \to R)$, but we have a canonical inclusion map $R \to D(R)$ of pairs.

2.8. **Proposition.** *Given chain complexes A and B there exists a map*

$$\Upsilon : D(A)\bar{\otimes}D(B) \to D(A \otimes B)$$

of pairs which is natural in A and B. Moreover, Υ commutes with the symmetry isomorphism T from 2.1 in the following sense: let $\bar{T}$ denote the interchange map $\bar{T} : D(A)\bar{\otimes}D(B) \to D(B)\bar{\otimes}D(A)$, then

$$\begin{array}{ccc} D(A)\bar{\otimes}D(B) & \xrightarrow{\Upsilon} & D(A \otimes B) \\ {\scriptstyle \bar{T}}\downarrow & & \downarrow{\scriptstyle D(T)} \\ D(B)\bar{\otimes}D(A) & \xrightarrow{\Upsilon} & D(A \otimes B) \end{array} \qquad \textit{commutes.}$$

In addition, if R is a field then Υ induces an isomorphism on the cokernel π_0 and an epimorphism on the kernel π_1.

Proof. We define $\Upsilon : D(A)\bar{\otimes}D(B) \to D(A \otimes B)$ as follows.

$$\begin{array}{ccc} \frac{(s^{-1}\operatorname{coker}(d_A)\otimes\ker(d_B))\oplus(\ker(d_A)\otimes s^{-1}\operatorname{coker}(d_B))}{\operatorname{Im}(d_2)} & \xrightarrow{\partial} & \ker(d_A)\otimes\ker(d_B) \\ {\scriptstyle \Upsilon_1}\downarrow & & \downarrow{\scriptstyle \Upsilon_0} \\ s^{-1}\operatorname{coker}(d_{A\otimes B}) & \xrightarrow{\partial_{A\otimes B}} & \ker(d_{A\otimes B}) \end{array}$$

The top row in the diagram corresponds to the $\bar{\otimes}$-product $D(A)\bar{\otimes}D(B)$ and the bottom row corresponds to $D(A \otimes B)$. The map $\Upsilon_0 : \ker(d_A) \otimes \ker(d_B) \to \ker(d_{A\otimes B})$ is defined by

$$\Upsilon_0(x_i \otimes y_j) = x_i \otimes y_j \qquad x_i \in (\ker(d_A))_i,\ y_j \in (\ker(d_B))_j$$

The map Υ_1 is defined as follows. For $s^{-1}(\overline{x_i}) \in (s^{-1}\operatorname{coker}(d_A))_i$ and $y_j \in (\ker(d_B))_j$ we define $\Upsilon_1(s^{-1}(\overline{x_i}) \otimes y_j)$ to be the element $s^{-1}(\overline{x_i \otimes y_j}) \in (s^{-1}\operatorname{coker}(d_{A\otimes B}))_{i+j}$. For $x_i \in (\ker(d_A))_i$ and $s^{-1}(\overline{y_j}) \in (s^{-1}\operatorname{coker}(d_B)_j$ we define $\Upsilon_1(x_i \otimes s^{-1}(\overline{y_j})) = (-1)^i s^{-1}(\overline{x_i \otimes y_j}) \in (s^{-1}\operatorname{coker}(d_{A\otimes B}))_{i+j}$.

We check that the map Υ_1 is well defined. Suppose $\overline{x_i} = \overline{0} \in (\operatorname{coker}(d_A))_i$, i.e., $x_i = d_A a_{i+1}$ for some $a_{i+1} \in A_{i+1}$. Then

$$\Upsilon_1(s^{-1}(\overline{x_i}) \otimes y_j) = s^{-1}(\overline{d_A a_{i+1} \otimes y_j}) = s^{-1}(\overline{d_{A\otimes B}(a_{i+1} \otimes y_j)}) = 0.$$

The same argument works for $\overline{y_j} = \overline{0} \in (\operatorname{coker}(d_B))_j$. For $z = (d_A x_i \otimes y_j - x_i \otimes d_B y_j) \in \operatorname{Im}(d_2)$ we have $\Upsilon_1(z) = d_{A\otimes B}((-1)^i x_i \otimes y_j)$. Thus Υ_1 is well defined.

If R is a field then Υ induces an isomorphism

$$\begin{aligned}\Upsilon_* : \pi_0(D(A)\bar{\otimes}D(B)) = \pi_0(D(A)) \otimes \pi_0(D(B)) = H_*(A) \otimes H_*(B) \to \pi_0(D(A \otimes B)) \\ = H_*(A \otimes B)\end{aligned}$$

and an epimorphism

$$\begin{aligned}\Upsilon_* : \pi_1(D(A)\bar{\otimes}D(B)) = s^{-1}H_*(A) \otimes H_*(B) \oplus H_*(A) \otimes s^{-1}H_*(B) \to \pi_1(D(A \otimes B)) \\ = s^{-1}H_*(A \otimes B)\end{aligned}$$

□

3. Operads

In this section we recall basic definitions and facts related to operads. Readers who are familiar with the notions of operads, their algebras and their modules might wish to skip this section and to continue with section 4.

Operads can be defined in any symmetric monoidal category, for example the categories $\mathcal{M}od$, $\mathcal{C}hain$, and $\mathcal{P}air(\mathcal{M}od)$ introduced above. We recall from [KM, Part I] the basic definitions and examples of operads, algebras over operads, ideals and modules over such algebras.

3.1. **Definition.** An *operad* $\mathcal{C}$ in a symmetric monoidal category $(\mathcal{M}, \otimes)$ consists of a family of objects $\mathcal{C}(j), j \geqslant 0$ together with a unit map $\nu : e \to \mathcal{C}(1)$ where e denotes the unit object for the symmetric monoidal structure, a right action by the symmetric group Σ_j on $\mathcal{C}(j)$ for every j and maps

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \to \mathcal{C}(j)$$

for $k \geqslant 1, j_s \geqslant 0$ and $j = \sum j_s$. The maps γ are required to be associative, unital and equivariant in the following sense.

(a) The following associativity diagrams commute, where $\sum j_s = j$ and $\sum i_t = i$. We set $g_s = j_1 + \ldots + j_s$ and $h_s = i_{g_{s-1}+1} + \ldots + i_{g_s}$ for $s = 1, \ldots, k$.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \Big(\bigotimes_{s=1}^{k} \mathcal{C}(j_s)\Big) \otimes \Big(\bigotimes_{r=1}^{j} \mathcal{C}(i_r)\Big) & \xrightarrow{\gamma \otimes \mathrm{id}} & \mathcal{C}(j) \otimes \Big(\bigotimes_{r=1}^{j} \mathcal{C}(i_r)\Big) \\
\Big\downarrow \text{shuffle} & & \Big\downarrow \gamma \\
& & \mathcal{C}(i) \\
& & \Big\uparrow \gamma \\
\mathcal{C}(k) \otimes \Bigg(\bigotimes_{s=1}^{k} \Big(\mathcal{C}(j_s) \otimes \Big(\bigotimes_{q=1}^{j_s} \mathcal{C}(i_{g_{s-1}+q})\Big)\Big)\Bigg) & \xrightarrow{\mathrm{id}\otimes(\otimes_s \gamma)} & \mathcal{C}(k) \otimes \Big(\bigotimes_{s=1}^{k} \mathcal{C}(h_s)\Big)
\end{array}$$

(b) The composition in the operad makes the unit in the operad act as identity:

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes e^{\otimes k} & \xrightarrow{\simeq} & \mathcal{C}(k) \\
\Big\downarrow \mathrm{id}\otimes\nu^{\otimes k} & \nearrow \gamma & \\
\mathcal{C}(k) \otimes \mathcal{C}(1)^{\otimes k} & &
\end{array}
\qquad
\begin{array}{ccc}
e \otimes \mathcal{C}(j) & \xrightarrow{\simeq} & \mathcal{C}(j) \\
\Big\downarrow \nu\otimes\mathrm{id} & \nearrow \gamma & \\
\mathcal{C}(1) \otimes \mathcal{C}(j) & &
\end{array}$$

(c) For $\sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}$, the permutation $\sigma(j_1, \dots, j_k) \in \Sigma_j$ permutes k blocks of letters as σ permutes k letters and $(\tau_1, \dots, \tau_k) \in \Sigma_j$ denotes the block sum, i.e., the image of $(\tau_1, \dots, \tau_k)$ in Σ_j under the natural inclusion $\Sigma_{j_1} \times \dots \times \Sigma_{j_k} \hookrightarrow \Sigma_j$. The following equivariance diagrams commute.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \dots \otimes \mathcal{C}(j_k) & \xrightarrow{\overline{\sigma}} & \mathcal{C}(k) \otimes \mathcal{C}(j_{\sigma(1)}) \otimes \dots \otimes \mathcal{C}(j_{\sigma(k)}) \\
\gamma \downarrow & & \downarrow \gamma \\
\mathcal{C}(j) & \xrightarrow{\sigma(j_{\sigma(1)}, \dots, j_{\sigma(k)})} & \mathcal{C}(j)
\end{array}$$

where $\overline{\sigma}(x_0 \otimes x_1 \otimes \dots \otimes x_k) = (x_0\sigma \otimes x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)})$ and

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \dots \otimes \mathcal{C}(j_k) & \xrightarrow{\overline{\tau}} & \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \dots \otimes \mathcal{C}(j_k) \\
\gamma \downarrow & & \downarrow \gamma \\
\mathcal{C}(j) & \xrightarrow{(\tau_1, \dots, \tau_k)} & \mathcal{C}(j)
\end{array}$$

where $\bar{\tau} = \mathrm{id} \otimes (\tau_1 \otimes \dots \otimes \tau_k)$.

3.2. **Definition.** Let $\mathcal{C}$ be an operad in the category $\mathcal{M}od$ (resp. $\mathcal{C}hain$). An *ideal* $\mathcal{J}$ *in* $\mathcal{C}$ consists of a sequence of graded $R[\Sigma_j]$-submodules (resp. sub-chain complexes) $\mathcal{J}(j)$ of $\mathcal{C}(j)$ such that $\gamma(c \otimes d_1 \otimes \dots \otimes d_k)$ is in $\mathcal{J}$ if either c or any of the d_s is in $\mathcal{J}$.

If $\mathcal{J}$ is an ideal in $\mathcal{C}$ then the quotient $\mathcal{C}/\mathcal{J}$ defined as $(\mathcal{C}/\mathcal{J})(j) = \mathcal{C}(j)/\mathcal{J}(j)$ is an operad.

3.3. **Definition.** Let $\mathcal{C}$ be an operad in a symmetric monoidal category. A $\mathcal{C}$-*algebra* is an object A together with maps

$$\theta : \mathcal{C}(j) \otimes A^{\otimes j} \to A$$

for $j \geqslant 0$, subject to the following conditions.

(a) The action of the operad on A is associative, thus for $j = \sum j_s$ the following two possible ways of composition agree.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \dots \otimes \mathcal{C}(j_k) \otimes A^{\otimes j} & \xrightarrow{\gamma \otimes \mathrm{id}} & \mathcal{C}(j) \otimes A^{\otimes j} \\
 & & \downarrow \theta \\
\text{shuffle} \downarrow & & A \\
 & & \uparrow \theta \\
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes A^{\otimes j_1} \otimes \dots \otimes \mathcal{C}(j_k) \otimes A^{\otimes j_k} & \xrightarrow{\mathrm{id} \otimes \theta^{\otimes k}} & \mathcal{C}(k) \otimes A^{\otimes k}
\end{array}$$

(b) The unit of the operad acts as the identity on A:

$$\begin{array}{ccc}
e \otimes A & \xrightarrow{\cong} & A \\
\nu \otimes \mathrm{id} \downarrow & \nearrow \theta & \\
\mathcal{C}(1) \otimes A & &
\end{array}$$

(c) The action θ is equivariant, i.e., for $\sigma \in \Sigma_j$ the composition $\theta \circ \overline{\sigma}$ is again θ

$$\begin{array}{ccc}
\mathcal{C}(j) \otimes A^{\otimes j} & \xrightarrow{\overline{\sigma}} & \mathcal{C}(j) \otimes A^{\otimes j} \\
 & \theta \searrow \quad \swarrow \theta & \\
 & A &
\end{array}$$

where $\overline{\sigma}(x \otimes a_1 \otimes \dots \otimes a_j)$ is $(x\sigma \otimes a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(j)})$.

