

Stable bundles over rig categories

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ABSTRACT

The point of this paper is to prove the conjecture that virtual 2-vector bundles are classified by $K(ku)$, the algebraic K -theory of topological K -theory. Hence, by the work of Ausoni and the fourth author, virtual 2-vector bundles give us a geometric cohomology theory of the same telescopic complexity as elliptic cohomology. The main technical step is showing that for well-behaved small rig categories \mathcal{R} (also known as bimonoidal categories), the algebraic K -theory space, $K(H\mathcal{R})$, of the ring spectrum $H\mathcal{R}$ associated to \mathcal{R} is equivalent to $\mathcal{K}(\mathcal{R}) \simeq \mathbb{Z} \times |BGL(\mathcal{R})|^+$, where $GL(\mathcal{R})$ is the monoidal category of weakly invertible matrices over \mathcal{R} . The title refers to the sharper result that $BGL(\mathcal{R})$ is equivalent to $BGL(H\mathcal{R})$. If $\pi_0\mathcal{R}$ is a ring, this is almost formal, and our approach is to replace \mathcal{R} by a ring completed version, $\bar{\mathcal{R}}$, provided by Baas, Dundas, Richter, and Rognes [*J. reine angew. Math.*, to appear] with $H\mathcal{R} \simeq H\bar{\mathcal{R}}$ and $\pi_0\bar{\mathcal{R}}$ the ring completion of $\pi_0\mathcal{R}$. The remaining step is then to show that ‘stable \mathcal{R} -bundles’ and ‘stable $\bar{\mathcal{R}}$ -bundles’ are the same, which is done by a hands-on contraction of a custom-built model for the difference between $BGL(\mathcal{R})$ and $BGL(\bar{\mathcal{R}})$.

1. Introduction and main result

In telescopic complexity 0, 1, and ∞ there are cohomology theories that possess a geometric definition: de Rham cohomology of manifolds is given in terms of differential forms, cohomology classes in real and complex K -theory are classes of virtual vector bundles, and complex cobordism has a geometric definition *per se*. In order to understand phenomena of intermediate telescopic complexity, it is desirable to have geometric interpretations for such cohomology theories as well.

In [5], it was conjectured that virtual 2-vector bundles provide a geometric interpretation of a cohomology theory of telescopic complexity 2, which qualifies as a form of elliptic cohomology. More precisely, it was conjectured that the algebraic K -theory of a commutative rig category \mathcal{R} is equivalent to the algebraic K -theory of the ring spectrum associated with \mathcal{R} . The case of virtual 2-vector bundles arises when \mathcal{R} is the category of finite-dimensional complex vector spaces, with \oplus and $\otimes_{\mathbb{C}}$ as sum and multiplication, respectively. This, together with the analysis of the K -theory of complex topological K -theory due to Ausoni and the fourth author [2, 1], and the Quillen–Lichtenbaum conjecture for the integers, gives the desired relation to elliptic cohomology.

In this paper, we prove the conjecture from [5]. This work was motivated by the study of extended topological quantum field theories and the search for a geometric definition of elliptic cohomology; see [5, 8, 12, 14].

Let \mathcal{R} be a rig category (also known as a bimonoidal category), that is, a category with two operations \oplus and \otimes satisfying the axioms of a rig (ring without negative elements) up to coherent natural isomorphisms. In analogy with Quillen’s definition of the algebraic K -theory space $K(A) = \Omega B(\coprod_n BGL_n(A))$ of a ring A , the algebraic K -theory of \mathcal{R} was defined in [5] as $\mathcal{K}(\mathcal{R}) = \Omega B(\coprod_n |BGL_n(\mathcal{R})|) \simeq \mathbb{Z} \times |BGL(\mathcal{R})|^+$, where B and GL_n are, respectively, versions of the bar construction and the general linear group appropriate for rig categories.

On the other hand, forgetting the multiplicative structure, \mathcal{R} has an underlying symmetric monoidal category, and so it makes sense to speak about its K -theory spectrum $H\mathcal{R}$ with respect to \oplus . The K -theory spectrum construction $H\mathcal{R}$ is a direct extension of the usual Eilenberg–Mac Lane construction, and can, as \mathcal{R} is a rig category, be endowed with the structure of a strict ring spectrum, for instance, through the model given by Elmendorf and Mandell in [7]. Hence, we may speak about its algebraic K -theory space $K(H\mathcal{R})$. We prove that, under certain mild restrictions on \mathcal{R} , there is an equivalence

$$\mathcal{K}(\mathcal{R}) \simeq K(H\mathcal{R}).$$

In the special situation where \mathcal{R} is a ring (that is, \mathcal{R} is discrete as a category and has negative elements), this is the standard assertion that the K -theory of a ring is equivalent to the K -theory of its associated Eilenberg–Mac Lane spectrum. The key difficulty in establishing the equivalence above lies in proving that the lack of negative elements makes no difference for algebraic K -theory, even for rig categories.

More precisely, we prove the following result.

THEOREM 1.1. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \otimes, 1_{\mathcal{R}})$ be a small topological rig category satisfying the following conditions.*

- (i) *All morphisms in \mathcal{R} are isomorphisms, that is, \mathcal{R} is a groupoid.*
- (ii) *For every object $X \in \mathcal{R}$ the translation functor $X \oplus (-)$ is faithful.*

Then $|BGL(\mathcal{R})|$ and $BGL(H\mathcal{R})$ are weakly equivalent. Hence, the algebraic K -theory space of \mathcal{R} as a rig category,

$$\mathcal{K}(\mathcal{R}) = \Omega B \left(\prod_{n \geq 0} |BGL_n(\mathcal{R})| \right) \simeq \mathbb{Z} \times |BGL(\mathcal{R})|^+,$$

is weakly equivalent to the algebraic K -theory space of the strict ring spectrum associated to \mathcal{R} :

$$K(H\mathcal{R}) = \Omega B \left(\prod_{n \geq 0} BGL_n(H\mathcal{R}) \right) \simeq \mathbb{Z} \times BGL(H\mathcal{R})^+.$$

ADDENDUM 1.2. *In particular, if \mathcal{R} is the category of finite-dimensional complex vector spaces, the theorem states that stable 2-vector bundles, in the sense of [5] are classified by $BGL(ku)$, where $ku = H\mathcal{R}$ is the connective complex K -theory spectrum with $\pi_* ku = \mathbb{Z}[u]$, $|u| = 2$.*

In contrast to $K(H\mathcal{R})$, which is built in a two-stage process, the K -theory of the (strictly) bimonoidal category \mathcal{R} is built using both monoidal structures at once, so in this sense $\mathcal{K}(\mathcal{R})$ is a model that is easier to understand and handle than $K(H\mathcal{R})$.

The conditions (i) and (ii) on \mathcal{R} are not restrictive for the applications we have in mind, and are associated with the fact that in [4] we chose to work with variants of the Grayson–Quillen model for K -theory. Probably, the restrictions can be removed if one uses another technological platform.

Among those rig categories that satisfy the requirements of Theorem 1.1 are the following ‘standard’ ones, usually considered in the context of K -theory constructions.

(i) If \mathcal{R} is the discrete category (having only identity morphisms) with objects the elements of a ring with unit, R , then $H\mathcal{R}$ is the Eilenberg–Mac Lane spectrum HR .

(ii) The sphere spectrum S is the algebraic K -theory spectrum of the small rig category of finite sets \mathcal{E} . The objects of \mathcal{E} are the finite sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geq 0$, with the

convention that $\mathbf{0}$ is the empty set. Morphisms from \mathbf{n} to \mathbf{m} only exist for $n = m$, and in this case they constitute the symmetric group on n letters. The algebraic K -theory of S is equivalent to Waldhausen’s A -theory of a point $A(*)$ (see [21]), and so gives information about diffeomorphisms of high-dimensional disks. Thus, we obtain that

$$A(*) \simeq K(S) \simeq \mathcal{K}(\mathcal{E}) \simeq \mathbb{Z} \times |\mathrm{BGL}(\mathcal{E})|^+.$$

(iii) For a commutative ring A we consider the following small rig category of finitely generated free A -modules, $\mathcal{F}(A)$. Objects of $\mathcal{F}(A)$ are the finitely generated free A -modules A^n for $n \geq 0$. The set of morphisms from A^n to A^m is empty unless $n = m$, and the morphisms from A^n to itself are the A -module automorphisms of A^n , that is, $\mathrm{GL}_n(A)$. Our result allows us to identify the two-fold iterated algebraic K -theory of A , $K(K(A))$, with $\mathbb{Z} \times |\mathrm{BGL}(\mathcal{F}(A))|^+$.

(iv) The case that started the project is the category of 2-vector spaces of Kapranov and Voevodsky [13], viewed as modules over the rig category \mathcal{V} of complex (Hermitian) vector spaces. Here \mathcal{V} has objects \mathbb{C}^n for $n \geq 0$, and the automorphism space of \mathbb{C}^n is the unitary group $U(n)$. This identifies $K(H\mathcal{V}) = K(ku)$ with $\mathcal{K}(\mathcal{V}) \simeq \mathbb{Z} \times |\mathrm{BGL}(\mathcal{V})|^+$, which was called the K -theory of the 2-category of complex 2-vector spaces in [5]. Ausoni’s calculations [1] show that $K(ku_p)$ has telescopic complexity 2 for every prime greater than or equal to 5, and thus qualifies as a form of elliptic cohomology.

