manuscripta math. 118, 99–119 (2005)

Andrew Baker · Birgit Richter

Invertible modules for commutative $\mathbb{S}$-algebras with residue fields

Received: 15 October 2004 / Revised version: 10 May 2005
Published online: 10 August 2005

Abstract. The aim of this note is to understand under which conditions invertible modules over a commutative $\mathbb{S}$-algebra in the sense of Elmendorf, Kriz, Mandell & May give rise to elements in the algebraic Picard group of invertible graded modules over the coefficient ring by taking homotopy groups. If a connective commutative $\mathbb{S}$-algebra R has coherent localizations $(R_*)_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \lhd R_*$, then for every invertible R-module U, $U_* = \pi_* U$ is an invertible graded R_*-module. In some non-connective cases we can carry the result over under the additional assumption that the commutative $\mathbb{S}$-algebra has 'residue fields' for all maximal ideals $\mathfrak{m} \lhd R_*$ if the global dimension of R_* is small or if R is 2-periodic with underlying Noetherian complete local regular ring R_0. We apply these results to finite abelian Galois extensions of Lubin-Tate spectra.

Key words. Commutative S-algebra, invertible module, Picard group, 55P15, 55P42, 55P60

Introduction

For an arbitrary symmetric monoidal category $\mathscr{C}$, one can ask which objects are invertible. The *Picard group*, $\mathrm{Pic}(\mathscr{C})$, is then the collection of isomorphism classes of such invertible objects in $\mathscr{C}$. This does not have to be a set in general, but if it is one, then $\mathrm{Pic}(\mathscr{C})$ is an abelian group in a natural way.

The notion of Picard group originates from algebraic geometry. The classical example is that of the Picard group of the category of A-modules for a commutative ring A. In recent years, topologists have introduced symmetric monoidal categories of spectra, the categories of modules over commutative $\mathbb{S}$-algebras, whose derived categories are also symmetric monoidal and provide natural generalizations of categories of modules over commutative rings.

The paper by Strickland [24], following a talk of Hopkins, introduced Picard groups in that framework. Examples have been considered and in some cases calculated in [24, 14, 19, 12].

In general, it is *not* clear if invertible modules over a commutative $\mathbb{S}$-algebra in the sense of [10] give rise to invertible modules over the coefficient ring when

A. Baker: Mathematics Department, University of Glasgow, Glasgow G12 8QW, Scotland. e-mail: a.baker@maths.gla.ac.uk

B. Richter: Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany. e-mail: richter@math.uni-hamburg.de

DOI: 10.1007/s00229-005-0582-1

one applies homotopy groups. In Example 44 we describe an explicit invertible module over a commutative $\mathbb{S}$-algebra R whose coefficient module is not an invertible graded R_*-module. We investigate some restricted classes of commutative $\mathbb{S}$-algebras for which we can prove such a transfer result. Similar questions about various other Picard groups were considered in [11, 12, 19], but we are unaware of published topological results of the form we describe. Our motivation comes in part from applications to work on Galois extensions in [3]. However, our results apply to classes of examples such as complex cobordism MU and complex K-theory.

We discuss mainly three classes of examples of commutative $\mathbb{S}$-algebras for which we can prove a transfer result from the Picard group of module spectra over these commutative $\mathbb{S}$-algebras to the Picard group of graded modules over its coefficients. Essentially these are as follows:

- connective commutative $\mathbb{S}$-algebras with coherent coefficient rings or coefficient rings satisfying Eilenberg's condition,
- commutative $\mathbb{S}$-algebras with coherent coefficients, multiplicative residue fields and small global dimension, and
- commutative $\mathbb{S}$-algebras whose coefficients R_* are of the form $R_0[u, u^{-1}]$ with R_0 a Noetherian complete local regular ring.

For the precise statements see Theorems 21, 25, 28, and 38.

In the non-connective case we add the restriction that the commutative $\mathbb{S}$-algebra R has 'residue fields' for all maximal ideals in its coefficients. Although we do not have a complete understanding of such $\mathbb{S}$-algebras, it appears that most standard examples satisfy this requirement; however, in Example 9 we draw attention to a spectrum which has no such residue field.