3.4. **Definition.** Let $\mathcal{C}$ be an operad and A be a $\mathcal{C}$-algebra. An *A-module* is an object M together with maps

$$\lambda : \mathcal{C}(j) \otimes A^{\otimes(j-1)} \otimes M \to M$$

for $j \geqslant 1$. These maps are required to be associative, unital and equivariant in the following sense.

(a) The following associativity diagrams commute, where $j = \sum j_s$.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \otimes A^{\otimes(j-1)} \otimes M & \xrightarrow{\gamma\otimes\mathrm{id}} & \mathcal{C}(j) \otimes A^{\otimes(j-1)} \otimes M \\
\Big\downarrow \text{shuffle} & & \Big\downarrow \lambda \\
 & & M \\
 & & \Big\uparrow \lambda \\
\mathcal{C}(k) \otimes \big(\bigotimes_{s=1}^{k-1} \mathcal{C}(j_s) \otimes A^{\otimes j_s}\big) \otimes \mathcal{C}(j_k) \otimes A^{\otimes(j_k-1)} \otimes M & \xrightarrow[\mathrm{id}\otimes\theta^{\otimes(k-1)}\otimes\lambda]{} & \mathcal{C}(k) \otimes A^{\otimes(k-1)} \otimes M
\end{array}$$

(b) The unit in the operad acts as identity on M:

$$\begin{array}{ccc}
e \otimes M & \xrightarrow{\cong} & M. \\
\Big\downarrow \nu\otimes\mathrm{id} & \nearrow \lambda & \\
\mathcal{C}(1) \otimes M & &
\end{array}$$

(c) The map λ is equivariant with respect to the action of the symmetric groups on A, i.e., for $\sigma \in \Sigma_{j-1} \subset \Sigma_j$ the following diagram commutes

$$\begin{array}{ccc}
\mathcal{C}(j) \otimes A^{\otimes(j-1)} \otimes M & \xrightarrow{\overline{\sigma}} & \mathcal{C}(j) \otimes A^{\otimes(j-1)} \otimes M \\
 & \searrow \lambda \quad \lambda \swarrow & \\
 & M &
\end{array}$$

where $\overline{\sigma}(x \otimes a_1 \otimes \ldots \otimes a_{j-1} \otimes m) = (x\sigma \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(j-1)} \otimes m)$.

A *map $f : M \to N$ of A-modules* is a map f in the category $\mathcal{M}$ such that the following diagram commutes for every $j \geqslant 1$.

$$\begin{array}{ccc}
\mathcal{C}(j) \otimes A^{\otimes(j-1)} \otimes M & \xrightarrow{\lambda} & M \\
\Big\downarrow \mathrm{id}\otimes\mathrm{id}\otimes f & & \Big\downarrow f \\
\mathcal{C}(j) \otimes A^{\otimes(j-1)} \otimes N & \xrightarrow{\lambda} & N
\end{array}$$

3.5. **Example.** If A is a $\mathcal{C}$-algebra then A is an A-module with the multiplication given by $\theta : \mathcal{C}(j) \otimes A^{\otimes j} \to A$.

3.6. **Example.** The operad Com is defined as follows. The module $\mathsf{Com}(j)$ is R for $j \geqslant 0$ and the Σ_j-actions are trivial. The unit map ν is the identity and the composition maps are the evident identifications.

3.7. **Example.** The operad Ass consists of the module $\mathsf{Ass}(j) = R[\Sigma_j]$ for $j \geqslant 0$; here $\mathsf{Ass}(0)$ has to be interpreted as the ground ring R. The unit map is the identity, the composition maps are defined by

$$\gamma(\sigma, \tau_1, \ldots, \tau_k) = \sigma(j_1, \ldots, j_k)(\tau_1, \ldots, \tau_k)$$

for $\sigma \in \Sigma_k$ and $\tau_s \in \Sigma_{j_s}$.

4. Secondary Operads

We now come to the type of operads which will be central to our work.

4.1. **Definition.** A *secondary operad* $\mathcal{D}$ is an operad in the symmetric monoidal category $(\mathcal{P}air(\mathcal{M}od), \bar{\otimes})$.

In order to detect for instance actions of a secondary operad on a certain pair of modules, it is crucial to have an explicit description of the data which determine a secondary operad. In work in progress [BMR] we use this to study the operadic structure of secondary cohomology.

A secondary operad can be defined explicitly in the following way.

4.2. **Proposition.** *A secondary operad consists of a family of pairs of graded modules*

$$\partial_j : \mathcal{D}_1(j) \to \mathcal{D}_0(j) \qquad j \geqslant 0$$

together with a unit map $\nu : R \to \mathcal{D}_0(1)$*, a right action by the symmetric group* Σ_j *on* $\mathcal{D}_i(j)$ *for* $i = 0, 1$ *and* $j \geqslant 0$ *such that* $\partial_j(x\sigma) = (\partial_j x)\sigma$ *for every* $\sigma \in \Sigma_j$ *and* $x \in \mathcal{D}_1(j)$ *and maps*

$$\begin{aligned}
\Gamma &: \mathcal{D}_0(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) \to \mathcal{D}_0(j)\\
\Gamma_0 &: \mathcal{D}_1(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) \to \mathcal{D}_1(j)\\
\Gamma_s &: \mathcal{D}_0(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_1(j_s) \otimes \ldots \otimes \mathcal{D}_0(j_k) \to \mathcal{D}_1(j)
\end{aligned}$$

for $k \geqslant 1, j_s \geqslant 0, \sum j_s = j, s = 1, \ldots, k$*, such that the following conditions (a),...,(e) hold.*

(a)

$$\Gamma(\mathrm{id} \otimes \partial_{j_s} \otimes \mathrm{id}) = \partial_j \Gamma_s \quad \text{and} \tag{1}$$

$$\Gamma(\partial_k \otimes \mathrm{id}) = \partial_j \Gamma_0. \tag{2}$$

Here we use the convention on signs above. Moreover for all $i, \ell \geqslant 0$, $y_i \in \mathcal{D}_1(j_i)$, $y_\ell \in \mathcal{D}_1(j_\ell)$ *and* $x_r \in \mathcal{D}_0(j_r), x_0 \in \mathcal{D}_0(k)$ *the following equation holds*

$$\Gamma_\ell(x_0 \otimes \ldots \otimes \partial_i y_i \otimes \ldots \otimes y_\ell \otimes \ldots \otimes x_{j_k}) - \Gamma_i(x_0 \otimes \ldots \otimes y_i \otimes \ldots \otimes \partial_\ell y_\ell \otimes \ldots \otimes x_{j_k}) = 0 \tag{3}$$

(b) $\mathcal{D}_0$ *together with the maps* $\Gamma : \mathcal{D}_0(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) \to \mathcal{D}_0(j)$ *is an operad in* $\mathcal{M}od$.

(c) *The mixed compositions* Γ_s *for* $s \geqslant 0$ *and the operad composition* Γ *interact nicely, i.e., the following associativity diagrams commute, where* $\sum j_s = j$ *and* $\sum i_t = i$. *We set* $g_s = j_1 + \ldots + j_s$ *and* $h_s = i_{g_{s-1}+1} + \ldots + i_{g_s}$ *for* $s = 1, \ldots, k$.

$$\begin{array}{ccc}
\mathcal{D}_1(k) \otimes \big(\bigotimes_{s=1}^{k} \mathcal{D}_0(j_s)\big) \otimes \big(\bigotimes_{r=1}^{j} \mathcal{D}_0(i_r)\big) & \xrightarrow{\Gamma_0 \otimes \mathrm{id}} & \mathcal{D}_1(j) \otimes \big(\bigotimes_{r=1}^{j} \mathcal{D}_0(i_r)\big)\\
 & & \downarrow{\scriptstyle \Gamma_0}\\
\big\downarrow{\scriptstyle \text{shuffle}} & & \mathcal{D}_1(i)\\
 & & \uparrow{\scriptstyle \Gamma_0}\\
\mathcal{D}_1(k) \otimes \Big(\bigotimes_{s=1}^{k} \Big(\mathcal{D}_0(j_s) \otimes \big(\bigotimes_{q=1}^{j_s} \mathcal{D}_0(i_{g_{s-1}+q})\big)\Big)\Big) & \xrightarrow{\mathrm{id} \otimes (\otimes_s \Gamma)} & \mathcal{D}_1(k) \otimes \big(\bigotimes_{s=1}^{k} \mathcal{D}_0(h_s)\big)
\end{array}$$

For every $s_0 = 1, \ldots, k$ *we have*

$$\begin{array}{ccc}
\mathcal{D}_0(k) \otimes \big(\bigotimes_{s=1}^{k} \mathcal{D}_{\delta_s}(j_s) \big) \otimes \big(\bigotimes_{r=1}^{j} \mathcal{D}_0(i_r) \big) & \xrightarrow{\Gamma_{s_0} \otimes \mathrm{id}} & \mathcal{D}_1(j) \otimes \big(\bigotimes_{r=1}^{j} \mathcal{D}_0(i_r) \big) \\
\Big\downarrow \text{shuffle} & & \Big\downarrow \Gamma_0 \\
 & & \mathcal{D}_1(i) \\
 & & \Big\uparrow \Gamma_{s_0} \\
\mathcal{D}_0(k) \otimes \Big(\bigotimes_{s=1}^{k} \Big(\mathcal{D}_{\delta_s}(j_s) \otimes \big(\bigotimes_{q=1}^{j_s} \mathcal{D}_0(i_{g_{s-1}+q}) \big) \Big) \Big) & \xrightarrow{\overline{\Gamma}} & \mathcal{D}_0(k) \otimes \big(\bigotimes_{s=1}^{k} \mathcal{D}_{\delta_s}(h_s) \big)
\end{array}$$

where $\delta_s = 0$ *for all* $s \neq s_0$ *and* $\delta_{s_0} = 1$ *and* $\overline{\Gamma} = \mathrm{id} \otimes \Gamma^{\otimes(s_0-1)} \otimes \Gamma_0 \otimes \Gamma^{\otimes(k-s_0)}$. *And for every* $r_0 = 1, \ldots, j$ *we have*

$$\begin{array}{ccc}
\mathcal{D}_0(k) \otimes \big(\bigotimes_{s=1}^{k} \mathcal{D}_0(j_s) \big) \otimes \big(\bigotimes_{r=1}^{j} \mathcal{D}_{\delta_r}(i_r) \big) & \xrightarrow{\Gamma \otimes \mathrm{id}} & \mathcal{D}_0(j) \otimes \big(\bigotimes_{r=1}^{j} \mathcal{D}_{\delta_r}(i_r) \big) \\
\Big\downarrow \text{shuffle} & & \Big\downarrow \Gamma_{r_0} \\
 & & \mathcal{D}_1(i) \\
 & & \Big\uparrow \Gamma_{s_1} \\
\mathcal{D}_0(k) \otimes \Big(\bigotimes_{s=1}^{k} \Big(\mathcal{D}_0(j_s) \otimes \big(\bigotimes_{q=1}^{j_s} \mathcal{D}_{\delta_{g_{s-1}+q}}(i_{g_{s-1}+q}) \big) \Big) \Big) & \xrightarrow{\tilde{\Gamma}} & \mathcal{D}_0(k) \otimes \big(\bigotimes_{s=1}^{k} \mathcal{D}_{\delta'_s}(h_s) \big)
\end{array}$$

where $\delta_r = 0$ *for all* $r \neq r_0$ *and* $\delta_{r_0} = 1$, $\delta'_s = 0$ *for* $s \neq s_1$ *and* $\delta'_{s_1} = 1$ *where* s_1 *is defined by* $j_1 + \ldots + j_{s_1-1} < r_0 \leqslant j_1 + \ldots + j_{s_1}$ *and* $\tilde{\Gamma} = \mathrm{id} \otimes \Gamma^{\otimes(s_1-1)} \otimes \Gamma_{r_0 - g_{s_1-1}} \otimes \Gamma^{\otimes(k-s_1)}$.