(v) Replacing the complex numbers by the reals yields an identification of $K(ko)$ with the K -theory of the 2-category of real 2-vector spaces.

(vi) Considering other subgroups of $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{R})$ as morphisms in a category with objects $\mathbf{n} = \{1, \dots, n\}$ with $n \geq 0$ gives a large variety of K -theory spectra that are in the range of our result. For a sample of such species have a look at [16, pp. 161–167].

1.1. *The spine of the argument giving a proof of Theorem 1.1*

Even though the proof contains some lengthy technical lemmas, it is possible to give the main flow of the argument in a few paragraphs, referring away the hard parts.

Remember that the group-like monoid $\mathrm{GL}_n(H\mathcal{R})$ is defined by the pullback

$$\begin{array}{ccc} \mathrm{GL}_n(H\mathcal{R}) & \longrightarrow & \mathrm{hocolim}_{\mathbf{m} \in I} \Omega^m M_n(H\mathcal{R}(S^m)) \\ \downarrow & & \downarrow \\ \mathrm{GL}_n(\pi_0 H\mathcal{R}) & \longrightarrow & M_n(\pi_0 H\mathcal{R}), \end{array}$$

where I is a strictly monoidal skeleton of the category of finite sets and injective functions, making $\mathrm{GL}_n(H\mathcal{R})$ a group-like monoid (here we have written $H\mathcal{R}$ in the form of a simplicial functor with associated symmetric spectrum $m \mapsto H\mathcal{R}(S^m)$).

Let $M_n(\bar{\mathcal{R}})$ be the monoidal category of $n \times n$ -matrices over $\bar{\mathcal{R}}$ (see Section 2). The set of components $\pi_0 M_n(\bar{\mathcal{R}})$ can be identified with $M_n(\pi_0 \bar{\mathcal{R}})$, and we let $\mathrm{GL}_n(\bar{\mathcal{R}})$ be the submonoidal category of $M_n(\bar{\mathcal{R}})$ consisting of the components that are invertible as matrices over the additive group completion of $\pi_0 \bar{\mathcal{R}}$ (see Definitions 2.3 and 2.4). As the effect of stabilization in $H\bar{\mathcal{R}}$ is exactly group completion, the natural isomorphism $|\bar{\mathcal{R}}| \rightarrow H\bar{\mathcal{R}}(S^0)$ together with stabilization induces a natural map $|\mathrm{GL}_n(\bar{\mathcal{R}})| \rightarrow \mathrm{GL}_n(H\bar{\mathcal{R}})$.

LEMMA 1.3. *If $\bar{\mathcal{R}}$ is a ring category, that is, a rig category with $\pi_0 \bar{\mathcal{R}}$ a ring, then*

$$|\mathrm{GL}_n(\bar{\mathcal{R}})| \xrightarrow{\sim} \mathrm{GL}_n(H\bar{\mathcal{R}})$$

is a homotopy equivalence.

Proof. By assumption $\pi_0\bar{\mathcal{R}}$ is isomorphic to its group completion $\mathrm{Gr}(\pi_0\bar{\mathcal{R}}) = \pi_0H\bar{\mathcal{R}}$, so it is enough to show that $|M_n(\bar{\mathcal{R}})|$ and $\mathrm{hocolim}_{\mathbf{m} \in I} \Omega^m M_n(H\bar{\mathcal{R}}(S^m))$ are equivalent. Both are n^2 -fold products, so it suffices to show that $|\bar{\mathcal{R}}|$ and $\mathrm{hocolim}_{\mathbf{m} \in I} \Omega^m H\bar{\mathcal{R}}(S^m)$ are equivalent. All the structure maps $\Omega^m H\bar{\mathcal{R}}(S^m) \rightarrow \Omega^{m'} H\bar{\mathcal{R}}(S^{m'})$ are equivalences for $\mathbf{m} \rightarrow \mathbf{m}'$ an injection of non-empty finite sets, and, as $\bar{\mathcal{R}}$ is already group-like, $|\bar{\mathcal{R}}| \simeq H\bar{\mathcal{R}}(S^0)$ maps by an equivalence to $\Omega H\bar{\mathcal{R}}(S^1)$. \square

We know from [4] that there is a chain of simplicial rig categories

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

such that

- (i) $\mathcal{R} \leftarrow Z\mathcal{R}$ becomes a weak equivalence on realization;
- (ii) $HZ\mathcal{R} \rightarrow H\bar{\mathcal{R}}$ is a stable equivalence; and
- (iii) $\pi_0\bar{\mathcal{R}}$ is a ring.

Consider the commutative diagram

$$\begin{array}{ccccc} |BGL(\mathcal{R})| & \xleftarrow{\sim} & |BGL(Z\mathcal{R})| & \longrightarrow & |BGL(\bar{\mathcal{R}})| \\ \downarrow & & \downarrow & & \downarrow \sim \\ BGL(H\mathcal{R}) & \xleftarrow{\sim} & BGL(HZ\mathcal{R}) & \xrightarrow{\sim} & BGL(H\bar{\mathcal{R}}) \end{array}$$

where the definition of GL of rig categories is given in Section 2 and the bar construction is recalled in Section 3. Both constructions preserve weak equivalences.

- (i) The leftward pointing horizontal maps are weak equivalences because $\mathcal{R} \leftarrow Z\mathcal{R}$ is one.
- (ii) The bottom rightward pointing arrow is a weak equivalence as $HZ\mathcal{R} \rightarrow H\bar{\mathcal{R}}$ is a stable equivalence.
- (iii) The right-hand vertical map is a weak equivalence by Lemma 1.3.

To show that the left-hand vertical arrow is a weak equivalence, it therefore suffices to prove that the upper right-hand horizontal map is a weak equivalence. By Proposition 3.8, the homotopy fibre of $BGL(Z\mathcal{R}) \rightarrow BGL(\bar{\mathcal{R}})$ is given by the one-sided bar construction $B(*, \mathrm{GL}(Z\mathcal{R}), \mathrm{GL}(\bar{\mathcal{R}}))$, and so we have reduced the problem to giving a contraction of the associated space.

Under the assumptions of Theorem 1.1, we know from [4] that there is a chain of weak equivalences

$$(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} Z(-\mathcal{R})\mathcal{R} \xrightarrow{\sim} \bar{\mathcal{R}}$$

of $Z\mathcal{R}$ -modules. Here, $(-\mathcal{R})\mathcal{R}$ is the Grayson–Quillen model [9], also discussed in Section 4.1, and so, by Remark 2.5, we get weak equivalences:

$$B(*, \mathrm{GL}(\mathcal{R}), \mathrm{GL}((-\mathcal{R})\mathcal{R})) \xleftarrow{\sim} B(*, \mathrm{GL}(Z\mathcal{R}), \mathrm{GL}(Z(-\mathcal{R})\mathcal{R})) \xrightarrow{\sim} B(*, \mathrm{GL}(Z\mathcal{R}), \mathrm{GL}(\bar{\mathcal{R}})).$$

This means that the somewhat complicated construction of $\bar{\mathcal{R}}$ from [4] may be safely forgotten once we know it exists.

Just one simplification remains: in Lemma 4.2 we display a weak equivalence $T\mathcal{R} \rightarrow (-\mathcal{R})\mathcal{R}$ of \mathcal{R} -modules. The \mathcal{R} -modules $T\mathcal{R}$ and $(-\mathcal{R})\mathcal{R}$ are generalizations of how to construct the integers from the natural numbers by considering equivalence classes of pairs of natural numbers.

We are left with showing that $B(*, \mathrm{GL}(\mathcal{R}), \mathrm{GL}(T\mathcal{R}))$ is contractible. This is done through a concrete contraction. It is an elaboration of the following path,

$$\begin{bmatrix} a-b & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} a-b & b \\ 0 & 1 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad a, b \in \mathbb{N}$$

in $B(*, \text{GL}_2(\mathbb{N}), \text{GL}_2(\mathbb{Z})) = \{[q] \mapsto \text{GL}_2(\mathbb{N})^{\times q} \times \text{GL}_2(\mathbb{Z})\}$, with 1-simplices given by

$$\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a-b & 0 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right) \in \text{GL}_2(\mathbb{N}) \times \text{GL}_2(\mathbb{Z})$$

respectively, showing that the inclusion of $B(*, \text{GL}_1(\mathbb{N}), \text{GL}_1(\mathbb{Z}))$ in $B(*, \text{GL}_2(\mathbb{N}), \text{GL}_2(\mathbb{Z}))$ is null-homotopic.

1.2. *Plan*

The structure of the paper is as follows. In Section 2, we discuss the monoidal category $\text{GL}_n(\mathcal{R})$ of (weakly) invertible matrices over a strictly bimonoidal category \mathcal{R} . Section 3 recalls the definition of the bar construction of monoidal categories as in [5] and introduces a version with coefficients in a module. In Section 4, we construct the promised contraction of $B(*, \text{GL}(\mathcal{R}), \text{GL}(T\mathcal{R}))$, thus completing the proof of the main theorem.