At the end of Section 8 we apply our results to Galois theory of commutative $\mathbb{S}$-algebras. If one adjoins enough roots of unity to a Lubin-Tate spectrum, we prove that there are no non-trivial abelian finite Galois extensions as long as the order of the group is invertible in the coefficient ring.

In Section 9 we summarize our results by providing a list of examples of commutative $\mathbb{S}$-algebras for which the topological and algebraic Picard groups agree. We close with an explicit counterexample, proving that $\mathrm{Pic}(R)$ might differ from $\mathrm{Pic}(R_*)$ in general.

1. The Picard group of a commutative $\mathbb{S}$-algebra

In this section we relate the Picard group of the commutative $\mathbb{S}$-algebra R to the Picard group of the coefficient ring R_*. Of course this is a special case of the more general notion for a symmetric monoidal category [11, 12, 19].

We follow [24, 14] in defining $\mathrm{Pic}(R)$ to be the collection of equivalence classes of invertible objects in the derived category of R-modules, $\mathscr{D}_R$. By Proposition 16, $\mathrm{Pic}(R)$ is a set in all cases that we will consider. Whenever choosing a representative for an equivalence class $[U] \in \mathrm{Pic}(R)$ we will pick a cofibrant R-module. Defining the product of equivalence classes $[U]$ and $[V]$ by

$$[U][V] = [U \wedge_R V],$$

$\operatorname{Pic}(R)$ becomes an abelian group. We also have an algebraic Picard group $\operatorname{Pic}(R_*)$ of invertible graded R_*-modules and our goal is to discuss the relationship of this with $\operatorname{Pic}(R)$.

Let M_* be an invertible graded R_*-module, say with inverse N_*, so

$$M_* \otimes_{R_*} N_* \cong R_*.$$

Then M_* and N_* are finitely generated projective of constant rank 1. Choosing a finitely generated free cover $F_* \longrightarrow M_*$, we see that M_* is a summand of F_*. Of course we have $F_* = \pi_* F$, where F is a finite wedge of suspensions of copies of R-spheres S_R. Furthermore, the associated splitting can be realized by a homotopy idempotent self-map $\varepsilon\colon F \longrightarrow F$. Now M_* is realized as the homotopy image of ε. (This argument was pointed out to us by the referee.)

Given the above realization of M_* as an R-module M, we obtain a map

$$\Phi\colon \operatorname{Pic}(R_*) \longrightarrow \operatorname{Pic}(R); \quad \Phi([M_*]) = [M] \tag{1}$$

which is a group homomorphism. If $[M_*] \in \ker \Phi = [R_*]$, then $M \simeq R$ and so $[M_*] = [R_*] = 0$. Thus Φ is a monomorphism. The main aim of this paper is to establish conditions under which Φ is an isomorphism.

2. Recollections on coherent rings and modules

The notion of a *coherent commutative ring* has proved important in topology, especially in connection with MU. We begin by reviewing the basic notions. As a convenient reference we cite Cohen [8], but the algebraic theory can be found in many places such as [6, 23].

Let A be a (possibly graded) commutative ring. Then an A-module M is *finitely presented* if there is a short exact sequence

$$0 \to K \longrightarrow F \longrightarrow M \to 0$$

in which F and K are finitely generated and F is free. Such a short exact sequence is called a *finite presentation of M*.

Lemma 1. *Let M be finitely presented and suppose that*

$$0 \to L \longrightarrow P \longrightarrow M \to 0$$

is a short exact sequence in which P is finitely generated and projective. Then L is finitely generated.

Proof. Let

$$0 \to K \longrightarrow F \longrightarrow M \to 0$$

be a finite presentation of M. By Schanuel's Lemma, there is an isomorphism

$$P \oplus K \cong F \oplus L.$$

Now since the left-hand side is finitely generated, L is also finitely generated. □

The A-module M is *coherent* if it and all its finitely generated submodules are finitely presented. A is *coherent* if it is coherent as an A-module.