(d) *The following unit diagrams commute*

$$\begin{array}{ccc}
\mathcal{D}_1(k) \otimes R^{\otimes k} & \xrightarrow{\simeq} & \mathcal{D}_1(k) \\
\Big\downarrow \mathrm{id} \otimes \nu^{\otimes k} & \nearrow \Gamma_0 & \\
\mathcal{D}_1(k) \otimes \mathcal{D}_0(1)^{\otimes k} & &
\end{array}
\qquad
\begin{array}{ccc}
R \otimes \mathcal{D}_1(j) & \xrightarrow{\simeq} & \mathcal{D}_1(j) \\
\Big\downarrow \nu \otimes \mathrm{id} & \nearrow \Gamma_1 & \\
\mathcal{D}_0(1) \otimes \mathcal{D}_1(j) & &
\end{array}$$

(e) *For* $\sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}$, *the permutation* $\sigma(j_1, \ldots, j_k) \in \Sigma_j$ *permutes* k *blocks of letters as* σ *permutes* k *letters and* $(\tau_1, \ldots, \tau_k) \in \Sigma_j$ *denotes the block sum. The mixed action maps* Γ_s *for* $s \geqslant 0$ *fulfill the usual equivariance conditions*

$$\begin{array}{ccc}
\mathcal{D}_1(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) & \xrightarrow{\overline{\sigma}} & \mathcal{D}_1(k) \otimes \mathcal{D}_0(j_{\sigma(1)}) \otimes \ldots \otimes \mathcal{D}_0(j_{\sigma(k)}) \\
\Big\downarrow \Gamma_0 & & \Big\downarrow \Gamma_0 \\
\mathcal{D}_1(j) & \xrightarrow{\sigma(j_{\sigma(1)}, \ldots, j_{\sigma(k)})} & \mathcal{D}_1(j)
\end{array}$$

where $\overline{\sigma}(x_0 \otimes x_1 \otimes \ldots \otimes x_k) = (x_0\sigma \otimes x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(k)})$ *and*

$$\begin{array}{ccc}
\mathcal{D}_1(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) & \xrightarrow{\overline{\tau}} & \mathcal{D}_1(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) \\
\Big\downarrow \Gamma_0 & & \Big\downarrow \Gamma_0 \\
\mathcal{D}_1(j) & \xrightarrow{(\tau_1, \ldots, \tau_k)} & \mathcal{D}_1(j)
\end{array}$$

where $\overline{\tau} = \mathrm{id} \otimes \tau_1 \otimes \ldots \otimes \tau_k$. *And for every* $s_0 = 1, \ldots, k$ *we have*

$$\begin{array}{ccc}
\mathcal{D}_0(k) \otimes \mathcal{D}_{\delta_1}(j_1) \otimes \ldots \otimes \mathcal{D}_{\delta_k}(j_k) & \xrightarrow{\overline{\sigma}} & \mathcal{D}_0(k) \otimes \mathcal{D}_{\delta_{\sigma(1)}}(j_{\sigma(1)}) \otimes \ldots \otimes \mathcal{D}_{\delta_{\sigma(k)}}(j_{\sigma(k)}) \\
\Gamma_{s_0}\downarrow & & \downarrow\Gamma_{\sigma^{-1}(s_0)} \\
\mathcal{D}_1(j) & \xrightarrow{\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})} & \mathcal{D}_1(j)
\end{array}$$

where $\overline{\sigma}(x_0 \otimes x_1 \otimes \ldots \otimes x_k) = (x_0\sigma \otimes x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(k)})$ *and*

$$\begin{array}{ccc}
\mathcal{D}_0(k) \otimes \mathcal{D}_{\delta_1}(j_1) \otimes \ldots \otimes \mathcal{D}_{\delta_k}(j_k) & \xrightarrow{\overline{\tau}} & \mathcal{D}_0(k) \otimes \mathcal{D}_{\delta_1}(j_1) \otimes \ldots \otimes \mathcal{D}_{\delta_k}(j_k) \\
\Gamma_{s_0}\downarrow & & \downarrow\Gamma_{s_0} \\
\mathcal{D}_1(j) & \xrightarrow{(\tau_1,\ldots,\tau_k)} & \mathcal{D}_1(j)
\end{array}$$

where $\delta_s = 0$ *for all* $s \neq s_0$ *and* $\delta_{s_0} = 1$.

Proof. By 2.3 a map

$$\gamma : \mathcal{D}(k)\bar{\otimes}\mathcal{D}(j_1)\bar{\otimes}\ldots\bar{\otimes}\mathcal{D}(j_k) \to \mathcal{D}(j)$$

corresponds to a commutative diagram

$$\begin{array}{ccc}
\frac{(\bigoplus_{i=1,\ldots,k} \mathcal{D}_0(k)\otimes\ldots\otimes\mathcal{D}_1(j_i)\otimes\ldots\otimes\mathcal{D}_0(j_k))}{Z} & \longrightarrow & \mathcal{D}_0(k) \otimes \ldots \otimes \mathcal{D}_0(j_k) \\
\downarrow & & \downarrow \\
\mathcal{D}_1(j) & \longrightarrow & \mathcal{D}_0(j)
\end{array}$$

Here Z is generated by the elements of the form

$$(x_0 \otimes x_1 \otimes \ldots \otimes \partial_i y_i \otimes \ldots \otimes y_r \otimes \ldots \otimes x_k) - (x_0 \otimes \ldots \otimes y_i \otimes \ldots \otimes \partial_r y_r \otimes \ldots \otimes x_k)$$

where $x_0 \in \mathcal{D}_0(k),\ x_i \in \mathcal{D}_0(j_i),\ y_r \in \mathcal{D}_1(j_r)$. The maps Γ, Γ_0 and Γ_s are induced by γ.

Conditions (a) and (b) follow from the diagram above and conditions (c),(d) and (e) follow from analogous conditions of 3.1. □

4.3. **Proposition.** *If* $\mathcal{D}$ *is a secondary operad then* $\mathrm{Im}(\partial)$ *defined by* $\mathrm{Im}(\partial)(j) = \mathrm{Im}(\partial_j) \subseteq \mathcal{D}_0(j)$ *is an ideal in the operad* $\mathcal{D}_0$. *In particular* $\pi_0\mathcal{D}$ *defined by* $\pi_0\mathcal{D}(j) = \mathcal{D}_0(j)/\mathrm{Im}(\partial_j)$ *with unit and right action of* Σ_j *induced by* $\mathcal{D}$ *is also an operad in* $\mathcal{M}od$.

Proof. Follows from equations (1) and (2) of (4.2). □

The inclusion functor $i : \mathcal{M}od \to \mathcal{P}air(\mathcal{M}od)$ defined by $i(M) = (0 : 0 \to M)$ takes operads $\mathcal{C}$ in $\mathcal{M}od$ to secondary operads $i(\mathcal{C}) = (0 \to \mathcal{C})$. In particular one can define the secondary operads $(0 \to \mathsf{Com})$ and $(0 \to \mathsf{Ass})$ from the corresponding operads in $\mathcal{M}od$.

4.4. *Remark.* In general, given an operad $\mathcal{C}$ in $\mathcal{M}od$ and an ideal $\mathcal{J}$ in $\mathcal{C}$ then the inclusion $\partial = \mathrm{inc} : \mathcal{J} \to \mathcal{C}$ yields a secondary operad. The unit and right action of the symmetric group is given by the structure of operad on $\mathcal{C}$ and the fact that $\mathcal{J}$ is an ideal in $\mathcal{C}$. The maps Γ and Γ_s's are induced by the maps $\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \to \mathcal{C}(j)$ and the inclusions.

The secondary homology functor $D : \mathcal{C}hain \to \mathcal{P}air(\mathcal{M}od)$ takes operads of chain complexes to secondary operads. Given an operad $\mathcal{C}$ of chain complexes we denote by $D(\mathcal{C})$ the families of pairs

$$D(\mathcal{C})(j) = D(\mathcal{C}(j)) = \Big(\partial_j : D_1\big(\mathcal{C}(j)\big) \to D_0\big(\mathcal{C}(j)\big)\Big)$$

An action of Σ_j on $D(\mathcal{C})(j)$ is induced by the action of Σ_j on $\mathcal{C}(j)$ and the unit of $\mathcal{C}$ induces a unit for $\mathcal{D}(\mathcal{C})$ via the inclusion map $R \to \mathcal{D}(R)$. The natural transformation Υ from (2.8) leads to the compositions

$$\Upsilon : \mathcal{D}(\mathcal{C})(j)\bar{\otimes}\mathcal{D}(\mathcal{C})(n_1)\bar{\otimes}\ldots\bar{\otimes}\mathcal{D}(\mathcal{C})(n_j) \longrightarrow \mathcal{D}\left(\mathcal{C}(j)\otimes\mathcal{C}(n_1)\otimes\ldots\otimes\mathcal{C}(n_j)\right).$$

Prolonging Υ with $\mathcal{D}$ applied to the composition maps γ for the operad $\mathcal{C}$, $\mathcal{D}(\gamma)$, gives composition maps for $\mathcal{D}(\mathcal{C})$.

4.5. **Proposition.** *If $\mathcal{C}$ is an operad in $\mathcal{C}hain$ then $D(\mathcal{C})$ is a secondary operad.*

Proof. The natural transformation Υ preserves the symmetric monoidal structures. The results from (2.8) guarantee that the composition $\mathcal{D}(\gamma)\circ\Upsilon$ is associative and equivariant. □

5. Crossed Modules over Operads

5.1. **Definition.** Let $\mathcal{D}$ be a secondary operad. A *crossed module $\partial : V_1 \to V_0$ over $\mathcal{D}$* is a $\mathcal{D}$-algebra in $\mathcal{P}air(\mathcal{M}od)$.

We can characterize crossed modules explicitly as follows.

5.2. **Proposition.** *A crossed module over $\mathcal{D}$ is a pair $\partial : V_1 \to V_0$ together with maps*

$$\begin{aligned}\theta &: \mathcal{D}_0(j)\otimes V_0^{\otimes j} \to V_0,\\ \beta &: \mathcal{D}_1(j)\otimes V_0^{\otimes j} \to V_1, \quad \text{and}\\ \lambda &: \mathcal{D}_0(j)\otimes V_0^{\otimes(j-1)}\otimes V_1 \to V_1\end{aligned}$$

such that the following properties (a),...,(f) hold. For now, let $\tau_{i,k}$ denote the permutation which exchanges i and k.

(a) *V_0 together with the maps $\theta : \mathcal{D}_0(j)\otimes V_0^{\otimes j} \to V_0$ is a $\mathcal{D}_0$-algebra in $\mathcal{M}od$.*
(b) *V_1 together with the maps $\lambda : \mathcal{D}_0(j)\otimes V_0^{\otimes(j-1)}\otimes V_1 \to V_1$ is a V_0-module. Moreover the map $\partial : V_1 \to V_0$ is a map of V_0-modules.*
(c) *The differential ∂ relates the action map β to θ (resp. λ) in the following sense. The map β satisfies*

$$\partial\beta = \theta(\partial_j \otimes \mathrm{id}_{V_0^{\otimes j}}) \tag{4}$$

where ∂_j is the map $\partial_j : \mathcal{D}_1(j) \to \mathcal{D}_0(j)$, and for $y_0 \in \mathcal{D}_1(j)$, $x_i \in V_0$ and $y_i \in V_1$ we have

$$\beta(y_0\otimes x_1\otimes\ldots\otimes x_{r-1}\otimes\partial y_r\otimes\ldots\otimes x_j) - \lambda((\partial_j y_0)\tau_{rj}\otimes x_1\ldots\otimes x_{r-1}\otimes x_j\otimes\ldots\otimes y_r) = 0 \tag{5}$$

(d) *For $s \neq r$, the map λ satisfies*

$$\lambda(x_0\tau_{sj}\otimes\ldots\otimes x_{s-1}\otimes x\otimes\ldots\otimes\partial y_r\otimes\ldots\otimes y) - \lambda(x_0\tau_{rj}\otimes\ldots\otimes x_{s-1}\otimes\partial y\otimes\ldots\otimes x\otimes\ldots y_r) = 0 \tag{6}$$

and

$$\lambda(x_0\otimes\ldots\otimes\partial y_s\otimes\ldots\otimes y) - \lambda(x_0\tau_{sj}\otimes\ldots\otimes\partial y\otimes\ldots\otimes y_s) = 0 \tag{7}$$

where $x_0 \in \mathcal{D}_0(j)$, $x, x_i \in V_0$ and $y, y_j \in V_1$.