This paper has circulated in preprint form under the title ‘Two-vector bundles define a form of elliptic cohomology’, which, although not highlighting the nuts and bolts of the paper perhaps better represented the reason for writing (or reading) it. The old preprint [3] also contained the main result of the paper [4].

Segal constructed a ring completion of the rig category of complex vector spaces [18, p. 300] (see also [17, Appendix]). His model is a topological category consisting of certain spaces of bounded chain complexes and spaces of quasi-isomorphisms. One can probably build a variant of his model that could replace the construction $\overline{\mathcal{R}}$ in our proof of Theorem 1.1, in the special case $\mathcal{R} = \mathcal{V}$.

A piece of notation: if \mathcal{C} is any small category, then the expression $X \in \mathcal{C}$ is short for ‘ X is an object of \mathcal{C} ’ and likewise for morphisms and diagrams. Displayed diagrams commute unless the contrary is stated explicitly. For basics on bipermutative and rig categories, we refer the reader to [4] Section 2.

2. *Weakly invertible matrices*

Let \mathcal{R} be a strictly bimonoidal category.

DEFINITION 2.1. The category of $n \times n$ -matrices over \mathcal{R} , $M_n(\mathcal{R})$, is defined as follows. The objects of $M_n(\mathcal{R})$ are matrices $X = (X_{i,j})_{i,j=1}^n$ of objects of \mathcal{R} and morphisms from $X = (X_{i,j})_{i,j=1}^n$ to $Y = (Y_{i,j})_{i,j=1}^n$ are matrices $F = (F_{i,j})_{i,j=1}^n$, where each $F_{i,j}$ is a morphism in \mathcal{R} from $X_{i,j}$ to $Y_{i,j}$.

LEMMA 2.2. For a strictly bimonoidal category $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus}, \otimes, 1_{\mathcal{R}})$, the category $M_n(\mathcal{R})$ is a monoidal category with respect to the matrix multiplication bifunctor

$$M_n(\mathcal{R}) \times M_n(\mathcal{R}) \longrightarrow M_n(\mathcal{R})$$

$$(X_{i,j})_{i,j=1}^n \cdot (Y_{i,j})_{i,j=1}^n = (Z_{i,j})_{i,j=1}^n \quad \text{with} \quad Z_{i,j} = \bigoplus_{k=1}^n X_{i,k} \otimes Y_{k,j}.$$

The unit of this structure is given by the unit matrix object E_n which has $1_{\mathcal{R}} \in \mathcal{R}$ as diagonal entries and $0_{\mathcal{R}} \in \mathcal{R}$ in the other places.

The property of \mathcal{R} being bimonoidal gives $\pi_0\mathcal{R}$ the structure of a rig, and its (additive) group completion $\text{Gr}(\pi_0\mathcal{R}) = (-\pi_0\mathcal{R})\pi_0\mathcal{R}$ is a ring.

DEFINITION 2.3. We define the *weakly invertible $n \times n$ -matrices over $\pi_0\mathcal{R}$* , $\mathrm{GL}_n(\pi_0\mathcal{R})$, to be the $n \times n$ -matrices over $\pi_0\mathcal{R}$ that are invertible as matrices over $\mathrm{Gr}(\pi_0\mathcal{R})$.

Note that we can define $\mathrm{GL}_n(\pi_0\mathcal{R})$ by the pullback square

$$\begin{array}{ccc} \mathrm{GL}_n(\pi_0\mathcal{R}) & \longrightarrow & \mathrm{GL}_n(\mathrm{Gr}(\pi_0\mathcal{R})) \\ \downarrow & & \downarrow \\ M_n(\pi_0\mathcal{R}) & \longrightarrow & M_n(\mathrm{Gr}(\pi_0\mathcal{R})). \end{array}$$

DEFINITION 2.4. The *category of weakly invertible $n \times n$ -matrices over \mathcal{R}* , $\mathrm{GL}_n(\mathcal{R})$, is the full subcategory of $M_n(\mathcal{R})$ with objects all matrices $X = (X_{i,j})_{i,j=1}^n \in M_n(\mathcal{R})$, whose matrix of π_0 -classes $[X] = ([X_{i,j}])_{i,j=1}^n$ is contained in $\mathrm{GL}_n(\pi_0\mathcal{R})$.

Matrix multiplication is of course compatible with the property of being weakly invertible. Thus, the category $\mathrm{GL}_n(\mathcal{R})$ inherits a monoidal structure from $M_n(\mathcal{R})$.

However, even if our base category is not bimonoidal, it still makes sense to talk about matrices and even weakly invertible matrices, as long as π_0 of that category is a rig.

REMARK 2.5. If \mathcal{M} is an \mathcal{R} -module, then matrix multiplication makes the category $M_n(\mathcal{M})$ into a module over the monoidal category $\mathrm{GL}_n(\mathcal{R})$. For our applications the following situation will be particularly important: let $\mathcal{M} \rightarrow \mathcal{N}$ be a map of \mathcal{R} -modules, where the map of $\pi_0\mathcal{R}$ -modules $\pi_0\mathcal{M} \rightarrow \pi_0\mathcal{N}$ comes from a rig map under $\pi_0\mathcal{R}$. Then we get a map $\mathrm{GL}_n(\mathcal{M}) \rightarrow \mathrm{GL}_n(\mathcal{N})$ of $\mathrm{GL}_n(\mathcal{R})$ -modules, which induces a weak equivalence on realization if $\mathcal{M} \rightarrow \mathcal{N}$ does.

There is a canonical stabilization functor $\mathrm{GL}_n(\mathcal{R}) \rightarrow \mathrm{GL}_{n+1}(\mathcal{R})$, which is induced by taking the block sum with $E_1 \in \mathrm{GL}_1(\mathcal{R})$. Let $\mathrm{GL}(\mathcal{R})$ be the sequential colimit of the categories $\mathrm{GL}_n(\mathcal{R})$.

3. The one-sided bar construction

In this section, we review some well-known facts about the one-sided bar construction of monoidal categories.

DEFINITION 3.1. Let $(\mathcal{M}, \cdot, 1)$ be a monoidal category and \mathcal{T} be a left \mathcal{M} -module. The *one-sided bar construction $B(*, \mathcal{M}, \mathcal{T})$* is the simplicial category whose category of q -simplices $B_q(*, \mathcal{M}, \mathcal{T})$ is as follows: consider the ordered set $[q]_+ = [q] \sqcup \{\infty\}$, that is, in addition to the numbers $0 < 1 < \dots < q$ there is a greatest element ∞ . An object a in $B_q(*, \mathcal{M}, \mathcal{T})$ consists of the following data.

- (i) For each $0 \leq i < j \leq q$ there is an object $a_{ij} \in \mathcal{M}$, and for each $0 \leq i \leq q$ an object $a_{i\infty} \in \mathcal{T}$.
- (ii) For each $0 \leq i < j < k \leq \infty$ there is an isomorphism

$$a_{ijk} : a_{ij} \cdot a_{jk} \longrightarrow a_{ik}$$

(in \mathcal{M} if $k < \infty$ and in \mathcal{T} if $k = \infty$) such that if $0 \leq i < j < k < l \leq \infty$, the following diagram commutes:

$$\begin{array}{ccc}
 (a_{ij} \cdot a_{jk}) \cdot a_{kl} & \xrightarrow{\text{struct. iso.}} & a_{ij} \cdot (a_{jk} \cdot a_{kl}) \\
 a_{ijk} \cdot \text{id} \downarrow & & \downarrow \text{id} \cdot a_{jkl} \\
 a_{ik} \cdot a_{kl} & \xrightarrow{a_{ikl}} a_{il} \xleftarrow{a_{ijl}} & a_{ij} \cdot a_{jl}
 \end{array}$$

A morphism $f : a \rightarrow b$ consists of morphisms $f_{ij} : a_{ij} \rightarrow b_{ij}$ (in \mathcal{M} if $j < \infty$ and in \mathcal{T} if $j = \infty$) such that if $0 \leq i < j < k \leq \infty$

$$f_{ik} a_{ijk} = b_{ijk} (f_{ij} \cdot f_{jk}) : a_{ij} \cdot a_{jk} \rightarrow b_{ik}.$$

The simplicial structure is obtained as follows: if $\phi : [q] \rightarrow [p] \in \Delta$, the functor $\phi^* : B_p(*, \mathcal{M}, \mathcal{T}) \rightarrow B_q(*, \mathcal{M}, \mathcal{T})$ is obtained by precomposing with $\phi_+ = \phi \sqcup \{\infty\}$. So, for instance, $d_1(a)$ is obtained by deleting all entries with indices containing 1 from the data giving a . In order to allow for degeneracy maps s_i , we use the convention that all objects of the form a_{ii} are the unit of the monoidal structure, and all isomorphisms of the form a_{iik} and a_{ikk} are identities.