Let A be a commutative local ring and M an A-module. Recall that a resolution $F_* \longrightarrow M$ is *minimal* if for each n, the differential $d_n\colon F_n \longrightarrow F_{n-1}$ has $\ker d_n \subseteq \mathfrak{m}F_n$ and so $\operatorname{im} d_n \subseteq \mathfrak{m}F_{n-1}$.

Proposition 2. *Let A be a commutative coherent local ring with maximal ideal $\mathfrak{m}$. If M is a coherent A-module, then M admits a minimal resolution $F_* \longrightarrow M \to 0$ which is by finitely generated free modules F_n.*

Proof. We begin by choosing a finitely generated free module F_0 with the property that its reduction modulo $\mathfrak{m}$ is isomorphic to the reduction of the module M modulo $\mathfrak{m}$; furthermore, this isomorphism factors through M, giving the following diagram.

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d, "p"'] \arrow[dr] & F_0 \otimes A/\mathfrak{m} \arrow[d, "\cong"] \\ M \arrow[r] & M \otimes A/\mathfrak{m} \end{tikzcd}$$

Note that by Nakayama's Lemma, the map p is an epimorphism. As p is a map between coherent modules, its kernel is finitely generated and coherent. It is obvious that the kernel of p is contained in $\mathfrak{m}F_0$. Following this pattern of argument we can inductively produce a minimal resolution as required. $\square$

In our work we will make use of the following result of [8, proposition 1.5].

Proposition 3. *Let A_α be a filtered direct system of coherent commutative rings such that $A = \operatorname{colim}_\alpha A_\alpha$ is flat over each A_α. Then A is coherent.*

Corollary 4. *Let A be a coherent commutative ring and Σ a multiplicative subset. Then the localization $A[\Sigma^{-1}]$ is a coherent ring. In particular, for every prime ideal $\mathfrak{p} \lhd A$, the localization $A_\mathfrak{p}$ is coherent.*

Proof. This follows from Proposition 3 since such localizations are filtered direct colimits and are exact. $\square$

Example 5.
1. Any commutative Noetherian ring is coherent.
2. Any countably generated polynomial ring over a Noetherian ring is coherent since it is a colimit as in Proposition 3.
3. In particular, MU_* and BP_* are coherent and so are all their localizations and quotients with respect to finitely generated ideals.

3. Local reductions of R-modules

Consider the following condition on a maximal ideal $\mathfrak{m} \lhd R_*$.

Condition (A). There is an R-module W for which the R_*-module $W_* = \pi_* W$ is isomorphic to the residue field $R_*/\mathfrak{m}$. If such a W exists, we will refer to it as a residue field. We will often choose one and denote it by $\kappa(\mathfrak{m})$.

Notice that $\kappa(\mathfrak{m})$ is clearly $R_\mathfrak{m}$-local, where $R_\mathfrak{m}$ denotes the commutative R-algebra associated with the algebraic localization

$$\pi_*(\quad)_\mathfrak{m} = (R_*)_\mathfrak{m} \otimes_{R_*} \pi_*(\quad)$$

of the homotopy functor on $\mathscr{M}_R$, for details see [10]. So the existence of such an R-module is equivalent to the existence of a corresponding $R_\mathfrak{m}$-module.

We will be interested in $\mathbb{S}$-algebras R for which Condition (A) is satisfied by *all* maximal ideals $\mathfrak{m} \lhd R_*$. Here are some examples.

Example 6. If R is connective then each maximal ideal $\mathfrak{m} \lhd R_*$ has for its residue field a quotient field $\Bbbk(\mathfrak{m}) = R_0/\mathfrak{m}_0$ of R_0. There is a corresponding Eilenberg-Mac Lane R-algebra $H\Bbbk(\mathfrak{m})$ which we may take for $\kappa(\mathfrak{m})$.

Example 7. Let $p > 0$ be a prime and $R = MU_{I_{p,n}}$, where

$$I_{p,n} = (p, v_1, \ldots, v_{n-1})$$

is the n-th invariant ideal for p. Then the Morava K-theory $K(n)$ with $1 \leqslant n < \infty$ is such a spectrum $\kappa(I_{p,n})$.