(e) *The following associativity diagrams commute, where* $j = \sum j_s$.

$$\begin{array}{ccc}
\mathcal{D}_1(k) \otimes \mathcal{D}_0(j_1) \otimes \ldots \otimes \mathcal{D}_0(j_k) \otimes V_0^{\otimes j} & \xrightarrow{\Gamma_0 \otimes 1} & \mathcal{D}_1(j) \otimes V_0^{\otimes j} \\
 & & \downarrow \beta \\
\downarrow \text{shuffle} & & V_1 \\
 & & \uparrow \beta \\
\mathcal{D}_1(k) \otimes \mathcal{D}_0(j_1) \otimes V_0^{\otimes j_1} \otimes \ldots \otimes \mathcal{D}_0(j_k) \otimes V_0^{\otimes j_k} & \xrightarrow{1 \otimes \theta^{\otimes k}} & \mathcal{D}_1(k) \otimes V_0^{\otimes k}
\end{array}$$

and for every $s_0 = 1, \ldots, k$

$$\begin{array}{ccc}
\mathcal{D}_0(k) \otimes \mathcal{D}_{\delta_1}(j_1) \otimes \ldots \otimes \mathcal{D}_{\delta_k}(j_k) \otimes V_0^{\otimes j} & \xrightarrow{\Gamma_{s_0} \otimes 1} & \mathcal{D}_1(j) \otimes V_0^{\otimes j} \\
 & & \downarrow \beta \\
\downarrow \text{shuffle} & & V_1 \\
 & & \uparrow \tilde{\lambda} \\
\mathcal{D}_0(k) \otimes \mathcal{D}_{\delta_1}(j_1) \otimes V_0^{\otimes j_1} \otimes \ldots \otimes \mathcal{D}_{\delta_k}(j_k) \otimes V_0^{\otimes j_k} & \xrightarrow{\alpha} & \mathcal{D}_0(k) \otimes V_0^{\otimes (s_0 - 1)} \otimes V_1 \otimes V_0^{\otimes (k - s_0)}
\end{array}$$

where $\delta_s = 0$ *for every* $s \neq s_0$ *and* $\delta_{s_0} = 1$, $\alpha = 1 \otimes \theta^{\otimes (s_0-1)} \otimes \beta \otimes \theta^{\otimes (k-s_0)}$ *and* $\tilde{\lambda}(x_0 \otimes \ldots \otimes y_{s_0} \otimes \ldots \otimes x_j) = \lambda(x_0 \tau_{s_0 j} \otimes \ldots \otimes x_j \otimes \ldots \otimes y_{s_0})$.

(f) *The map* β *is equivariant with respect to the action of the symmetric group, i.e., for* $\sigma \in \Sigma_j$, $\beta \circ \overline{\sigma} = \beta$

$$\begin{array}{ccc}
\mathcal{D}_1(j) \otimes V_0^{\otimes j} & \xrightarrow{\overline{\sigma}} & \mathcal{D}_1(j) \otimes V_0^{\otimes j} \\
 & {}_{\beta}\searrow \quad \swarrow_{\beta} & \\
 & V_1 &
\end{array}$$

where $\overline{\sigma}(x_0 \otimes \ldots \otimes x_j) = (x_0 \sigma \otimes x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(j)})$.

Proof. A map $\tilde{\theta} : \mathcal{D}(j) \bar{\otimes} V^{\bar{\otimes} j} \to V$ can be described as a commutative diagram

$$\begin{array}{ccc}
\dfrac{(\mathcal{D}_1(j) \otimes V_0^{\otimes j}) \bigoplus_{i=1,\ldots,k} (\mathcal{D}_0(j) \otimes V_0^{(i-1)} \otimes V_1 \otimes V_0^{(j-i)})}{Z} & \longrightarrow & \mathcal{D}_0(j) \otimes V_0^{\otimes j} \\
\downarrow & & \downarrow \\
V_1 & \longrightarrow & V_0
\end{array}$$

Here Z is generated by the elements of the form

$$(y_0 \otimes x_1 \otimes \ldots \otimes \partial_r y_r \otimes \ldots \otimes x_j) - (\partial_0 y_0 \otimes \ldots \otimes y_r \otimes \ldots \otimes x_j)$$

and

$$(x_0 \otimes x_1 \otimes \ldots \otimes \partial_i y_i \otimes \ldots \otimes y_r \otimes \ldots \otimes x_j) - (x_0 \otimes \ldots \otimes y_i \otimes \ldots \otimes \partial_r y_r \otimes \ldots \otimes x_j)$$

where $x_0 \in \mathcal{D}_0(j)$, $y_0 \in \mathcal{D}_1(j)$ and $x_i \in V_0$, $y_i \in V_1$. The maps θ, β and λ are induced by $\tilde{\theta} : \mathcal{D}(j) \bar{\otimes} V^{\bar{\otimes} j} \to V$. Conditions (a),(b),(c) and (d) follow from the commutative diagram above and conditions (e) and (f) follow from analogous conditions in 3.3. □

The following two results now follow directly from the definitions.

5.3. Proposition. *Given a crossed module* $V = (\partial : V_1 \to V_0)$ *over* $\mathcal{D}$ *then* $\pi_0(V)$ *is a* $\pi_0\mathcal{D}$*-algebra and* $\pi_1(V)$ *is a* $\pi_0(V)$*-module.*

5.4. **Proposition.** *Let $D : \mathcal{C}hain \to \mathcal{P}air(\mathcal{M}od)$ be the secondary homology functor. If $\mathcal{C}$ is an operad in $\mathcal{C}hain$ and A is a $\mathcal{C}$-algebra then $D(A)$ is a crossed module over the secondary operad $D(\mathcal{C})$.*

Here the results (2.8) guarantee that $D(A)$ has well-defined action maps because D respects the monoidal structure.

Let R be a commutative ring with unit. Following [BM, 3.1] and [B2, 6.1] a *crossed module* over a graded associative R-algebra A consist of an A-bimodule V and a map $\partial : V \to A$ of A-bimodules, such that $\partial(v)w = v\partial(w)$ for any elements $v, w \in V$.

5.5. **Lemma.** *Consider the secondary operad* $(0 \to \mathsf{Ass})$. *A crossed module $\partial : V_1 \to V_0$ over* $(0 \to \mathsf{Ass})$ *is the same as a structure of a crossed module over V_0 on $V_1 \to V_0$ in the sense of* [BM, 3.1] *or* [B2, 6.1].

Proof. If $\partial : V_1 \to V_0$ is a crossed module over $(0 \to \mathsf{Ass})$ then V_0 is a graded associative algebra with unit by setting

$$a_1 \cdots a_j = \theta(e \otimes a_1 \otimes \ldots \otimes a_j)$$

where $e \in \Sigma_j$ is the identity. The graded R-module V_1 is a V_0-bimodule where the bimodule structure is defined as follows. For $a \in V_0$ and $m \in V_1$ we set

$$a.m = \lambda(e \otimes a \otimes m)$$
$$m.a = \lambda(\tau \otimes a \otimes m)$$

where $\lambda : R[\Sigma_2] \otimes V_0 \otimes V_1 \to V_1$ and $\tau \in \Sigma_2$ is the transposition. It is clear that $\partial : V_1 \to V_0$ is a map of bimodules. For $v, w \in V_1$ we have

$$\partial(v).w = \lambda(e \otimes \partial(v) \otimes w) = \lambda(\tau \otimes \partial(w) \otimes v) = v.\partial(w).$$

This proves that $\partial : V_1 \to V_0$ is a crossed module in the sense of [B2] and [BM]. Conversely given a crossed module $\partial : V_1 \to V_0$ over a graded associative algebra V_0 we define

$$\theta : R[\Sigma_j] \otimes V_0^{\otimes j} \to V_0$$

by $\theta(\sigma \otimes a_1 \otimes \ldots \otimes a_j) = \mathrm{sign}(\sigma) a_{\sigma^{-1}(1)} \cdots a_{\sigma^{-1}(j)}$. In the same way we define

$$\lambda : R[\Sigma_j] \otimes V_0^{\otimes(j-1)} \otimes V_1 \to V_1$$

The equations (6) and (7) in the definition of crossed modules follow from the fact that $\partial(v).w = v.\partial(w)$. Thus $\partial : V_1 \to V_0$ is a crossed module over $(0 \to \mathsf{Ass})$. □

5.6. **Lemma.** *A crossed module $\partial : V_1 \to V_0$ over* $(0 \to \mathsf{Com})$ *is determined by the following data. The graded module V_0 is a graded commutative algebra, V_1 is a module over V_0 and ∂ is a map of modules such that $\partial(v).w = v.\partial(w)$ for all $v, w \in V_1$.*

5.7. **Definition.** Let $\mathcal{D}$ be a secondary operad. We define the category $\mathrm{Cross}_{\mathcal{D}}$ as follows. Objects are crossed modules over $\mathcal{D}$. A morphism f of crossed modules over $\mathcal{D}$ is a morphism of algebras over $\mathcal{D}$. Explicitly, such a morphism f is a pair of maps (f_0, f_1) which makes the following diagrams of graded modules commute:

$$\begin{array}{ccc} V_1 & \xrightarrow{\partial} & V_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ V_1' & \xrightarrow{\partial'} & V_0' \end{array}$$

such that

(a) $f_0 : V_0 \to V_0'$ is a map of $\mathcal{D}_0$-algebras, i.e., for every $j \geqslant 0$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}_0(j) \otimes V_0^{\otimes j} & \xrightarrow{\theta} & V_0 \\ \downarrow{\scriptstyle \mathrm{id}\otimes f_0^j} & & \downarrow{\scriptstyle f_0} \\ \mathcal{D}_0(j) \otimes (V_0')^{\otimes j} & \xrightarrow{\theta'} & V_0'. \end{array}$$

(b) The map $f_1 : V_1 \to V_1'$ is f_0-equivariant, i.e., the diagrams

$$\begin{array}{ccc} \mathcal{D}_1(j) \otimes V_0^{\otimes j} & \xrightarrow{\beta} & V_1 \\ \downarrow{\scriptstyle \mathrm{id}\otimes f_0^j} & & \downarrow{\scriptstyle f_1} \\ \mathcal{D}_1(j) \otimes (V_0')^{\otimes j} & \xrightarrow{\beta'} & V_1' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{D}_0(j) \otimes V_0^{\otimes (j-1)} \otimes V_1 & \xrightarrow{\lambda} & V_1 \\ \downarrow{\scriptstyle \mathrm{id}\otimes f_0^{(j-1)}\otimes f_1} & & \downarrow{\scriptstyle f_1} \\ \mathcal{D}_0(j) \otimes (V_0')^{\otimes (j-1)} \otimes V_1' & \xrightarrow{\lambda'} & V_1' \end{array} \quad \text{commute.}$$

5.8. **Definition.** Given a secondary operad $\mathcal{D}$, a $\pi_0\mathcal{D}$-algebra A and an A-module M we define the category $\mathrm{Cross}_{\mathcal{D}}(A, M)$ as follows. Objects are crossed modules over $\mathcal{D}$ with cokernel A and kernel M. Morphisms in $\mathrm{Cross}_{\mathcal{D}}(A, M)$ are morphisms of crossed modules in $\mathrm{Cross}_{\mathcal{D}}$ which induce the identity on A and M.