REMARK 3.2. A good way to think about this comes from the discrete case when \mathcal{M} is a monoid and \mathcal{T} is an \mathcal{M} -set. Then an object $a \in B_q(*, \mathcal{M}, \mathcal{T})$ is uniquely given by the ‘superdiagonal’ $(a_{01}, a_{12}, \dots, a_{q-1q}, a_{q\infty})$, and $B(*, \mathcal{M}, \mathcal{T})$ is isomorphic to the nerve of the category with objects \mathcal{T} and morphisms $a_{1\infty} \rightarrow a_{01} \cdot a_{1\infty} = a_{0\infty}$ corresponding to $(a_{01}, a_{1\infty})$.

The reason why we have to include all of the ‘upper triangular’ elements is that associativity may not be strict. For instance, it is not strict in our main example: matrix multiplication over a rig category is in general not strictly associative. Hence, $a_{01} \cdot (a_{12} \cdot a_{23})$ may be different from $(a_{01} \cdot a_{12}) \cdot a_{23}$, and so the superdiagonal elements do not carry enough information to turn the obvious choice of face maps into a simplicial structure. We remedy this by adding choices for all faces in our simplices. This just adds more elements in each isomorphism class in every simplicial degree, and is a standard trick used in many places, for instance, by Waldhausen in his S_\bullet -construction.

EXAMPLE 3.3.

(i) If \mathcal{T} is the one-point category $*$, then $B(*, \mathcal{M}, *)$ is isomorphic to the bar construction $B\mathcal{M}$ of [5].

(ii) If $F : \mathcal{M} \rightarrow \mathcal{M}'$ is a lax monoidal functor, then \mathcal{M}' may be considered as an \mathcal{M} -module, and we write without further ado $B(*, \mathcal{M}, \mathcal{M}')$ for the corresponding bar construction (with F suppressed). In case F is an isomorphism, $B(*, \mathcal{M}, \mathcal{M}')$ is contractible.

We think of elements of $B_q(*, \mathcal{M}, \mathcal{T})$ in terms of strictly upper triangular arrays of objects, suppressing the isomorphisms, so that a typical element in $B_2(*, \mathcal{M}, \mathcal{T})$ is written as

$$\begin{array}{ccc}
 a_{01} & a_{02} & a_{0\infty} \\
 & a_{12} & a_{1\infty} \\
 & & a_{2\infty}
 \end{array}$$

with d_1 given by

$$\begin{array}{cc}
 a_{02} & a_{0\infty} \\
 & a_{2\infty} .
 \end{array}$$

The one-sided bar construction is functorial in ‘natural modules’. A natural module is a pair $(\mathcal{M}, \mathcal{T})$, where \mathcal{M} is a monoidal category and \mathcal{T} is an \mathcal{M} -module. A morphism

$(\mathcal{M}, \mathcal{T}) \rightarrow (\mathcal{M}', \mathcal{T}')$ consists of a pair (F, G) where $F: \mathcal{M} \rightarrow \mathcal{M}'$ is a lax monoidal functor and $G: \mathcal{T} \rightarrow F^*\mathcal{T}'$ is a map of \mathcal{M} -modules, where $F^*\mathcal{T}'$ is \mathcal{T}' endowed with the \mathcal{M} -module structure given by restricting along F .

LEMMA 3.4. *For each q there is an equivalence of categories between $B_q(*, \mathcal{M}, \mathcal{T})$ and the product category $\mathcal{M}^{\times q} \times \mathcal{T}$.*

Proof. The equivalence is given by the forgetful functor

$$F : B_q(*, \mathcal{M}, \mathcal{T}) \longrightarrow \mathcal{M}^{\times q} \times \mathcal{T}$$

sending a to the ‘superdiagonal’ $F(a) = (a_{01}, \dots, a_{q-1q}, a_{q\infty})$. The inverse is obtained by sending $(a_1, \dots, a_q, a_\infty)$ to the a with $a_{ij} = a_{i+1} \cdot (\dots (a_{j-1} \cdot a_j) \dots)$ and a_{ijk} given by the structural isomorphisms. □

COROLLARY 3.5. *Let $(F, G): (\mathcal{M}, \mathcal{T}) \rightarrow (\mathcal{M}', \mathcal{T}')$ be a map of natural modules such that F and G are equivalences of categories. Then the induced map*

$$B(*, F, G) : B(*, \mathcal{M}, \mathcal{T}) \longrightarrow B(*, \mathcal{M}', \mathcal{T}')$$

is a degreewise equivalence of simplicial categories.

REMARK 3.6. The same result holds, if instead of equivalences of categories we consider weak equivalences.

Usually $\mathcal{M}^{\times q} \times \mathcal{T}$ is not functorial in $[q]$, but if $(\mathcal{M}, \mathcal{T})$ is strict, the monoidal structure gives a simplicial category

$$B^{\text{strict}}(*, \mathcal{M}, \mathcal{T}) = \{[q] \mapsto \mathcal{M}^{\times q} \times \mathcal{T}\}.$$

In this situation Lemma 3.4 reads as follows.

COROLLARY 3.7. *Let \mathcal{M} be a strict monoidal category and \mathcal{T} be a strict \mathcal{M} -module. Then there is a degreewise equivalence between the simplicial categories $B(*, \mathcal{M}, \mathcal{T})$ and $B^{\text{strict}}(*, \mathcal{M}, \mathcal{T})$.*

PROPOSITION 3.8. *Let $F: \mathcal{M} \rightarrow \mathcal{G}$ be a strong monoidal functor such that the monoidal structure on \mathcal{G} induces a group structure on $\pi_0\mathcal{G}$. Then*

$$\begin{array}{ccc} B(*, \mathcal{M}, \mathcal{G}) & \longrightarrow & B\mathcal{M} \\ \downarrow & & \downarrow \\ B(*, \mathcal{G}, \mathcal{G}) & \longrightarrow & B\mathcal{G} \end{array}$$

is homotopy cartesian, meaning that it induces a homotopy cartesian diagram upon applying the nerve functor in every degree. The (nerve of the) lower left-hand corner is contractible.

Proof. By Joyal and Street [11] there is a diagram of monoidal categories

$$\begin{array}{ccc} \mathbf{st}\mathcal{M} & \xrightarrow{\sim} & \mathcal{M} \\ \mathbf{st}F \downarrow & & F \downarrow \\ \mathbf{st}\mathcal{G} & \xrightarrow{\sim} & \mathcal{G} \end{array}$$

such that the horizontal maps are monoidal equivalences, and $\mathbf{st}F$ is a strict monoidal functor between strict monoidal categories. Together with Corollaries 3.5 and 3.7, this tells us that we may just as well consider the strict situation, and use the strict bar construction. However, note that the nerve of the strict monoidal category $\mathbf{st}\mathcal{M}$ is a simplicial monoid, and that reversal of priorities gives a natural isomorphism

$$B(*, N\mathbf{st}\mathcal{M}, N\mathbf{st}\mathcal{G}) \cong NB^{\text{strict}}(*, \mathbf{st}\mathcal{M}, \mathbf{st}\mathcal{G}),$$

so that our statement reduces to the statement that

$$B(*, N\mathbf{st}\mathcal{M}, N\mathbf{st}\mathcal{G}) \longrightarrow B(N\mathbf{st}\mathcal{M}) \longrightarrow B(N\mathbf{st}\mathcal{G})$$

is a fibre sequence up to homotopy, which is a classical result [15] given that $N\mathbf{st}\mathcal{G}$ is group-like. \square

4. Contracting the one-sided bar construction

4.1. A model for K -theory of \mathcal{R} as an \mathcal{R} -module

In order to construct concrete homotopies, we offer a slight variant of the Grayson–Quillen model where morphisms are not entire equivalence classes. The price is as usual that the resulting object is a 2-category. As there was some confusion about this point while the paper was still at a preprint stage, we emphasize that this is not the construction of Thomason [19, 4.3.2] and Jardine [10].

DEFINITION 4.1. Let $(\mathcal{M}, \oplus, 0_{\mathcal{M}}, \tau_{\mathcal{M}})$ be a permutative category written additively. Let $T\mathcal{M}$ be the following 2-category. The objects of $T\mathcal{M}$ are pairs $(A^+, A^-) =: A$ of objects in \mathcal{M} , thought of as plus and minus objects in \mathcal{M} . Given two objects $A, B \in T\mathcal{M}$, the category of morphisms $T\mathcal{M}(A, B)$ has objects the pairs (X, α) , where X is an object in \mathcal{M} and α is a pair of morphisms $\alpha^{\pm} : A^{\pm} \oplus X \rightarrow B^{\pm}$ in \mathcal{M} . A morphism from (X, α) to (Y, β) is an isomorphism $\phi : X \rightarrow Y$ such that $\beta^{\pm}(1 \oplus \phi) = \alpha^{\pm}$. Composition $T\mathcal{M}(B, C) \times T\mathcal{M}(A, B) \rightarrow T\mathcal{M}(A, C)$ is given by sending $((Y, \beta), (X, \alpha))$ to the pair consisting of $X \oplus Y$ and the composite maps

$$A^{\pm} \oplus (X \oplus Y) = (A^{\pm} \oplus X) \oplus Y \xrightarrow{\alpha^{\pm} \oplus \text{id}} B^{\pm} \oplus Y \xrightarrow{\beta^{\pm}} C^{\pm}.$$

Composition on morphisms is simply given by addition. Composition is strictly associative because \mathcal{M} is permutative; if \mathcal{M} is merely symmetric monoidal, standard modifications are necessary. Symmetry allows for a symmetric monoidal structure on $T\mathcal{M}$: if we define $(A^+, A^-) \oplus (B^+, B^-) := (A^+ \oplus B^+, A^- \oplus B^-)$, then we need the symmetry in order to turn that prescription into a bifunctor.