Example 8. $KO[1/2]$ satisfies Condition (A) for every maximal ideal in

$$KO[1/2]_* = \mathbb{Z}[1/2, y, y^{-1}].$$

The only maximal ideal containing an odd prime p is (p) and we may take $\kappa(p) = KO \wedge M(p)$ where $M(p)$ is the usual mod p Moore spectrum.

Example 9. To see a non-example, we refer to [5, example 7.6]: the topological significance of this is that the Tate spectrum $\widehat{\mathbb{H}}(BC_3, \mathbb{F}_3)$ is a commutative $\mathbb{S}$-algebra, but the maximal ideal generated by the exterior generator in degree one does not give rise to a residue field. We learned of this example from John Greenlees.

Example 8 leads us to introduce another condition on a commutative $\mathbb{S}$-algebra R and a maximal ideal $\mathfrak{m} \lhd R_*$ that turns out to be useful in our work.

Condition (B). There is an R-algebra R' where R'_* is a local ring with maximal ideal $\mathfrak{m}' \lhd R'_*$ which satisfies Condition (A) and whose unit induces a local homomorphism $(R_*)_\mathfrak{m} \longrightarrow R'_*$ which makes R'_* a flat $(R_*)_\mathfrak{m}$-module.

If $\mathfrak{m}$ satisfies Condition (A), then the localization map $R \longrightarrow R_\mathfrak{m}$ satisfies Condition (B).

Remark 10. Suppose that B is a commutative ring and $A \subseteq B$ is a subring so that B is a finite A-algebra. It is standard that if A is Noetherian then so is B. Conversely, if B is Noetherian then A is Noetherian by the Eakin-Nagata theorem [17, theorem 3.7]. In these cases B is automatically flat over A.

Example 11. Consider KO and the maximal ideals

$$\mathfrak{m} \lhd KO_* = \mathbb{Z}[\eta, y, w, w^{-1}]/(2\eta, \eta^3, \eta y, y^2 - 4w).$$

If p is an odd prime, then the only maximal ideal containing p is (p) and as in Example 8 we may take $\kappa(p) = KO \wedge M(p)$. The only maximal ideal containing 2 is $(2, \eta, y)$ and we may take the obvious map $KO \longrightarrow KU_{(2)}$.

Thus every maximal ideal containing an odd prime $\mathfrak{m} \lhd KO_*$ satisfies Condition (B).

The notions in the next definition extend to graded rings. For the Noetherian case, see [17].

Definition 12. *Let A be a commutative ring.*

- *A is a* regular local ring *if it is a local ring whose maximal ideal $\mathfrak{m}$ is generated by a sequence $u_1, u_2, \ldots, u_n$, where n is the Krull dimension of A; such a sequence is regular.*
- *A is a* regular ring *if for every maximal ideal $\mathfrak{m} \lhd A$, the localization $A_\mathfrak{m}$ is a regular local ring.*
- *A is a* non-Noetherian regular local ring *if its maximal ideal is generated by an infinite countable regular sequence $u_1, u_2, \ldots$.*
- *A is a* non-Noetherian regular ring *if for every maximal ideal $\mathfrak{m} \lhd A$, the localization $A_\mathfrak{m}$ is a (possibly non-Noetherian) regular local ring.*

Proposition 13. *If R_* is a (possibly non-Noetherian) regular local ring with maximal ideal $\mathfrak{m} \lhd R_*$, then there is an R-module $\kappa(\mathfrak{m})$ for which $\kappa(\mathfrak{m})_* = R_*/\mathfrak{m}$. Therefore $\mathfrak{m}$ satisfies* Condition (A).

Proof. Given a (possibly infinite) regular sequence $u_1, u_2, \ldots$ which generates $\mathfrak{m}$, we may follow the approach of [10, section V.1] to construct an R-module $\kappa(\mathfrak{m})$ which realizes $R_*/\mathfrak{m}$ as $\kappa(\mathfrak{m})_* = \pi_*\kappa(\mathfrak{m})$. □

Corollary 14. *If R_* is a regular ring then every maximal ideal $\mathfrak{m} \lhd R_*$ satisfies* Condition (A).