6. Operadic Cohomology

Let $\mathcal{D} = (\mathcal{D}_1 \to \mathcal{D}_0)$ be a secondary operad in the category of graded R-modules, and let A be a $\pi_0\mathcal{D}$-algebra. For any algebra $(\partial : B_1 \to B_0)$ over the secondary operad $(\mathcal{D}_1 \to \mathcal{D}_0)$, the cokernel $B_0/\partial B_1$ of ∂ is such an algebra. In addition let M be an A-module, i.e., there are structure maps $\nu : \pi_0\mathcal{D}(n) \otimes A^{\otimes n-1} \otimes M \to M$.

6.1. **Definition.** A *$\mathcal{D}$-extension of length $n, n \geqslant 2$, of A by M* is an exact sequence of graded R-modules

$$\mathcal{E} : 0 \longrightarrow M \longrightarrow C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\pi} A \longrightarrow 0$$

Here, the $C_{n-1}, \ldots, C_2$ are A-modules, the pair $\partial_1 : C_1 \to C_0$ is a crossed module over $\mathcal{D}$, and all maps in the sequence are maps of C_0-modules, where the C_0-module structure on $C_{n-1}, \ldots, C_2$ is induced by the quotient map $\pi : C_0 \to A$.

Note that a $\mathcal{D}$-extension of length 2 is an element of the category $\mathrm{Cross}_{\mathcal{D}}(A, M)$ as in 5.8. Note that the kernel of ∂_1 is an A-module: the module structure of C_1 given by $\lambda : \mathcal{D}_0(n) \otimes C_0^{\otimes n-1} \otimes C_1 \to C_1$ sends the kernel of ∂_1 to itself, because ∂_1 is a map of C_0-modules. Furthermore it is well-defined on $\pi_0\mathcal{D}$, because of $\partial\beta = \theta(\partial_n \otimes 1_{C_0^{\otimes n}})$ (compare 5.2 (4)). Thanks to relation 5.2 (6), λ is well-defined on the cokernel of π as well.

6.2. **Definition.** An *equivalence of two $\mathcal{D}$-extensions of A by M* is a sequence of maps $f_{n-1}, \ldots, f_0$

$$\begin{array}{ccccccccccccc} \mathcal{E} : 0 & \longrightarrow & M & \longrightarrow & C_{n-1} & \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\ \mathcal{E}' : 0 & \longrightarrow & M & \longrightarrow & C_{n-1}' & \xrightarrow{\partial_{n-1}'} \cdots \xrightarrow{\partial_2'} & C_1' & \xrightarrow{\partial_1'} & C_0' & \xrightarrow{\pi'} & A & \longrightarrow & 0, \end{array}$$

where the pair (f_1, f_0) is a map of $\mathcal{D}$-algebras, and the f_i are maps of $\pi_0\mathcal{D}_0$-A-modules for $2 \leq i \leq n-1$.

6.3. **Definition.** We now define for any $n \geqslant 2$, the *$\mathcal{D}$-cohomology of A with coefficients in M*, $H^{n+1}_{\mathcal{D}}(A, M)$, to be the set of equivalence classes of $\mathcal{D}$-extensions of A by M of length n.

We prove in section 8 below that such equivalence classes form a well-defined set which has the structure of an abelian group.

Work of MacLane and Whitehead ([MLW] and [ML3]) identified elements in the third cohomology of groups as equivalence classes of crossed modules of groups. Later, Huebschmann [Hue] extended this result and proved an isomorphism between the cohomology classes of groups and equivalence classes of certain crossed extensions of groups. In [BM], Hochschild cohomology of algebras over fields was shown to be isomorphic to equivalence classes of crossed extensions of algebras. In particular, result [BM, 4.3] together with our lemma 5.5 proves that operadic cohomology $H^*_{(0\to\mathsf{Ass})}(A, M)$ coincides with usual Hochschild cohomology of an associative k-algebra A (for k a field) with coefficients in an A-bimodule M.

For any map $\alpha : \mathcal{D} \to \mathcal{D}'$ of secondary operads, any crossed module over $\mathcal{D}'$ is a crossed module over $\mathcal{D}$ via α and similarly algebras over $\pi_0\mathcal{D}'$ are algebras over $\pi_0\mathcal{D}$. A $\mathcal{D}'$-extension of length n can be considered as a $\mathcal{D}$-extension of the same length.

6.4. **Definition.** For any map of secondary operads $\alpha : \mathcal{D} \to \mathcal{D}'$ we define the induced map in cohomology

$$\alpha^* : H^{n+1}_{\mathcal{D}'}(A, M) \longrightarrow H^{n+1}_{\mathcal{D}}(A, M)$$

on an equivalence class $[\mathcal{E}]_{\mathcal{D}'}$ of a $\mathcal{D}'$-extension of length n to be the equivalence class of $\mathcal{E}$ considered as a $\mathcal{D}$-extension:

$$\alpha^*([\mathcal{E}]_{\mathcal{D}'}) := [\mathcal{E}]_{\mathcal{D}}.$$

We point out, that α^* does not have to be an isomorphism, if α is a weak equivalence.

Recall that we denote the secondary cohomology functor from chain complexes to pairs of graded modules by D (see definition 2.7).

6.5. **Lemma.** *Let* Ass *denote the operad in chain complexes given by* Ass *concentrated in degree zero. A crossed module* $V = (\partial : V_1 \to V_0)$ *over* $D(\mathsf{Ass})$ *is the same as a crossed module over* $(0 \to \mathsf{Ass})$ *together with a map* $\bar{\beta} : \pi_0(V) \to \pi_1(V)$ *of degree* -1*. In particular, the natural map given by the inclusion of operads* $(0 \to \mathsf{Ass}) \subset D(\mathsf{Ass})$ *yields a surjective map from* $H^*_{D(\mathsf{Ass})}$ *to* $H^*_{(0\to\mathsf{Ass})}$ *for all graded associative algebras* A *and all* A*-modules* M *for all degrees* $* \geqslant 3$*. In degree three, we obtain a splitting*

$$H^3_{D(\mathsf{Ass})}(A, M) \cong H^3_{(0\to\mathsf{Ass})}(A, M) \oplus \mathrm{Hom}(A, s^{-1}M).$$

Proof. To prove this, note that $D_1(\mathsf{Ass})(j)$ is the R-module $R[\Sigma_j]$ concentrated in degree -1, $s^{-1}(R[\Sigma_j])$. The map $\bar{\beta}$ is induced by $\beta : s^{-1}(R[\Sigma_1]) \otimes V_0 \cong s^{-1}(R) \otimes V_0 \to V_1$. It is easy to check that $\beta \circ \partial = 0$ and $\partial\tilde{\beta} = 0$.

Conversely, given $\bar{\beta} : \pi_0(V) \to \pi_1(V)$ we consider $\beta : V_0 \to V_1$ induced by $\bar{\beta}$ and define the maps $\beta : D_1(\mathsf{Ass})(j) \otimes V_0^{\otimes j} \to V_1$ by $\beta(\sigma \otimes a_1 \otimes \ldots \otimes a_j) = \beta(\mathrm{sign}(\sigma) a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(j)})$.

Consider the cohomology corresponding to the secondary operad $D(\mathsf{Ass})$ of a graded associative algebra A with coefficients in M. An n-fold extension of A by M is

$$0 \to M \longrightarrow C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow A \to 0$$

where $\partial : C_1 \to C_0$ is a crossed module over $D(\mathsf{Ass})$. This projects to an extension with respect to $(0 \to \mathsf{Ass})$, but in $H^*_{D(\mathsf{Ass})}$ maps between extensions have to respect $\beta : C_0 \to C_1$ of degree -1, that is they have to preserve $\bar\beta : A \to s^{-1}\mathrm{coker}(\partial)$. In degree three, maps between $D(\mathsf{Ass})$-crossed extensions are the identity on A and M, therefore β is automatically preserved and splits off as an additional datum. □

A similar result holds for the operad Com:

6.6. **Lemma.** *Let* Com *denote the operad in chain complexes given by* Com *concentrated in degree zero. A crossed module* $V = (\partial : V_1 \to V_0)$ *over* $D(\mathsf{Com})$ *is the same as a crossed module over* $(0 \to \mathsf{Com})$ *together with a map* $\bar\beta : \pi_0(V) \to \pi_1(V)$ *of degree* -1. *In particular, the operadic cohomology* $H^*_{D(\mathsf{Com})}$ *has a canonical projection map to* $H^*_{(0\to\mathsf{Com})}$ *for* $* \geqslant 3$ *and we have a splitting*

$$H^3_{D(\mathsf{Com})}(A, M) \cong H^3_{(0\to\mathsf{Com})}(A, M) \oplus \mathrm{Hom}(A, s^{-1}M).$$

6.7. *Remark.* If $\mathcal{O}$ is an arbitrary operad in the category of R-modules, then we get a similar projection map

$$H^*_{D(\mathcal{O})}(A, M) \twoheadrightarrow H^*_{(0\to\mathcal{O})}(A, M)$$

for any $\mathcal{O}$-algebra A and any A-module M, because any extension with respect to $D(\mathcal{O})$ is a $(0 \to \mathcal{O})$-extension and the choice $\beta = 0$ is allowed. But the contribution to the $D(\mathcal{O})$ structure from $s^{-1}\mathcal{O}$ in $D_1(\mathcal{O})$ might be bigger than in the cases above and we cannot expect to get a splitting for the third cohomology group in general.

The functor D sends a differential (commutative) graded algebra A_* to $\partial : D_1(A_*) \to D_0(A_*)$ with the homology of A_* as cokernel and the shifted homology of A_* as kernel. Therefore, every differential (commutative) graded algebra A_* gives a canonical cohomology class in $H^3_{D(\mathsf{Ass})}$ (resp. $H^3_{D(\mathsf{Com})}$)

$$0 \to s^{-1}H_*(A_*) \longrightarrow D_1(A_*) \longrightarrow D_0(A_*) \longrightarrow H_*(A_*) \to 0. \tag{6.8}$$

Here, the map $\bar\beta$ is the natural shift map from $H_*(A_*)$ to $s^{-1}H_*(A_*)$. The $H_*(A_*)$-bimodule structure of $s^{-1}H_*(A_*)$ is given by

$$a.s^{-1}(x).b = (-1)^{|a|}s^{-1}(axb).$$

For any differential graded algebra A_* the cohomology class represented by $D(A_*)$ was studied in [BM, 3.6] and termed the *characteristic class* of A_*; the corresponding class in Hochschild cohomology of $H_*(A_*)$ was studied in [BD]. We now extend the concept of characteristic classes to algebras over an E_∞ operad.

7. Mandell's operad and characteristic classes

In [M, §1], Mandell uses a specific E_∞ operad which acts on cochain complexes $C^*(X)$ of topological spaces. In the following we call a unital operad $\mathcal{O}$ an E_∞ operad, if each $\mathcal{O}(n)$ is free over $R[\Sigma_n]$ and if the augmentations $\mathcal{O}(n) \otimes \mathcal{O}(0)^{\otimes n} \to \mathcal{O}(0)$ are quasi-isomorphisms.

Mandell takes the Eilenberg-Zilber operad $\mathcal{Z}$ from [HS], takes its good truncation to $\tilde{\mathcal{Z}}$ (i.e., $\tilde{\mathcal{Z}}_n = \mathcal{Z}_n$ for negative n, $\tilde{\mathcal{Z}}_0$ are the cycles of $\mathcal{Z}_0$ and all the other $\tilde{\mathcal{Z}}_n$ are trivial) and tensors it with an arbitrary E_∞ operad $\mathcal{O}$. Let us denote the resulting operad by $\mathcal{M}(\infty)$ – although $\mathcal{M}(\infty)$ depends of course on the choice of the operad $\mathcal{O}$. The Eilenberg-Zilber operad is the endomorphism operad of the normalized cochain functor, i.e., $\mathcal{Z}(n)$ is the cochain complex of natural transformations from $N^*(-)^{\otimes n}$ to $N^*(-)$. As the normalized cochains $N^*(X)$ are a direct summand of all cochains $C^*(X)$, one obtains an action of $\mathcal{Z}$ on $C^*(X)$.