The Grayson–Quillen model for the K -theory of \mathcal{M} is the category $(-\mathcal{M})\mathcal{M}$ with the same objects as $T\mathcal{M}$ and with morphism sets the path components $(-\mathcal{M})\mathcal{M}(A, B) = \pi_0 T\mathcal{M}(A, B)$. If all morphisms in \mathcal{M} are isomorphisms and if additive translation in \mathcal{M} is faithful, $(-\mathcal{M})\mathcal{M}$ is shown in [9] to be a group completion of \mathcal{M} . Under these hypotheses, there is at most one morphism between two given objects (X, α) and (Y, β) in $T\mathcal{M}(A, B)$. Consequently, the

morphism spaces are homotopy discrete: the projection $T\mathcal{M}(A, B) \rightarrow \pi_0 T\mathcal{M}(A, B)$ is a weak equivalence. In the case of a topological category, we interpret π_0 as the coequalizer of the source and target maps from the morphism space to the object space. The assignment that is the identity on objects, and otherwise is induced by the projection $T\mathcal{M}(A, B) \rightarrow \pi_0 T\mathcal{M}(A, B)$, gives a 2-functor $T\mathcal{M} \rightarrow (-\mathcal{M})\mathcal{M}$ (considering $(-\mathcal{M})\mathcal{M}$ as a 2-category with only identity 2-morphisms).

LEMMA 4.2. *Let \mathcal{M} be a permutative category with all morphisms in \mathcal{M} isomorphisms and faithful additive translation. The 2-functor $T\mathcal{M} \rightarrow (-\mathcal{M})\mathcal{M}$ is a weak equivalence and the standard inclusion $\mathcal{M} \rightarrow T\mathcal{M}$ is a group completion.*

Note that if \mathcal{R} is a rig category, $T\mathcal{R}$ will not be a rig category (essentially because of the non-strict symmetry in quadratic terms, as in [20, p. 572]), but it will still be an \mathcal{R} -module.

LEMMA 4.3. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus}, \otimes, 1_{\mathcal{R}})$ be a strictly bimonoidal category. The map*

$$\mathcal{R} \times T\mathcal{R} \longrightarrow T\mathcal{R}$$

given on objects by $(A, (B^+, B^-)) \mapsto (A \otimes B^+, A \otimes B^-)$, and on morphisms by sending $\phi : A \rightarrow B \in \mathcal{R}$ and $(X, \alpha) \in T\mathcal{R}(C, D)$ to the pair consisting of $A \otimes X$ and the map

$$A \otimes C^{\pm} \oplus A \otimes X \longrightarrow A \otimes (C^{\pm} \oplus X) \xrightarrow{\phi \otimes \alpha^{\pm}} B \otimes D^{\pm}$$

(where the first map is the left distributivity isomorphism) induces an \mathcal{R} -module structure on $T\mathcal{R}$.

We consider $T\mathcal{R}$ as a simplicial category by taking the nerve of each category of morphisms; thus in simplicial degree ℓ , the objects of $T_{\ell}\mathcal{R}$ are the objects of $T\mathcal{R}$. The morphisms in $T_{\ell}\mathcal{R}$ from (A^+, A^-) to (B^+, B^-) consist of objects X^0, \dots, X^{ℓ} , a 1-morphism $\alpha^{\pm} : A^{\pm} \oplus X^0 \rightarrow B^{\pm}$, and isomorphisms $\phi^r : X^r \rightarrow X^{r-1}$ for $r = 1, \dots, \ell$. The simplicial structure is given by composing and forgetting isomorphisms ϕ^r and inserting identity maps.

4.2. Subdivisions

We use the following variant of edgewise subdivision to make room for an explicit simplicial contraction, whose construction begins in Subsection 4.4. Consider the shear functor $z : \Delta \times \Delta \rightarrow \Delta \times \Delta$ given by sending (S, T) to $(T \sqcup S, T)$, where $T \sqcup S$ is the disjoint union with the ordering obtained from T and S with the extra declaration that every object in S is greater than every object in T . If B is a bisimplicial object, then we let $z^*B = B \circ z$. The standard inclusion $S \rightarrow T \sqcup S$ gives a natural transformation η in $\Delta \times \Delta$ from the identity to z , and hence a natural transformation in bisimplicial sets $\eta^* : z^* \rightarrow \text{id}$. Let Ens denote the category of sets and functions.

LEMMA 4.4. *For any bisimplicial set X the map $\eta^* : z^*X \rightarrow X$ becomes a weak equivalence after realization.*

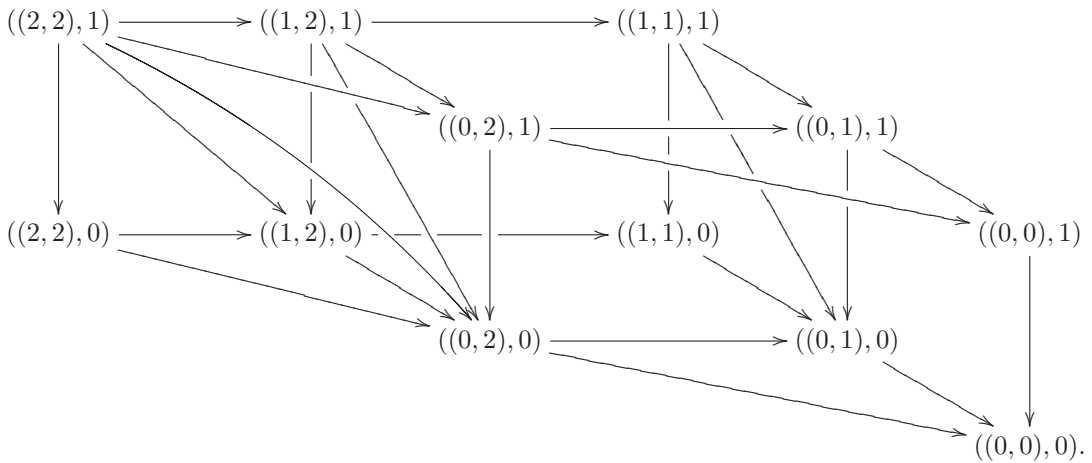
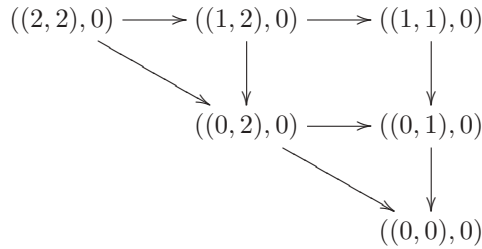
Proof. The diagonal of z^*X is equal to the evaluation of X on the opposite of the composite

$$\Delta \xrightarrow{S \mapsto (S, S)} \Delta \times \Delta \xrightarrow{(S, T) \mapsto (S \sqcup S, T)} \Delta \times \Delta,$$

so as a map of bisimplicial sets is an equivalence if it is one in every (vertical) degree, it is enough to know that, for each fixed $T \in \Delta$, the natural map $\{S \mapsto X(S \sqcup S, T)\} \rightarrow \{S \mapsto X(S, T)\}$ is a weak equivalence. But this is a standard weak equivalence from the (second) edgewise subdivision, which is known to be homotopic to a homeomorphism after realization. See [6, Lemma 1.1] and the proof of [6, Proposition 2.5]. \square

Vertices in $z^*(\Delta[p] \times \Delta[q])$ (where products of simplicial sets are viewed as bisimplicial sets, and vertices are $(0,0)$ -simplices) are, for instance, indexed by tuples $((a, b), c)$ where $0 \leq a \leq b \leq p$ and $0 \leq c \leq q$.

Here are pictures of $z^*(\Delta[2] \times \Delta[0])$ and $z^*(\Delta[2] \times \Delta[1])$:



Note that, for any bisimplicial set X ,

$$X_{(p,q)} \cong \text{Hom}_{\text{bisimp. sets}}(\Delta[p] \times \Delta[q], X) = \int_{([s],[t])} \text{Ens}(\Delta([s], [p]) \times \Delta([t], [q]), X_{(s,t)})$$

as a categorical end. Therefore, the right adjoint of z^* , z_* , is given by

$$(z_* X)_{(p,q)} = \text{Hom}_{\text{bisimp. sets}}(\Delta[p] \times \Delta[q], z_* X) \cong \text{Hom}_{\text{bisimp. sets}}(z^*(\Delta[p] \times \Delta[q]), X),$$

and thus

$$\begin{aligned}
 z_* X &= \{[p], [q] \mapsto \{\text{bisimp. maps } z^*(\Delta[p] \times \Delta[q]) \rightarrow X\}\} \\
 &= \{[p], [q] \mapsto \int_{([s],[t])} \text{Ens}(\Delta([t] \sqcup [s], [p]) \times \Delta([t], [q]), X_{(s,t)})\}.
 \end{aligned}$$

Let $\eta_* : X \rightarrow z_* X$ be the natural transformation associated with η . Note that η_* maps $X_{(0,q)}$ isomorphically to $(z_* X)_{(0,q)}$ for all $q \geq 0$, so $(z_* X)_{(0,q)} \cong X_{(0,q)}$.