There is an associated *Koszul complex*

$$K_{*,*} = \Lambda_{R_*}(e_i : i \geqslant 1),$$

with e_i in bidegree $(1, |u_i|)$ and which is a differential graded algebra with differential d given by

$$de_i = u_i.$$

This provides a free resolution

$$K_{\bullet,*} \longrightarrow R_*/\mathfrak{m} \to 0.$$

Following the method of construction of the Künneth spectral sequence in [10, section IV.5], we can use this to define a cell structure on the R-module $\kappa(\mathfrak{m})$, with the cells corresponding to the distinct monomials in the e_i.

Later, we will need multiplicative structures on our residue fields. These are provided by Angeltveit's result [1, theorem 4.2].

Theorem 15. *If R is a commutative $\mathbb{S}$-algebra whose coefficients are concentrated in even degrees and if an ideal $I \lhd R_*$ is generated by a regular sequence, then there is an $\mathbb{S}$-algebra structure on R/I and $R \longrightarrow R/I$ is central.*

4. A finiteness result

General treatments of invertible objects in derived categories and Picard groups may be found in [11, 12, 19]. It is standard that an invertible object in $\mathscr{D}_R$ is strongly dualizable [15]. The following result on strongly dualizable objects in $\mathscr{D}_R$ is taken from [12, proposition 2.1] (also see [18, 19]).

Proposition 16. *Let X be an R-module. Then X is strongly dualizable in $\mathscr{D}_R$ if and only if it is weakly equivalent to a retract of a finite cell R-module.*

Let U be an *invertible* R-module, *i.e.*, U is a cofibrant R-module and there is a cofibrant R-module V for which $U \wedge_R V \simeq R$. Then V is a strong dual for U and by Proposition 16, U and V are retracts of finite cell R-modules.

The following Lemma allows us to apply coherence conditions to topological settings.

Lemma 17. *Let R be a commutative $\mathbb{S}$-algebra with coherent coefficient ring R_*.*

(a) Any finite cell R-module M gives rise to a finitely generated coherent R_*-module M_*.
(b) Every retract N of a finite R-cell module M has finitely generated coherent coefficients N_*.

Proof. As M is built in finitely many steps via cofibre sequences of the form

$$\Sigma^n R \longrightarrow X \longrightarrow Y,$$

its coefficients M_* are built up out of exact couples of the form

$$\begin{tikzcd} X_* \arrow[rr] & & Y_* \arrow[dl] \\ & \Sigma^n R_* \arrow[ul] & \end{tikzcd}$$

Applying [8, theorem 3.1], we see that M_* is finitely generated coherent.

As retracts of cell R-modules correspond to finitely generated submodules of R_*-modules M_* as above, the claim follows. $\square$

Corollary 18. *Suppose that R_* is coherent and U is an invertible R-module. Then U_* is a coherent R_*-module and hence it has a resolution by finitely generated projective R_*-modules.*

5. The connective case

Let R be a connective commutative $\mathbb{S}$-algebra and let $\mathfrak{m} \lhd R_*$ be a maximal ideal. Recall from [10] that there is an Eilenberg-Mac Lane object $H(R_0/\mathfrak{m}_0)$ which is also a commutative R-algebra. We will view this as a residue field $\kappa(\mathfrak{m})$.

Lemma 19. *Let R be a connective commutative $\mathbb{S}$-algebra and let $\mathfrak{m} \lhd R_*$ be a maximal ideal for which $(R_*)_\mathfrak{m}$ is coherent. If M, N are R-modules with $(N_*)_\mathfrak{m}$ coherent as an $(R_*)_\mathfrak{m}$-module, then the E^2-term of the Künneth spectral sequence has the form*

$$\mathrm{E}^2_{p,*} = \mathrm{Tor}^{(R_\mathfrak{m})_*}_{p,*}(\kappa(\mathfrak{m})^R_* M_\mathfrak{m}, (N_\mathfrak{m})_*) \cong \kappa(\mathfrak{m})^R_* M_\mathfrak{m} \otimes_{R_*} Q_{p,*}, \tag{2}$$

where $Q_{\bullet,*}$ is a minimal resolution of $(N_\mathfrak{m})_*$.