7.1. **Proposition.** *There is a choice of an E_∞ operad $\mathcal{O}$, such that the resulting Mandell operad $\mathcal{M}(\infty)$ is part of a commutative diagram*

$$\begin{array}{ccc} & \mathcal{M}(\infty) & \\ \nearrow & & \searrow^{\sim} \\ \mathsf{Ass} & \longrightarrow & \mathsf{Com}. \end{array}$$

Proof. In degree zero, the Eilenberg-Zilber operad contains the Alexander-Whitney transformations AW^n from the n-fold tensor product of normalized cochain complexes $N^*(-)^{\otimes n}$ to the normalized cochain complex $N^*(-)$. We define a map from $\mathsf{Ass}(n)$ to $\mathcal{Z}(n)$ by sending a permutation $\sigma \in \Sigma_n$ to $AW^n \circ \sigma$. We have to check that the composition in the operad of associative graded algebras $\gamma(\sigma; \tau_1, \ldots, \tau_n)$ of elements $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{k_i}$ for $1 \leqslant i \leqslant n$ gives a well-defined element in $\mathcal{Z}(\sum k_i)$. The image of $\gamma(\sigma; \tau_1, \ldots, \tau_n)$ in $\mathcal{Z}(\sum k_i)$ is $AW^{\sum k_i} \circ \gamma(\sigma; \tau_1, \ldots, \tau_n)$ which is $AW^{\sum k_i} \circ \sigma(k_1, \ldots, k_n) \circ (\tau_1, \ldots, \tau_n)$ where $\sigma(k_1, \ldots, k_n)$ denotes the block permutation in $\Sigma_{k_1+\cdots+k_n}$ corresponding to σ and $(\tau_1, \ldots, \tau_n)$ is the block sum of the τ_i. Thus we have to show that this transformation coincides with $\gamma(AW^n \circ \sigma; AW^{k_1} \circ \tau_{k_1}, \ldots, AW^{k_n} \circ \tau_{k_n})$. Using the symmetry isomorphism in the category of cosimplicial modules and cochain complexes, we see, that $(AW^n \circ \sigma) \circ AW^{k_1} \otimes \cdots \otimes AW^{k_n}$ agrees with $AW^n \circ AW^{k_{\sigma^{-1}(1)}} \otimes \cdots \otimes AW^{k_{\sigma^{-1}(n)}}$ and therefore the claim follows. As each AW^n gives rise to cochain maps, the image of the map from Ass lies in $\tilde{\mathcal{Z}}$.

Now choose any E_∞ operad $\mathcal{O}$ which allows a map from Ass to $\mathcal{O}$. For instance one can take the cochain complex of the Barratt-Eccles operad, which has the classifying spaces of the translation category of the symmetric groups as building blocks. The operad Ass has a diagonal in each part $\Delta : \mathsf{Ass}(n) \longrightarrow \mathsf{Ass}(n) \otimes \mathsf{Ass}(n)$ which is a map of operads (that turns Ass into a so-called Hopf-operad). The map Δ is the linear extension of the group-like diagonal, i.e., $\Delta(\sigma) = \sigma \otimes \sigma$.

We obtain the desired map from Ass to the operad $\mathcal{M}(\infty)$ by composing the two maps from Ass to the operads $\tilde{\mathcal{Z}}$ resp. $\mathcal{O}$:

$$\mathsf{Ass} \xrightarrow{\Delta} \mathsf{Ass} \otimes \mathsf{Ass} \longrightarrow \tilde{\mathcal{Z}} \otimes \mathcal{O} = \mathcal{M}(\infty).$$

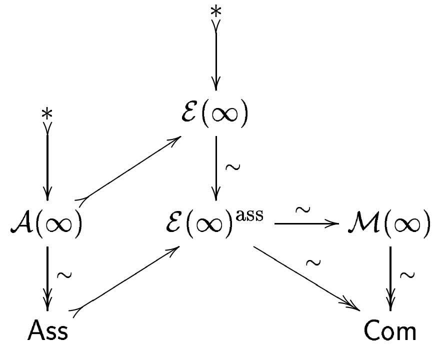
□

In [S, Theorem 3.2] Spitzweck defines a semi model category structure for operads in the category of (unbounded) chain complexes which we will use from now on. Note, that fibrations in his structure are given by surjective maps, in particular, every operad is fibrant. For people who are not familiar with semi model category structures we note, that cofibrations in such a structure have the left lifting property with respect to fibrations which are weak equivalences (see [S, 2.3]) and that any morphism can be factored as a cofibration followed by a fibration which is as well a weak equivalence – these are the only facts which we will use about semi model structures.

Let $\mathcal{A}(\infty)$ be a cofibrant replacement of the operad Ass, $* \rightarrowtail \mathcal{A}(\infty) \xrightarrow{\sim} \mathsf{Ass}$, and consider a factorization of the composite $f : \mathcal{A}(\infty) \to \mathsf{Ass} \to \mathsf{Com}$ into a cofibration followed by an acyclic fibration $\mathcal{A}(\infty) \rightarrowtail \mathcal{E}(\infty) \xrightarrow{\sim} \mathsf{Com}$. Therefore $\mathcal{E}(\infty)$ is a cofibrant replacement of the operad Com. Furthermore, the canonical map from Ass to Com has such a factorization as well

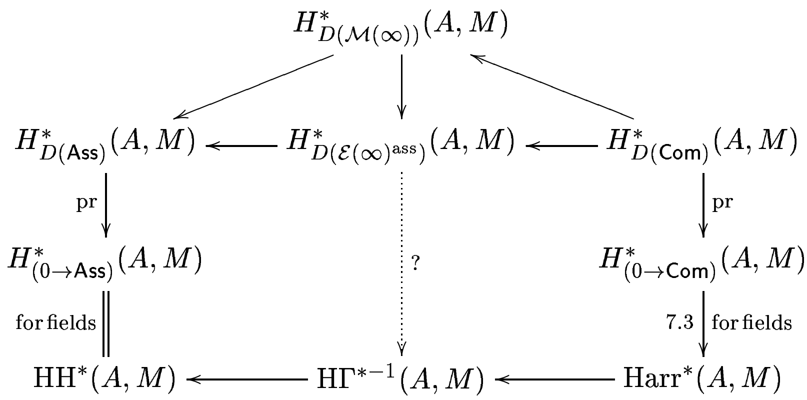
$$\mathsf{Ass} \rightarrowtail \mathcal{E}(\infty)^{\text{ass}} \xrightarrow{\sim} \mathsf{Com}.$$

Note that the homotopy types of $\mathcal{A}(\infty)$ and $\mathcal{E}(\infty)$ are well-defined, and the homotopy type of $\mathcal{E}(\infty)^{\text{ass}}$ is well-defined in the category of operads under Ass. The augmentation map from Mandell's operad $\mathcal{M}(\infty)$ to Com is surjective and a quasi-isomorphism, hence it is an acyclic fibration. Therefore there exists a map from $\mathcal{E}(\infty)^{\text{ass}}$ to $\mathcal{M}(\infty)$ which lifts the map from $\mathcal{E}(\infty)^{\text{ass}}$ to Com which is a weak equivalence. Altogether, we obtain the following diagram of operads and operad maps



Applying the functor D from chain complexes to pairs of modules, we obtain a similar diagram of secondary operads which leads to maps between cohomology groups.

Let A be a graded commutative algebra and let M be an A-module. There is a cohomology theory for (graded) commutative algebras, namely Gamma cohomology, $\mathrm{H}\Gamma^*$, defined by A. Robinson and S. Whitehouse in [RW]. For that cohomology theory one obtains a chain of maps from Harrison cohomology, $\mathrm{Harr}^*(A, M)$, to Gamma cohomology and further on to Hochschild cohomology as in the bottom row of the following diagram.



As the diagram indicates there might be a comparison map from the cohomology with respect to the secondary operad $D(\mathcal{E}(\infty)^{\text{ass}})$ to Gamma cohomology.

7.2. **Definition.** Let X be an arbitrary space. The cochains $C^*(X)$ on X are an algebra over Mandell's operad; therefore $D(C^*(X))$ is a crossed module over $D(\mathcal{M}(\infty))$ with cokernel $H^*(X)$ and kernel $s^{-1}H^*(X)$. We call the resulting cohomology class in $H^3_{D(\mathcal{M}(\infty))}(H^*(X), s^{-1}H^*(X))$ given by the extension

$$0 \to s^{-1}H^*(X) \longrightarrow D_1(C^*(X)) \longrightarrow D_0(C^*(X)) \longrightarrow H^*(X) \to 0$$

the *characteristic class of the* $\mathcal{M}(\infty)$*-algebra* $C^*(X)$.

In general, for an arbitrary algebra A^* over the operad $\mathcal{M}(\infty)$, one can define the *characteristic class of* A^* to be the canonical extension

$$0 \to s^{-1}H^*(A^*) \longrightarrow D_1(A^*) \longrightarrow D_0(A^*) \longrightarrow H^*(A^*) \to 0.$$

Then for any $\mathcal{M}(\infty)$-algebra the characteristic class of A^* lifts the characteristic class of A^* as defined in [BM, 3.5], where one views A^* as a differential associative graded algebra, see (6.8) above.

For the secondary operad $(0 \to \mathsf{Com})$ one gets a map from $H^*_{(0\to\mathsf{Com})}$ to Harrison cohomology, Harr* as follows.

7.3. **Proposition.** *Let k be a field, A a graded commutative algebra and M an A-module. Then there is a map*

$$H^*_{(0\to\mathsf{Com})}(A, M) \longrightarrow \mathrm{Harr}^*(A, M).$$

Proof. Let

$$0 \longrightarrow M \stackrel{i}{\longrightarrow} A_1 \stackrel{\partial}{\longrightarrow} A_0 \stackrel{\pi}{\longrightarrow} A \to 0$$

be a crossed extension of A by M. We can choose a k-linear section s of π, $\pi \circ s = \mathrm{id}_A$, and a k-linear section t from the image of ∂ (which is the kernel of π) to A_1. As π is multiplicative, we obtain that the difference $s(a)s(b) - s(ab)$ is in the kernel of π. We define

$$g : A \otimes A \to A_1, \quad g(a, b) := t(s(a)s(b) - s(ab)).$$

Note, that g is symmetric in both arguments because A and A_0 are graded commutative algebras, i.e.,

$$g(b, a) = t(s(b)s(a) - s(ba)) = (-1)^{|a||b|}t(s(a)s(b) - s(ab)) = (-1)^{|a||b|}g(a, b).$$

With the help of the section s and the map g we can now associate a Hochschild cocycle to the crossed extension above: let $\theta : A^{\otimes 3} \to A_1$ be

$$\theta(a, b, c) := s(a)g(b, c) - g(ab, c) + g(a, bc) - g(a, b)s(c).$$

In fact, $\theta(a, b, c)$ is annihilated by ∂, so θ is actually a map to M. Furthermore, the symmetry of g translates to a Harrison condition for θ: we claim that

$$\theta(a, b, c) - (-1)^{|b||c|}\theta(a, c, b) + (-1)^{|c|(|a|+|b|)}\theta(c, a, b) = 0.$$

The proof of this fact is a direct calculation. Therefore, this cocylce θ is in fact a cocycle for the third Harrison cohomology (compare [Ha, p 192] adapted to the graded case).

In order to prove that the cohomology class of this cocycle does not depend on the choice of the sections we assume there is a different section s' of π. As the difference map $s - s'$ is clearly in the kernel of π, there is a map $h : A \to A_1$ with $\partial h = s - s'$. Let $g' : A \otimes A \to M$ be the analogue of g for the section s and let θ' be the corresponding cocycle. Then, similar to the case of usual Hochschild cohomology, one can prove that that the difference $\theta - \theta'$ can be expressed as the coboundary of $g - g' - \psi$ where $\psi(a, b) = s(a)h(b) - h(ab) + h(a)s(b) - h(a)\partial h(b)$. We already saw that g and therefore g' is symmetric; similarly the map ψ satisfies $\psi(a, b) = (-1)^{|a||b|}\psi(b, a)$ because $A_1 \to A_0$ is a crossed module, A is graded commutative and A_1 is a graded A_0-module. Therefore up to a Harrison coboundary, the cocycle θ does not depend on the choice of the section s.