LEMMA 4.5. *In the homotopy category (with respect to maps that become weak equivalences on realization), $\eta_* : X \rightarrow z_*X$ is a split monomorphism.*

Proof. By formal considerations the diagram

$$\begin{array}{ccc} z^*X & \xrightarrow{z^*\eta_*} & z^*z_*X \\ & \searrow \eta^* & \downarrow \text{counit} \\ & & X \end{array}$$

commutes, and η^* is a weak equivalence after realization. Hence, $z^*\eta_*$ (and so η_*) is a split monomorphism in the homotopy category. \square

4.3. *The bar construction on matrices*

Let \mathcal{R} be a strictly bimonoidal category such that all morphisms are isomorphisms and each translation functor is faithful.

Consider the one-sided bar construction $B(*, \text{GL}_n(\mathcal{R}), \text{GL}_n(T\mathcal{R}))$. Viewing $T\mathcal{R}$ as a simplicial category, we get that $B(*, \text{GL}_n(\mathcal{R}), \text{GL}_n(T\mathcal{R}))$ is a bisimplicial category. We are going to show that

$$B(*, \text{GL}(\mathcal{R}), \text{GL}(T\mathcal{R})) \cong \text{colim}_n B(*, \text{GL}_n(\mathcal{R}), \text{GL}_n(T\mathcal{R}))$$

is contractible, and it is enough to show that $B(*, \text{GL}(\mathcal{R}), \text{GL}(T_\ell\mathcal{R}))$ is contractible for each ℓ .

To ease readability, we abandon the cumbersome \oplus and \otimes in favour of the more readable $+$ and $-$ —reminding us of the matrix nature of our efforts.

Fix ℓ once and for all, and let $B^n = B(*, \text{GL}_n(\mathcal{R}), \text{GL}_n(T_\ell\mathcal{R}))$. An object in B^n is a collection m_{ij} of $n \times n$ matrices in \mathcal{R} for $0 \leq i < j \leq n$, and for each $0 \leq i < n$ a matrix $m_{i\infty}$ in $T_\ell\mathcal{R}$, together with suitably compatible structural isomorphisms $m_{ijk} : m_{ij} \cdot m_{jk} \rightarrow m_{ik}$. The matrices are drawn from the ‘weakly invertible components’. The matrices $m_{i\infty}$ and the structural isomorphisms relating these need special attention. Each entry is in $T_\ell\mathcal{R}$, so $m_{i\infty}$ can be viewed as a pair $m_{i\infty}^\pm$ of matrices, and the structural isomorphism $m_{ij\infty} : m_{ij} \cdot m_{j\infty} \rightarrow m_{i\infty}$ is a tuple $(m_{ij\infty}^\pm, \phi_{ij\infty}^1, \dots, \phi_{ij\infty}^\ell)$, where the $\phi_{ij\infty}^r : x_{ij\infty}^r \rightarrow x_{i\infty}^{r-1} \in M_n(\mathcal{R})$ for $r = 1, \dots, \ell$ are matrices of isomorphisms and the $m_{ij\infty}^\pm : m_{ij} \cdot m_{j\infty}^\pm + x_{ij\infty}^0 \rightarrow m_{i\infty}^\pm$ are isomorphisms.

The assumed commutativity of

$$\begin{array}{ccc} (m_{ij} \cdot m_{jk}) \cdot m_{k\infty} & \xrightarrow{\cong} & m_{ij} \cdot (m_{jk} \cdot m_{k\infty}) \\ m_{ijk} \cdot \text{id} \downarrow & & \downarrow \text{id} \cdot m_{jk\infty} \\ m_{ik} \cdot m_{k\infty} & \xrightarrow{m_{ik\infty}} m_{i\infty} \xleftarrow{m_{ij\infty}} & m_{ij} \cdot m_{j\infty} \end{array}$$

says that two morphisms from $(m_{ij} \cdot m_{jk}) \cdot m_{k\infty}$ agree: one is an isomorphism with source $(m_{ij} \cdot m_{jk}) \cdot m_{k\infty} + x_{ik\infty}^r$; the other is an isomorphism with source $(m_{ij} \cdot m_{jk}) \cdot m_{k\infty} + m_{ij} \cdot x_{jk\infty}^r + x_{ij\infty}^r$. Therefore, we obtain the following equality.

LEMMA 4.6. *In the situation above one has the identity*

$$x_{ik\infty}^r = m_{ij} \cdot x_{jk\infty}^r + x_{ij\infty}^r,$$

for $r = 0, 1, \dots, \ell$, and the diagram

$$\begin{array}{ccc}
 m_{ij} \cdot (m_{jk} \cdot m_{k\infty}^\pm) + x_{ik\infty}^0 & \xlongequal{\quad} & m_{ij} \cdot (m_{jk} \cdot m_{k\infty}^\pm) + m_{ij} \cdot x_{jk\infty}^0 + x_{ij\infty}^0 \\
 \cong \uparrow & & \downarrow \text{id} \cdot m_{jk\infty}^\pm + \text{id} \\
 (m_{ij} \cdot m_{jk}) \cdot m_{k\infty}^\pm + x_{ik\infty}^0 & & m_{ij} \cdot m_{j\infty}^\pm + x_{ij\infty}^0 \\
 \downarrow m_{ijk} \cdot \text{id} + \text{id} & & \downarrow m_{ij\infty}^\pm \\
 m_{ik} \cdot m_{k\infty}^\pm + x_{ik\infty}^0 & \xrightarrow{m_{ik\infty}^\pm} & m_{i\infty}^\pm
 \end{array}$$

commutes.

Here the map $\text{id} \cdot m_{jk\infty}^\pm$ already incorporates the distributivity isomorphism, as specified in Lemma 4.3.

A morphism $\alpha : m \rightarrow \tilde{m}$ in B_q^n consists of an $n \times n$ matrix of maps $\alpha_{ij} : m_{ij} \rightarrow \tilde{m}_{ij}$ in \mathcal{R} for $0 \leq i < j \leq q$, and of morphisms $(\alpha_{i\infty}^\pm, \psi_{i\infty}^1, \dots, \psi_{i\infty}^\ell) : m_{i\infty}^\pm \rightarrow \tilde{m}_{i\infty}^\pm$ in $T_\ell \mathcal{R}$ for $0 \leq i \leq q$, all compatible with the structure maps of m and \tilde{m} . Thus, there are matrices of objects $\xi_{i\infty}^r$ of \mathcal{R} for $0 \leq r \leq \ell$, each $\alpha_{i\infty}^\pm$ is a map $m_{i\infty}^\pm + \xi_{i\infty}^0 \rightarrow \tilde{m}_{i\infty}^\pm$, and the $\psi_{i\infty}^r$ for $r = 1, \dots, \ell$ are maps $\xi_{i\infty}^r \rightarrow \xi_{i\infty}^{r-1}$ of matrices in \mathcal{R} .

The compatibility condition $\alpha_{i\infty} m_{ij\infty} = \tilde{m}_{ij\infty} (\alpha_{ij} \cdot \alpha_{j\infty})$ allows us to draw the following conclusion.

LEMMA 4.7. *In the situation above one has the identity*

$$x_{ij\infty}^r + \xi_{i\infty}^r = m_{ij} \cdot \xi_{j\infty}^r + \tilde{x}_{ij\infty}^r,$$

for each $r = 0, \dots, \ell$.

4.4. *Start of the proof that $B(*, \text{GL}(\mathcal{R}), \text{GL}(T_\ell \mathcal{R}))$ is contractible*

In the following, 0 and 1 are short for zero and unit matrices over \mathcal{R} of varying size respectively. We will show that

$$\text{colim}_n B^n = B(*, \text{GL}(\mathcal{R}), \text{GL}(T_\ell \mathcal{R}))$$

is contractible by showing that each matrix stabilization functor $\text{in} : B^n \rightarrow B^{2n}$ is trivial in the homotopy category. Here $\text{in}(m) = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}$.

We regard the simplicial categories B^n and B^{2n} as bisimplicial sets, by way of their respective nerves NB^n and NB^{2n} . To be precise, the (p, q) -simplices of NB^{2n} are $N_p B_q^{2n}$. By Lemma 4.5, it then suffices to show that the composite map $\text{inc} = \eta_* \circ \text{in} : NB^n \rightarrow z_* NB^{2n}$ is trivial in the homotopy category. As remarked above, $z_*(NB^{2n})_{(0,q)} \cong (NB^{2n})_{(0,q)} = N_0 B_q^{2n}$, so the subdivision operator z_* does not make any difference before we start to consider positive-dimensional simplices ($p > 0$) in the nerve direction.