Proof. As $\pi_*(\kappa(\mathfrak{m}) \wedge_R M)$ is an $(R_\mathfrak{m})_*$-module, we can replace R by $R_\mathfrak{m}$ and N by its localization $N_\mathfrak{m}$. Thus we might as well assume that R_* is a coherent local ring for the remainder of this proof.

We begin by choosing a free resolution $Q_{\bullet,*} \longrightarrow N_* \to 0$ of N_* by finitely generated free R_*-modules. Using Proposition 2, we can arrange this to be minimal.

Following [10], the E^2-term for the Künneth spectral sequence (2) can be constructed using the above resolution, giving

$$\mathrm{E}^2_{p,*} = \mathrm{Tor}^{R_*}_{p,q}(\kappa(\mathfrak{m})^R_* M, N_*) = \mathrm{H}_p(\kappa(\mathfrak{m})^R_* M \otimes_{R_*} Q_{\bullet,*}, \mathrm{id} \otimes d_\bullet).$$

By minimality this yields

$$\mathrm{E}^2_{p,*} = \kappa(\mathfrak{m})^R_* M \otimes_{R_*} Q_{p,*}. \quad \square \tag{3}$$

We begin with a local result. Recall that a finitely generated module over a local ring is projective if and only if it is free.

Proposition 20. *Suppose that R is connective and that R_* is coherent and local with maximal ideal $\mathfrak{m} \lhd R_*$. If U is an invertible R-module, then for some $k \in \mathbb{Z}$, $U \simeq \Sigma^k R$ and U_* is an invertible graded R_*-module.*

Proof. We follow the ideas and notation in the proof of Lemma 19, taking $M = U$ and $N = V$ where $U \wedge_R V \simeq R$.

In order to shorten notation we will write $\kappa = H(R_0/\mathfrak{m}_0)$. By Corollary 18 we can choose a minimal free resolution $Q_{\bullet,*} \longrightarrow V_* \to 0$. Using (3) we get as E^2-term of the Künneth spectral sequence

$$\mathrm{E}^2_{p,*} = \mathrm{Tor}^{R_*}_{p,*}(\kappa^R_* U, V_*) \cong \kappa^R_* U \otimes Q_{p,*}.$$

Without loss of generality we can assume that U is connective and $\pi_0(U) \neq 0$. The Künneth isomorphism

$$\kappa_*^R(U) \otimes_{\kappa_*} \kappa_*^R(V) \cong \kappa_*^R(R) \cong \kappa_*$$

forces $\kappa_*^R U$ to be free of rank one over $\kappa_* \cong R_0/\mathfrak{m}_0$. Thus it must be concentrated in degree zero and has to be isomorphic to $R_0/\mathfrak{m}_0 = \kappa_0$. The whole spectral sequence is concentrated in the first quadrant. No differential can hit the entry in the (0, 0)-coordinate. Therefore

$$(\kappa_* \otimes_{R_*} Q_{0,*})_0 = \kappa_0 \otimes_{R_0} Q_{0,0} \cong \kappa_0.$$

As $Q_{0,0}$ is the zeroth homotopy group of some sum of R-spheres, this forces $Q_{0,0}$ to be equal to R_0, in particular it is free over R_0. The minimality of the resolution ensures that $Q_{p,0}$ is zero for $p > 0$. Therefore the zero-line $\mathrm{E}^2_{p,0}$ vanishes except for κ_0 at $p = 0$.

Inductively we assume that $Q_{p,i} = 0$ for all $p > 0$, $i \leqslant n$, and that $Q_{0,i} \cong R_i$ for all $i \leqslant n$. Then the $(0, n+1)$-entry in the E^2-term cannot be hit by any differential, therefore it must be an infinite cycle. As nothing else survives in total degree $n+1$, we obtain

$$(\kappa_* \otimes_{R_*} Q_{0,*})_{n+1} = \kappa_0 \otimes Q_{0,n+1} = 0. \tag{4}$$

This means that the whole module $Q_{0,n+1}$ gets killed by the relations in the tensor product over R_*. We know that $R_{n+1} \subseteq Q_{0,n+1}$ because $Q_{\bullet,*}$ is an R_*-resolution and $Q_{0,0} \cong R_0$. From (4) we know that there cannot be more in $Q_{0,n+1}$.