In a similar fashion one shows, that an equivalence of crossed extensions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & A_1 & \xrightarrow{\partial} & A_0 & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{i'} & A_1' & \xrightarrow{\partial'} & A_0' & \xrightarrow{\pi'} & A & \longrightarrow & 0 \end{array}$$

yields equivalent cocycles. Therefore we obtain a well-defined map from equivalence classes of extensions to the third Harrison cohomology of A with coefficients in M.

Similar arguments to those in [BM] show that short exact sequences of A-modules $0 \to M' \to M \to M'' \to 0$ lead to long exact sequences of cohomology groups $H^*_{(0\to\mathsf{Com})}(A,-)$. In addition, it is clear that cohomology with respect to the secondary operad $(0 \to \mathsf{Com})$ vanishes for every injective module M, because then every extension $0 \to M \to C_{n-1} \to \ldots \to C_1 \to C_0 \to A \to 0$ can be seen to be equivalent to the extension

$$0 \to M \to M \to 0 \to \ldots \to 0 \to A \to A \to 0.$$

Thus inductively, if we have a map up to degree $n \geqslant 3$ and we embed the A-module M into an injective module I, the long exact sequence associated to $0 \to M \to I \to I/M \to 0$ gives us an isomorphism between $H^{n+1}_{(0\to\mathsf{Com})}(A,M)$ and $H^{n}_{(0\to\mathsf{Com})}(A,I/M)$ which yields the desired map

$$H^{n+1}_{(0\to\mathsf{Com})}(A,M) \cong H^{n}_{(0\to\mathsf{Com})}(A,I/M) \to \mathrm{Harr}^n(A,I/M) \to \mathrm{Harr}^{n+1}(A,M).$$

□

7.4. *Remark.* Note that the map constructed above cannot be an isomorphism in general, because Harrison cohomology does not vanish on free (graded) commutative algebras. Barr proves in [Ba, 4.4] (following an example of André) that for a field k of characteristic p and all n of the form $n = 2p^m$ the Harrison cohomology groups $\mathrm{Harr}^n(k[x],k)$ do not vanish, whereas we will later show in (8.14) that $H^*_{(0\to\mathsf{Com})}$ vanishes on free algebras.

But in characteristic zero, Harrison cohomology agrees with André-Quillen cohomology up to a degree shift. Here both $H^*_{(0\to\mathsf{Com})}$ and André-Quillen cohomology vanish on free graded commutative algebras and they are connected by the map above at least if the ground ring is a field.

8. The Homotopy Category of Crossed Modules

In the following, $\mathcal{D}$ will always be a secondary operad. We now redefine operadic cohomology by using cofibrant resolutions.

8.1. **Definition.** A morphism $f: V = (\partial : V_1 \to V_0) \to V' = (\partial' : V_1' \to V_0')$ of crossed modules over $\mathcal{D}$ is a *weak equivalence* if f induces isomorphisms on kernel and cokernel:

$$\pi_0(f):\ \pi_0(V) \xrightarrow{\simeq} \pi_0(V') \qquad \text{and} \qquad \pi_1(f):\ \pi_1(V) \xrightarrow{\simeq} \pi_1(V').$$

Let $\mathcal{H}o(\mathrm{Cross}_{\mathcal{D}})$ be the localization of the category of crossed modules over $\mathcal{D}$ with respect to weak equivalences. The results in this section imply that this localized category exists. We need to talk about free objects.

8.2. **Definition.** Let $\mathcal{C}$ be an operad in $\mathcal{M}od$. A $\mathcal{C}$-algebra A is called *free (with basis W)* if a graded module $W \subseteq A$ is given, such that $\mathrm{Hom}_{\mathcal{C}\text{-}alg}(A,B) = \mathrm{Hom}_{R\text{-}mod}(W,B)$ for every $\mathcal{C}$-algebra B.

8.3. **Definition.** A crossed module over $\mathcal{D}$, $V = (\partial : V_1 \to V_0)$, is called *free with basis* (W, d) if $d : W \to V_0$ is an R-linear map from the R-module W to V_0 and the set $\mathrm{Hom}_{\mathrm{Cross}_\mathcal{D}}(V, V')$ of morphisms of crossed modules from V to V' is in 1-1 correspondence with the set of pairs (f_0, f_1) of maps such that $f_1 : W \to V_1'$ is R-linear, $f_0 : V_0 \to V_0'$ is a map of $\mathcal{D}_0$-algebras and the diagram

$$\begin{array}{ccc} W & \xrightarrow{d} & V_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ V_1' & \xrightarrow{\partial'} & V_0' \end{array}$$

commutes. A free crossed module $\partial : V_1 \to V_0$ is called *totally free* if in addition V_0 is free as a $\mathcal{D}_0$-algebra.

8.4. **Definition.** Given a secondary operad $\mathcal{D}$ we say that $\mathcal{D}$ *satisfies the freeness condition* if for every R-linear map $d : W \to V_0$ from an R-module W to V_0 there exists a free crossed module over $\mathcal{D}$ with basis (W, d).

Freyd's adjoint functor theorem (see for instance [ML1, V.6, Theorem 2]) ensures the existence of free crossed modules. We will give explicit constructions in two particular cases.

8.5. **Example.** An explicit construction for the free crossed module over the secondary operad $(0 \to \mathsf{Ass})$ is given in [B2, p.58].

8.6. **Example.** The secondary operad $(0 \to \mathsf{Com})$ satisfies the freeness condition and the free crossed module over $(0 \to \mathsf{Com})$ is constructed as follows. Let A be a graded commutative k-algebra and let W be a graded k-vector space. Suppose there is a k-linear map $d : W \to A$ of degree zero. The free crossed module associated to A and $d : W \to A$ is given as follows: Consider the free symmetric A-module $A \otimes W$ generated by W. Here the symmetric bimodule structure is given by $b.(a \otimes w) := ba \otimes w$ and $(a \otimes w).b := (-1)^{|b||w|} ab \otimes w = (-1)^{|b|(|a|+|w|)} ba \otimes w$ for a, b in A and w in W.

Let $\varphi : A \otimes W \longrightarrow A$ be defined as

$$\varphi(a \otimes w) := ad(w)$$

for a in A and w in W. We want to turn $A \otimes W$ into a crossed module. To this end, we define $d' : A \otimes W \otimes A \otimes W \to A \otimes W$ as $d'(a \otimes v \otimes b \otimes w) := advb \otimes w - (-1)^{|w||b||v|} adwb \otimes v$.

We define $A_1 = A \otimes W / \mathrm{im}(d')$ and $\partial : A_1 \to A$ to be $\partial([a \otimes v]) := adv$. Then ∂ satisfies

$$\partial([a \otimes v]).[b \otimes w] = adv.[b \otimes w] = [advb \otimes w] = (-1)^{|w||b||v|}[adwb \otimes v] = [a \otimes v].bdw.$$

The pair (A_1, A, ∂) is a crossed module and every map of crossed modules from (A_1, A, ∂) to another crossed module $B_1 \to B_0$ over $0 \to \mathsf{Com}$ corresponds uniquely to a linear map from W to B_1 and a map of Com-algebras from A to B_0.

8.7. **Theorem.** *If the secondary operad $\mathcal{D}$ satisfies the freeness condition then the category $\mathcal{H}o(\mathrm{Cross}_\mathcal{D})$ exists.*

We show the existence of the localized category by using the homotopy category of a cofibration category as in [BHA].

8.8. **Definition.** A *$\mathcal{D}$-cross chain complex* is a sequence of graded R-modules

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\pi} A \longrightarrow 0$$

where $\partial_{i-1}\partial_i = 0$ for all $i \geqslant 2$, $\partial_1 : C_1 \to C_0$ is a $\mathcal{D}$-algebra with cokernel A, the $C_i, i \geqslant 2$ are A-modules and the maps ∂_i are maps of A-modules for $i \geqslant 2$.

8.9. **Definition.** A *morphism of $\mathcal{D}$-cross chain complexes* C_* and C'_* is a sequence of maps $f_n : C_n \to C'_n$ such that

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\pi} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \bar{f}_0} & & \\
\cdots & \longrightarrow & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \xrightarrow{\partial'_{n-1}} & \cdots & \xrightarrow{\partial'_2} & C'_1 & \xrightarrow{\partial'_1} & C'_0 & \xrightarrow{\pi} & A' & \longrightarrow & 0
\end{array}$$

commutes. Here the f_i are $\bar{f}_0$-equivariant in the sense that the diagram

$$\begin{array}{ccc}
\pi_0\mathcal{D}(n) \otimes A^{\otimes n-1} \otimes C_i & \longrightarrow & C_i \\
\downarrow{\scriptstyle \mathrm{id}\otimes \bar{f}_0^{\otimes n-1} \otimes f_i} & & \downarrow{\scriptstyle f_i} \\
\pi_0\mathcal{D}(n) \otimes A'^{\otimes n-1} \otimes C'_i & \longrightarrow & C'_i
\end{array}$$

commutes and the pair (f_1, f_0) is a map of $\mathcal{D}$-algebras.

8.10. **Definition.** A morphism f of $\mathcal{D}$-cross chain complexes is called a *weak equivalence* if it induces an isomorphism on the corresponding homology groups.

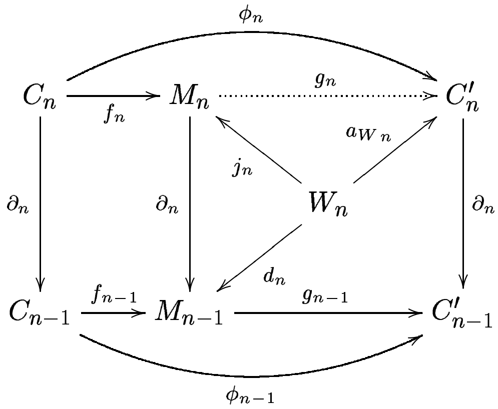
The cofibrations will consist of morphisms which are built by adding free generators. For a $\mathcal{D}$-cross chain complex C_* let $C_*^{(n)}$ denote its truncation at degree n, that is

$$C_\ell^{(n)} := \begin{cases} C_\ell & \text{if} \quad \ell \leqslant n \\ 0 & \text{for} \quad \ell > n. \end{cases}$$

8.11. **Definition.** A morphism $f : C_* \to M_*$ of $\mathcal{D}$-cross chain complexes is called a *cofibration* if it is a free extension in each degree, i.e., if for any $n \geqslant 0$ there is a free graded R-module W_n together with maps

$$\begin{array}{ccc}
W_n & \xrightarrow{j_n} & M_n \\
{\scriptstyle d_n}\searrow & & \swarrow{\scriptstyle \partial_n} \\
 & M_{n-1} &
\end{array}$$

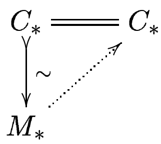
such that for any map from the n-truncation of C_* to some $\mathcal{D}$-cross chain complex C'_*, $\phi : C_*^{(n)} \to C'_*$, for any map g from the $(n-1)$-truncation of M to C'_*, $g : M_*^{(n-1)} \to C'_*$, and for any choice of a R-linear map a_{W_n} from W_n to C'_n which is compatible with the structure maps up to degree $n-1$ and which extends g on W_n there is a unique map of $\mathcal{D}$-cross chain complexes g from $M_*^{(n)}$ to C'_* which extends a_{W_n} and which is compatible with ϕ up to degree n:



So cofibrations $i : C_* \rightarrowtail M_*$ can be characterized in such a way that there are free R-modules W_n in every degree and M_* is freely built out of C_* by the W_n.