Seeing that the image lies in a single path component is easy: if $m \in N_0 B_0^n = \text{ob GL}_n(T_\ell \mathcal{R})$, then there is a path

$$\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} m & m^- \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} m & m^- \\ (1,1) & 1 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 0 \\ (0,1) & 1 \end{bmatrix}.$$

The first arrow represents the 1-simplex in the bar direction given by the matrix multiplication

$$\begin{bmatrix} 1 & m^- \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & m^- \\ 0 & 1 \end{bmatrix} \in \text{GL}_{2n}(T_\ell \mathcal{R}).$$

The second arrow represents the 1-simplex in the nerve direction induced by the morphism $0 = (0, 0) \rightarrow (1, 1) \in T_0\mathcal{R} \subset T_\ell\mathcal{R}$. The third map represents the 1-simplex in the bar direction given by multiplication by

$$\begin{bmatrix} m^+ & m^- \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}_{2n}(\mathcal{R}).$$

The rest of this section extends this path to a full homotopy, from *inc* via maps *jnc* and *knc* to a constant map *inc*.

4.5. *The homotopic maps inc and jnc*

Recall that $\ell \geq 0$ is fixed, $B^n = B(*, \mathrm{GL}_n(\mathcal{R}), \mathrm{GL}_n(T_\ell\mathcal{R}))$ is the simplicial category given by the one-sided bar construction, and $NB^n : [p], [q] \mapsto N_p B_q^n$ is the bisimplicial set given by its degreewise nerve. We already let $\mathrm{inc} : NB^n \rightarrow z_* NB^{2n}$ be the composite of the matrix stabilization map $\mathrm{in} : NB^n \rightarrow NB^{2n}$ and the natural map $\eta_* : NB^{2n} \rightarrow z_* NB^{2n}$.

There is another map $\mathrm{jnc} : NB^n \rightarrow z_* NB^{2n}$, which is homotopic to *inc*. On $N_0 B_q^n$ it is easy to describe: if $m \in N_0 B_q^n$, we declare that $X(m)$ is given by

$$X(m)_{ij} = \begin{cases} \begin{bmatrix} 1 & m_{i\infty}^- \\ 0 & 1 \end{bmatrix} & \text{if } i < j = \infty, \\ \begin{bmatrix} 1 & x_{ij\infty}^0 \\ 0 & 1 \end{bmatrix} & \text{if } i < j < \infty, \end{cases}$$

and let

$$\mathrm{jnc}(m) = X(m) \cdot \mathrm{inc}(m) \in N_0 B_q^{2n} = z_*(NB^{2n})_{(0,q)}.$$

Here

$$\mathrm{jnc}(m)_{ij} = \begin{cases} \begin{bmatrix} m_{i\infty} & m_{i\infty}^- \\ 0 & 1 \end{bmatrix} & \text{if } i < j = \infty, \\ \begin{bmatrix} m_{ij} & x_{ij\infty}^0 \\ 0 & 1 \end{bmatrix} & \text{if } i < j < \infty \end{cases}$$

with $\mathrm{jnc}(m)_{ijk} : \mathrm{jnc}(m)_{ij} \cdot \mathrm{jnc}(m)_{jk} \rightarrow \mathrm{jnc}(m)_{ik}$ being the isomorphisms induced by m_{ijk} as follows: for $k < \infty$ we use the identity $x_{ik\infty}^0 = m_{ij} \cdot x_{jk\infty}^0 + x_{ij\infty}^0$ from Lemma 4.6 and obtain

$$\begin{bmatrix} m_{ijk} & \mathrm{id} \\ \mathrm{id} & \mathrm{id} \end{bmatrix} : \begin{bmatrix} m_{ij} & x_{ij\infty}^0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_{jk} & x_{jk\infty}^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{ij} \cdot m_{jk} & m_{ij} \cdot x_{jk\infty}^0 + x_{ij\infty}^0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} m_{ik} & x_{ik\infty}^0 \\ 0 & 1 \end{bmatrix},$$

and for $k = \infty$ we use the string of isomorphisms

$$\begin{bmatrix} x_{ij\infty}^\ell & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \dots \longrightarrow \begin{bmatrix} x_{ij\infty}^0 & 0 \\ 0 & 0 \end{bmatrix}$$

together with the isomorphism

$$\begin{aligned} \begin{bmatrix} m_{ij\infty} & m_{ij\infty}^- \\ \mathrm{id} & \mathrm{id} \end{bmatrix} : \begin{bmatrix} m_{ij} & x_{ij\infty}^0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_{j\infty} & m_{j\infty}^- \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x_{ij\infty}^0 & 0 \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} m_{ij} \cdot m_{j\infty} + x_{ij\infty}^0 & m_{ij} \cdot m_{j\infty}^- + x_{ij\infty}^0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} m_{i\infty} & m_{i\infty}^- \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

We note that the T_ℓ -direction does not add any complications other than notational. This continues to be true in general, so we simplify notation by considering only the case $\ell = 0$.

The relevant complications arise when one starts moving in the nerve direction. As the construction of the map jnc is quite involved, we give some examples first. The impatient reader can skip this part and restart reading in Subsection 4.6 where the formula in the general case is given.

As an illustration, let $\ell = 0, p = 2$ and $q = 0$ so that

$$m \longleftarrow (m^0 \xleftarrow{(\xi^1, \alpha^1)} m^1 \xleftarrow{(\xi^2, \alpha^2)} m^2) \in N_2 B_0^n = N_2 \text{GL}_n(T_0 \mathcal{R}).$$

Then $\text{jnc}(m)$ is captured by the picture

$$\begin{array}{ccccc} \begin{bmatrix} m^2 & (m^2)^- \\ 0 & 1 \end{bmatrix} & \xrightarrow{\begin{bmatrix} 1 & \xi^2 \\ 0 & 1 \end{bmatrix}} & \begin{bmatrix} m^2 & (m^2)^- + \xi^2 \\ 0 & 1 \end{bmatrix} & \longrightarrow & \begin{bmatrix} m^1 & (m^1)^- \\ 0 & 1 \end{bmatrix} \\ & \searrow \begin{bmatrix} 1 & \xi^2 + \xi^1 \\ 0 & 1 \end{bmatrix} & \downarrow \begin{bmatrix} 1 & \xi^1 \\ 0 & 1 \end{bmatrix} & & \downarrow \begin{bmatrix} 1 & \xi^1 \\ 0 & 1 \end{bmatrix} \\ & & \begin{bmatrix} m^2 & (m^2)^- + \xi^2 + \xi^1 \\ 0 & 1 \end{bmatrix} & \longrightarrow & \begin{bmatrix} m^1 & (m^1)^- + \xi^1 \\ 0 & 1 \end{bmatrix} \\ & & & \searrow & \downarrow \\ & & & & \begin{bmatrix} m^0 & (m^0)^- \\ 0 & 1 \end{bmatrix} \end{array}$$

where the bar direction is written in the ' $g \xrightarrow{m} mg$ ' form, and the unlabelled arrows correspond to the nerve direction, with entries consisting of the appropriate maps α .

An even more complicated example, essentially displaying all the complexity of the general case: $\ell = 0, p = q = 1$, and $\alpha : m^1 \rightarrow m^0 \in N_1 B_1^n$ with $(m^u)_{01\infty}^\pm : (m^u)_{01} \cdot (m^u)_{1\infty}^\pm + (x^u)_{01\infty}^0 \rightarrow (m^u)_{0\infty}^\pm$ (for $u = 0, 1$ running in the nerve direction), $\alpha_{01} : m_{01}^1 \rightarrow m_{01}^0$ and $\alpha_{i\infty}^\pm : (m^1)_{i\infty}^\pm + \xi_{i\infty} \rightarrow (m^0)_{i\infty}^\pm$.

Then $\text{jnc}(m)$ is the map from

$$\begin{array}{ccccc} z^*(\Delta[1] \times \Delta[1]) = & ((0, 0), 0) & \longleftarrow & ((0, 1), 0) & \longleftarrow & ((1, 1), 0) \\ & \uparrow & & \uparrow & \swarrow & \uparrow \\ & ((0, 0), 1) & \longleftarrow & ((0, 1), 1) & \longleftarrow & ((1, 1), 1) \end{array}$$

sending the $((0, 1), 0) \leftarrow ((1, 1), 0) \leftarrow ((1, 1), 1)$ -simplex to

$$\begin{bmatrix} 1 & \xi_{0\infty} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{01}^1 & x_{01\infty}^1 + \xi_{0\infty} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{0\infty}^1 & (m^1)_{0\infty}^- + \xi_{0\infty} \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} m_{01}^1 & x_{01\infty}^1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{0\infty}^1 & (m^1)_{0\infty}^- \\ 0 & 1 \end{bmatrix} \in N_0 B_2^{2n}, \\ \begin{bmatrix} m_{1\infty}^1 & (m^1)_{1\infty}^- \\ 0 & 1 \end{bmatrix}$$

the $((0, 1), 0) \leftarrow ((0, 1), 1) \leftarrow ((1, 1), 1)$ -simplex to

$$\begin{bmatrix} m_{01}^1 & x_{01\infty}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{01}^1 & x_{01\infty}^1 + \xi_{0\infty} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{0\infty}^1 & (m^1)_{0\infty}^- + \xi_{0\infty} \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & \xi_{1\infty} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{1\infty}^1 & (m^1)_{1\infty}^- + \xi_{1\infty} \\ 0 & 1 \end{bmatrix} \in N_0 B_2^{2n}, \\ \begin{bmatrix} m_{1\infty}^1 & (m^1)_{1\infty}^- \\ 0 & 1 \end{bmatrix}$$

(here the identity from Lemma 4.7 is used), and the $(1, 1)$ -simplex $((0, 0), 0) \longleftarrow ((0, 1), 0)$

$$\begin{array}{ccc} & & \uparrow \\ & & ((0, 0), 1) \longleftarrow ((0, 1), 1) \\ & & \uparrow \end{array}$$

to

$$\begin{bmatrix} m_{01}^0 & x_{01\infty}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{0\infty}^0 & (m^0)_{0\infty}^- \\ 0 & 1 \end{bmatrix} \xleftarrow{\begin{bmatrix} \alpha_{01} & \text{id} \\ \text{id} & \text{id} \end{bmatrix} \begin{bmatrix} \alpha_{0\infty} & \alpha_{0\infty}^- \\ \text{id} & \text{id} \end{bmatrix}} \begin{bmatrix} m_{01}^1 & x_{01\infty}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{0\infty}^1 & (m^1)_{0\infty}^- + \xi_{0\infty} \\ 0 & 1 \end{bmatrix}$$

in $N_1 B_1^{2n}$.