As $Q_{0,n+1} \cong R_{n+1}$, minimality ensures again that $Q_{p,n+1} = 0$ for all $p > 0$ and the induction is continued.

Hence we obtain that $Q_{0,q} \cong R_q$ for all $q \geqslant 0$ and the higher terms in the resolution satisfy $Q_{p,q} \cong 0$ for $p > 0$. This gives $V_* \cong R_*$, proving the claim. □

We now use our local information to obtain a global result.

Theorem 21. *If R is a connective commutative $\mathbb{S}$-algebra, such that for every maximal ideal $\mathfrak{m} \lhd R_*$ the localization is coherent, then every invertible R-module spectrum U has invertible coefficients U_*.*

Proof. Let V be an inverse for U. For each maximal ideal $\mathfrak{m} \lhd R_*$,

$$\mathrm{Tor}^{R_*}_{s,*}(U_*, V_*)_{\mathfrak{m}} \cong \mathrm{Tor}^{(R_*)_{\mathfrak{m}}}_{s,*}((U_*)_{\mathfrak{m}}, (V_*)_{\mathfrak{m}})$$

and also

$$(U_{\mathfrak{m}} \wedge_{R_{\mathfrak{m}}} V_{\mathfrak{m}}) \simeq (U \wedge_R V)_{\mathfrak{m}} \simeq R_{\mathfrak{m}}.$$

By our local result Proposition 20 we have

$$U_{\mathfrak{m}} \simeq \Sigma^k R_{\mathfrak{m}}, \quad V_{\mathfrak{m}} \simeq \Sigma^{-k} R_{\mathfrak{m}}.$$

Hence for $s > 0$,

$$\mathrm{Tor}^{R_*}_{s,*}(U_*, V_*)_{\mathfrak{m}} = 0.$$

Now by a standard result on localizations [17],

$$\mathrm{Tor}^{R_*}_{s,*}(U_*, V_*) = 0 \quad \text{for } s > 0.$$

So we find that the edge homomorphism $U_* \otimes_{R_*} V_* \longrightarrow R_*$ of the Künneth spectral sequence is an isomorphism. Therefore U_* is an invertible graded R_*-module with inverse V_*. □

6. Eilenberg's condition

For some important examples of spectra the coherence requirement is too much to ask for. In [9], Eilenberg introduced conditions which ensure the existence of minimal resolutions. We recall a particular case which then applies to $\mathbb{S}$ and other commutative $\mathbb{S}$-algebras, leading to important topological results.

Recall that a graded group M_* is *connective* if $M_n = 0$ whenever $n < 0$. Also, if A_* is a connective graded commutative local ring, then its unique maximal ideal $\mathfrak{m} \lhd A_*$ has components

$$\mathfrak{m}_n = \begin{cases} \mathfrak{m}' & \text{if } n = 0, \\ A_n & \text{otherwise,} \end{cases}$$

where A_0 is local with maximal ideal $\mathfrak{m}' \lhd A_0$.

Proposition 22. *Let A_* be a connective graded commutative local ring for which A_0 is Noetherian and each A_n is a finitely generated A_0-module. Then every finitely generated A_*-module admits a minimal resolution by free A_*-modules.*

Proof. See [9, proposition 14]. □

Example 23. The maximal ideals in the graded ring $\mathbb{S}_*$ have the form

$$\mathfrak{m}(p)_n = \begin{cases} (p) \lhd \mathbb{Z} & \text{if } n = 0, \\ \mathbb{S}_n & \text{otherwise,} \end{cases}$$

for rational primes $p > 0$. On localizing we obtain the graded local rings $(\mathbb{S}_*)_{(p)}$ which satisfy the requirements of Proposition 22.