Note that a $\mathcal{D}$-cross chain complex $\cdots \to C_n \to \cdots \to C_1 \to C_0 \to A$ which consists of a totally free $\mathcal{D}$-cross module $C_1 \to C_0$ and free A-modules $C_n, n \geqslant 2$ is a cofibrant object. In analogy to 8.3 we call such complexes *totally free.*

We claim that these data suffice to get the desired homotopy category. We will mark cofibrations with a tailed arrow $\rightarrowtail$ and $\xrightarrow{\sim}$ will denote the weak equivalences. Recall that an object C_* is 'fibrant' in a cofibration category, if any trivial cofibration $i : C_* \to M_*$ has a retract:



8.12. **Theorem.** *The category of $\mathcal{D}$-cross chain complexes together with the classes of weak equivalences and cofibrations as above is a cofibration category.*

Proof. We have to check the axioms (C1) – (C4) of [BHA, p.6]. For (C1) one has to show that the weak equivalences fulfill the 2-out-of-3-property, that isomorphisms are weak equivalences and that cofibrations are closed under composition. But these properties are obvious in our setting.

We will first check (C3), i.e., we have to prove that every map $f : C_* \to C'_*$ of $\mathcal{D}$-cross chain complexes possesses a factorization as $f = q \circ i$ where i is a cofibration and q is a weak equivalence. Consider $f_0 : C_0 \to C'_0 \twoheadrightarrow \pi_0 C'_*$ and choose enough $\mathcal{D}_0$-algebra generators in a free R-module W_0 and build $C_0 \coprod \mathcal{D}_0(W_0) =: M_0$ together with a map $M_0 \xrightarrow{q_0} C'_0$ which becomes surjective after projecting down to $\pi_0 C'_*$. In order to continue the construction we have to kill any superfluous elements in π_0 and have to find a map of $\mathcal{D}$-crossed modules in the next step. Consider a free R-module W'_1 together with a R-linear map ∂'_1 to M_0. We extend this map over the coproduct with C_1 and the free $\mathcal{D}$-crossed module in degree 1, $\tilde{W}_1$ generated by W'_1. We choose W'_1 and a map d_1 in such a way that the image of that coproduct under d_1 is exactly the kernel of the composition $M_0 \to C'_0 \to \pi_0 C'_*$. Thus q_0 sends the image of d_1 to the image of ∂'_1 and we can choose a map $q'_1 : \tilde{W}_1 \to C'_1$ such that the diagram

$$\begin{array}{ccccc} C_1 & \xrightarrow{i_1} & \tilde{W}_1 & \overset{q'_1}{\dashrightarrow} & C'_1 \\ \downarrow{\scriptstyle \partial_1} & & \vdots{\scriptstyle d_1} & & \downarrow{\scriptstyle \partial'_1} \\ C_0 & \xrightarrow{i_0} & M_0 & \xrightarrow{q_0} & C'_0 \end{array}$$

commutes and such that the composition $q'_1 \circ i_1$ is f_1. But still the restriction of q'_1 to the kernel of d_1 to the kernel of ∂'_1 does not have to be surjective. So we have to correct the middle term $\tilde{W}_1$ and have to choose additional generators in some free R-module U_1, so that we can map these elements surjectively to the kernel of ∂'_1. In order to get a well-defined map we send them to zero under the vertical map d_1. Along the same line of argumentation, the factorization in higher degrees can be proven.

Instead of proving (C4), namely that every object has a fibrant model, we prove that every object is indeed fibrant, and in addition to the retract r to $i : C_* \xrightarrow{\sim} M_*$ we will construct a homotopy $\alpha : i \circ r \simeq \mathrm{id}_{M_*}$.

Let $(M_*^{(n)}|C_*^{>n+1})$ be the subcomplex of M_* which consists of M_i in degrees $i \leq n$ and is C_i for $i > n$ and let ι_n be the inclusion of $(M_*^{(n)}|C_*^{>n+1})$ in M_*

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ C_{n+1} & \xrightarrow{\iota_{n+1}=i_{n+1}} & M_{n+1} \\ {\scriptstyle \partial_{n+1}}\downarrow & & \downarrow{\scriptstyle \partial_{n+1}} \\ M_n & \xrightarrow{\iota_n=\mathrm{id}} & M_n \\ {\scriptstyle \partial_n}\downarrow & & \downarrow{\scriptstyle \partial_n} \\ \vdots & & \vdots \end{array}$$

Note that $i : C_* \to M_*$ gives rise to a morphism $C_* \to (M_*^{(n)}|C_*^{>n+1})$ and by abuse of notation we will call this map i again. We construct retractions

$$C_* \underset{r}{\overset{i}{\rightleftarrows}} (M_*^{(n)}|C_*^{>n+1}) \xrightarrow{\iota} M_*$$

and retracting homotopies $\alpha_n : M_n \to M_{n+1}$.

We start in degree zero. The zeroth part M_0 is built out of C_0 by adding a free algebra on some free module W_0. We have to define the retraction r_0 and the retracting homotopy α_0 on this free module. Consider

$$\begin{array}{ccc} & & W_0 \\ & & \downarrow{\scriptstyle j_0} \\ C_0 & \xrightarrow{i} & M_0 \end{array}$$

As every element $j_0(w_0)$ is a cycle in degree zero, we can choose linear maps $\chi_0 : W_0 \to C_0$ and $\rho_0 : W_0 \to M_1$ such that $j_0(w_0)$ is hit by $i \circ \chi_0$ up to a boundary, i.e., we can fill in the above diagram to obtain:

$$\begin{array}{ccc} & M_1 & \\ & \uparrow{\scriptstyle \rho_0} & \searrow{\scriptstyle \partial_1} \\ & W_0 & \\ {\scriptstyle \chi_0}\swarrow & \downarrow{\scriptstyle j_0} & \\ C_0 & \xrightarrow[i]{} & M_0 \end{array} \qquad \text{with} \quad i \circ \chi_0 + \partial_1 \circ \rho = j_0.$$

Assume such retractions together with a deformation homotopy α exist on $(M_*^{(n)}|C_*^{>n+1})$ up to a fixed degree n. In order to construct an extension up to degree $n+1$, note that r and α satisfy

$$\partial\alpha + \alpha\partial = \iota - i \circ r$$

and therefore

$$ir\partial = \iota\partial - \partial\alpha\partial = \partial(\iota - \alpha\partial).$$

As i is a weak equivalence, there is an element that hits this boundary. So we can choose a map $\psi : W_{n+1} \to C_{n+1}$ with $\partial\psi = r\partial$. Note that $\partial(\iota - \alpha\partial - i\psi)$ is zero, i.e., $\iota - \alpha\partial - i\psi$ sends the module W_{n+1} to cycles. As i is a weak equivalence, these cycles have to be in the image of i up to a boundary. So there are maps $\chi_{n+1} : W_{n+1} \to C_{n+1}$ and $\rho_{n+1} : W_{n+1} \to M_{n+2}$ such that $i\chi_{n+1} + \partial\rho_{n+1} = \iota - i\psi - \alpha\partial$. Now we can define our retraction in degree $n+1$ as $\psi + \chi$ on the module W_{n+1} and α can be extended to degree $n+1$ by defining it to be ρ on W_{n+1}. This gives inductively the desired retraction and homotopy.

Finally we have to prove (C2), (b), namely that for any cofibration $i : C_* \rightarrowtail M_*$ and any morphism $f : C_* \to C'_*$ the push-out

$$\begin{array}{ccc} C_* & \xrightarrow{f} & C'_* \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ M_* & \xrightarrow{\bar{f}} & M_* \coprod_{C_*} C'_* \end{array}$$

exists, and $\bar{i}$ is a weak equivalence if i is one.

To see that such push-outs exists, consider a cofibration $i : C_* \rightarrowtail M_*$; so M_* is built out of C_* and some free R-modules W_n. Each W_n has a structure map $d_n : W_n \to M_{n-1}$. Take the image of the W_n under f to get the corresponding modules for the push-out: We use $f_{n-1} \circ d_n$ to map W_n to $M_* \coprod_{C_*} {C'_*}_{n-1}$. The map $\bar{f}$ is induced by f on C_* and the identity on W_n.

If $i : C_* \rightarrowtail M_*$ is in addition a weak equivalence, then by the preceeding argument it is a strong deformation retract with retraction r and homotopy α relative to C_*. We define a retract for $\bar{i}$ to be $f \circ r$ and the homotopy relative to C'_* to be $\bar{\alpha} = \bar{f} \circ \alpha$ on the generators of W_n. □

Having a cofibration category at hand, we can now define operadic cohomology of a $\pi_0\mathcal{D}$-algebra A with coefficients in an A-module M to be the cohomology of the homomorphisms from a cofibrant resolution of A into M, $H^*\mathrm{Hom}_A(C_*, M)$ for $* > 2$. Note that an appropriate notion of a homotopy in our context is the following.

8.13. **Definition.** Given two morphisms f and g of $\mathcal{D}$-cross chain complexes, a *homotopy* between f and g, $\alpha : f \simeq g$, is a sequence of maps $\alpha_n : C_n \to C'_{n+1}$ such that

$$\begin{aligned} g_0 - f_0 &= \partial'_1 \alpha_0 \\ g_n - f_n &= \partial'_n \alpha_n + \alpha_{n-1}\partial_n \end{aligned}$$

where all the α_n are morphisms of $\pi_0\mathcal{D}_0$-A-modules for $n \geqslant 1$ and where α_0 is a derivation in the following operadic sense: The diagram

$$\begin{array}{ccc} \mathcal{D}_0(n) \otimes C_0^{\otimes n} & \xrightarrow{\theta} & C_0 \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\alpha_0} \\ \mathcal{D}_0(n) \otimes {C'_0}^{\otimes n-1} \otimes C'_1 & \xrightarrow{\lambda} & C'_0 \end{array}$$

has to commute where Φ is the sum $\sum_{i=0}^{n-1} \mathrm{id}_{\mathcal{D}_0(n)} \otimes t_n^{-i} \circ (\mathrm{id}_{\mathcal{D}_0(n)} \otimes f_0^{\otimes n-1} \otimes \alpha_0) \circ \mathrm{id}_{\mathcal{D}_0(n)} \otimes t_n^i$ with t_n denoting the cyclic permutation of n letters.

One can see that the two definitions coincide by mimicking the usual proof in homological algebra, that Yoneda extension groups and Ext-groups in terms of projective resolutions coincide (compare for instance [ML2, III.6]). Let us just indicate, how the corresponding maps are defined: If

$$\ldots \to C_{n+1} \to C_n \to \ldots \to C_1 \to C_0$$

is a cofibrant replacement of A, then by definition the above sequence prolonged to A is exact. Now assume a map $f : C_n \to M$ is given with $f \circ \partial_{n+1} = 0$. We can truncate the resolution at the stage n and consider

$$0 \to M \to f_*(C_{n-1}) \stackrel{\bar{\partial}_{n-1}}{\to} C_{n-2} \to \ldots \to C_1 \to C_0 \to A$$

where $f_*(C_{n-1})$ is the push-out $M \coprod_{C_n/C_{n-1}} C_{n-1}$ and where $\bar{\partial}_{n-1} : M \coprod_{C_n/C_{n-1}} C_{n-1} \to C_{n-2}$ is the induced differential. Then it is easy to check that we have exactness at $f_*(C_{n-1})$ and therefore get an n-fold extension. The proof that this gives an isomorphism between equivalence classes of n-fold extensions and $H^n_{\mathcal{D}}(A, M)$ is then almost verbatim the one in [ML2, III.6].

Following MacLane's remarks in [ML2, p.393] one could obtain a direct reformulation of cohomology in terms of equivalence classes of extensions; but we prefer the explicit reformulation in terms of cofibration categories.

In particular, the above comparison implies that the cohomology $H^*_{\mathcal{D}}(A, M)$ is a set, and an abelian group structure can be imposed as usual by the addition of extensions. If A is free, then $\ldots \to 0 \to 0 \to A$ is a cofibrant resolution of A. Therefore we obtain the following important fact.

8.14. **Theorem.** *If $\mathcal{D}$ is a secondary operad of the form $(0 \to \mathcal{O})$ then the cohomology groups $H^*_{(0\to\mathcal{O})}(A, M)$ vanish for all $* \geqslant 3$ if A is a free $\mathcal{O}$-algebra.*

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Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: baues@mpim-bonn.mpg.de

Dep. de Matemática Pabellón I, Ciudad Universitaria 1428, Buenos Aires, Rep. Argentina
E-mail address: gminian@dm.uba.ar

Mathematisches Institut der Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany
E-mail address: richter@math.uni-bonn.de