Here we have employed the formula $x_{01\infty}^1 + \xi_{0\infty} = m_{01}^1 \cdot \xi_{1\infty} + x_{01\infty}^0$ of Lemma 4.7.

4.6. General version of *jnc*

Consider

$$m = (m^0 \xleftarrow{\alpha^1} m^1 \xleftarrow{\alpha^2} \dots \xleftarrow{\alpha^p} m^p) \in N_p B_q^n. \tag{4.1}$$

Then $(\alpha^u)_{i\infty}$ is given by the tuple

$$((\alpha^u)_{i\infty}^\pm : (m^u)_{i\infty}^\pm + (\xi^u)_{i\infty}^0 \rightarrow (m^{u-1})_{i\infty}^\pm, \{(\psi^u)_{i\infty}^r : (\xi^u)_{i\infty}^r \rightarrow (\xi^u)_{i\infty}^{r-1}\}),$$

for $u = 1, \dots, p, r = 1, \dots, \ell$, but we simplify notation by setting $\xi_{i\infty}^u = (\xi^u)_{i\infty}^0, x_{ij\infty}^u = (x^u)_{ij\infty}^0$, and ignoring the maps ψ . Then *jnc* sends an m as in (4.1) to the simplex $\text{jnc}(m) \in z_*(NB^n)_{pq}$ with value at the $((a, b), c)$ -vertex in $z^*(\Delta[p] \times \Delta[q])$ given by

$$\begin{bmatrix} (m^b)_{c\infty} & (m^b)_{c\infty}^- + \xi_{c\infty}^b + \dots + \xi_{c\infty}^{a+1} \\ 0 & 1 \end{bmatrix} \in \text{GL}_{2n}(T\ell\mathcal{R})$$

with the convention that the ξ -terms only occur if $a + 1 \leq b$. Higher simplices are given by the structural isomorphisms in m . Note that the elements in the off-diagonal blocks are actually all in \mathcal{R} .

More precisely, a triple (ϕ, b, ψ) , where $\phi : [r] \rightarrow [p]$ and $\psi : [r] \rightarrow [q]$ are in Δ and $\phi(r) \leq b \leq p$, determines a $(0, r)$ -simplex in $z^*(\Delta[p] \times \Delta[q])$, because $z^*(\Delta[p] \times \Delta[q])_{(0,r)} = \Delta([r+1], [p]) \times \Delta([r], [q])$ and ϕ together with b determine an element in the first factor. We see that $\text{jnc}(m)(\phi, b, \psi) \in N_0 B_r^{2n}$ is the element whose $(0 \leq i < j \leq r)$ - and $(0 \leq i \leq r < j = \infty)$ -entries are

$$\begin{bmatrix} (m^b)_{\psi(i)\psi(j)} & x_{\psi(i)\psi(j)\infty}^{\phi(j)} + \xi_{\psi(i)\infty}^{\phi(j)} + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} (m^b)_{\psi(i)\infty} & (m^b)_{\psi(i)\infty}^- + \xi_{\psi(i)\infty}^b + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1} \\ 0 & 1 \end{bmatrix}$$

respectively. (As before, the ξ -terms only occur when $\phi(i) + 1 \leq \phi(j)$ and $\phi(i) + 1 \leq b$ respectively.)

Moving in the (nerve =) b -direction is easy because it amounts to connecting two values $\text{jnc}(m)(\phi, b, \psi)$ and $\text{jnc}(m)(\phi, b', \psi)$ by morphisms. As this is determined by the 1-skeleton, it is enough to describe the case $b' = b - 1 < b$. On the $(0 \leq i < j \leq r)$ -entries it is induced by $(\alpha^b)_{\psi(i)\psi(j)} : (m^b)_{\psi(i)\psi(j)} \rightarrow (m^{b-1})_{\psi(i)\psi(j)}$ (in the upper left-hand corner, and otherwise the identity), and on the $(0 \leq i \leq r < j = \infty)$ -entries it is given by $(\xi_{\psi(i)\infty}^b, (\alpha^b)_{\psi(i)\infty}) :$

$(m^b)_{\psi(i)\infty} \rightarrow (m^{b-1})_{\psi(i)\infty}$ and $(\alpha^b)_{\psi(i)\infty}^- : (m^b)_{\psi(i)\infty}^- + \xi_{\psi(i)\infty}^b \rightarrow (m^{b-1})_{\psi(i)\infty}^-$ (in the upper row, and otherwise the identity).

Checking that this is well defined and simplicial amounts to the same kind of checking as we have already encountered, using the same identities. One should note that at no time during the verifications is the symmetry of addition used. It is used, however, for the isomorphism that renders matrix multiplication associative up to isomorphism.

The simplicial homotopy from *inc* to *jnc* is obtained by multiplications (in the bar direction) by matrices of the form

$$\begin{bmatrix} 1 & x_{\psi(i)\psi(j)\infty}^{\phi(j)} + \xi_{\psi(i)\infty}^{\phi(j)} + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & (m^b)_{\psi(i)\infty}^- + \xi_{\psi(i)\infty}^b + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1} \\ 0 & 1 \end{bmatrix}.$$

4.7. *The homotopic maps knc and lnc*

Consider the following variant *knc* of the map *jnc*: using the same notation as for *jnc*, when evaluated on (ϕ, b, ψ) , where $\phi : [r] \rightarrow [p]$ and $\psi : [r] \rightarrow [q]$ are in Δ and $\phi(r) \leq b \leq p$, $\text{knc}(m)(\phi, b, \psi) \in N_0 B_r^{2n}$ is the element whose $(0 \leq i < j \leq r)$ -entries are

$$\begin{bmatrix} (m^b)_{\psi(i)\psi(j)} & x_{\psi(i)\psi(j)\infty}^{\phi(j)} + \xi_{\psi(i)\infty}^{\phi(j)} + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1} \\ 0 & 1 \end{bmatrix},$$

and whose $(0 \leq i \leq r < j = \infty)$ -entries are

$$\begin{bmatrix} (m^b)_{\psi(i)\infty} + (\Xi, \Xi) & (m^b)_{\psi(i)\infty}^- + \Xi \\ (1, 1) & 1 \end{bmatrix},$$

where we use the abbreviation $\Xi = \xi_{\psi(i)\infty}^b + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1}$. The entries for finite j are the same as for *jnc*.

There is a natural map (in the nerve direction) from *jnc* to *knc* (of the form $(X, \text{id}) : (A, B) \rightarrow (A + X, B + X) \in T_\ell \mathcal{R}$, induced by the identity) giving a homotopy.

Finally, let *lnc* : $NB^n \rightarrow z_* NB^{2n}$ be induced by the constant map sending any matrix to $\begin{bmatrix} 1 & 0 \\ (0,1) & 1 \end{bmatrix}$. Matrix multiplication yields

$$\begin{bmatrix} (m^b)_{\psi(i)\infty}^+ + \Xi & (m^b)_{\psi(i)\infty}^- + \Xi \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ (0, 1) & 1 \end{bmatrix} = \begin{bmatrix} (m^b)_{\psi(i)\infty} + (\Xi, \Xi) & (m^b)_{\psi(i)\infty}^- + \Xi \\ (1, 1) & 1 \end{bmatrix}$$

with the same abbreviation $\Xi = \xi_{\psi(i)\infty}^b + \dots + \xi_{\psi(i)\infty}^{\phi(i)+1}$ as above. We obtain a homotopy from *lnc* to *knc*.

Hence, *inc* is connected by a chain of homotopies to a constant map. As $\eta_* : \text{id} \rightarrow z_*$ is a monomorphism in the homotopy category, this means that the stabilization map $\text{in} : B^n \rightarrow B^{2n}$ is homotopically trivial, and so $B(*, \text{GL}(\mathcal{R}), \text{GL}(T_\ell \mathcal{R})) = \text{colim}_n B^n$ is contractible for each $\ell \geq 0$. Hence, $B(*, \text{GL}(\mathcal{R}), \text{GL}(T\mathcal{R}))$ is also contractible. This concludes the proof of Theorem 1.1.

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