Lemma 24. *Let R be a commutative $\mathbb{S}$-algebra with connective homotopy ring R_* and let $\mathfrak{m} \lhd R_*$ be a maximal ideal. If $(R_\mathfrak{m})_* = (R_*)_\mathfrak{m}$ satisfies the requirements of* Proposition 22*, then for any retract W of a finite cell $R_\mathfrak{m}$-module, there is a minimal resolution of W_* by free $(R_\mathfrak{m})_*$-modules.*

Proof. For a finite cell module W, this involves a straightforward inductive verification that W_* is finitely generated. But any retract of such a W has the same property. □

The key point in the proof of Proposition 20 and Theorem 21 was the construction of a minimal free resolution and a local-to-global argument. We can apply the same strategy to obtain the following result.

Theorem 25. *Let R be a commutative $\mathbb{S}$-algebra whose homotopy ring R_* is connective. Suppose that for every maximal ideal $\mathfrak{m} \lhd R_*$, the localization $(R_{\mathfrak{m}})_* = (R_*)_{\mathfrak{m}}$ has $(R_0)_{\mathfrak{m}}$ Noetherian and each $(R_n)_{\mathfrak{m}}$ is a finitely generated $(R_0)_{\mathfrak{m}}$-module. If U and V are invertible R-modules for which $U \wedge_R V \simeq R$, then we have*

$$U_* \otimes_{R_*} V_* \cong R_*$$

and so U_* is an invertible graded R_*-module. In particular, there is a $k \in \mathbb{Z}$ for which $U_m = 0 = V_n$ whenever $m < k$ and $n < -k$ and then

$$U_k \otimes_{R_0} V_{-k} \cong R_0,$$

so U_k is an invertible R_0-module.

Using this, we obtain the following well-known result of [24, 14].

Example 26. Taking $R = \mathbb{S}$ and recalling Example 23, we see that $U \wedge V \simeq \mathbb{S}$ implies that for some $k \in \mathbb{Z}$ as in the Theorem, U_k is an invertible $\mathbb{Z}$-module and $U_k \cong \mathbb{Z} \cong V_{-k}$, hence $U_{*+k} \cong \mathbb{S}_* \cong V_{*-k}$. It follows that $\Sigma^{-k}U \simeq \mathbb{S} \simeq \Sigma^k V$.

Other examples include MU, MSp, ku, ko and the connective spectrum of topological modular forms tmf. The last example is known to be a commutative $\mathbb{S}$-algebra and its homotopy ring is computed in [4, 20]; it has $\pi_0 tmf = \mathbb{Z}$ and satisfies the conditions of Proposition 22.

So far we did not give any proof in the case of Eilenberg-Mac Lane spectra over arbitrary commutative rings. For the sake of completeness we add this result here, although it is probably well-known.

Proposition 27. *Let A be a commutative ring with unit. Then for every invertible HA-module spectrum U, U_* is an invertible graded A-module.*

Proof. The proof we give here is an elementary adaption of Fausk's proof in [11, 3.2,3.3] to our setting. Without loss of generality we can assume that U has its first non-vanishing homotopy group in degree zero.

Assume first that A is a local ring. Let V be an inverse of U over HA. We know that $U_0 \otimes_A V_0 \cong A$ because nothing else can hit the zeroth homotopy group in

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^A_{p,q}(U_*, V_*) \Longrightarrow \pi_0(HA) = A.$$

As A is local, the only invertible A-modules are the ones which are isomorphic to A. In particular U_0 and V_0 are free. Therefore the rest of the $(p, 0)$-line vanishes. This forces the $(1, 0)$-entry to survive, so it has to be trivial, which means that $U_1 = 0 = V_1$. Iteratively, we can clear out the whole E^2-page except for the $(0, 0)$-entry. In particular, for all $p > 0$,

$$\mathrm{Tor}^A_{p,q}(U_*, V_*) = 0.$$

A local-to-global argument then proves the result in general. □

7. Small global dimension

The crucial point in our proofs is the collapsing of the Künneth spectral sequence. For commutative $\mathbb{S}$-algebras with residue fields we gain an analogue of Theorem 21 as long as we can exclude non-trivial differentials.

Theorem 28. *Let R be a commutative $\mathbb{S}$-algebra, such that for every maximal ideal $\mathfrak{m} \lhd R_*$ the ring $(R_*)_{\mathfrak{m}}$ is coherent and assume that R satisfies* Condition (A) *and has a structure of a ring spectrum on each of its residue fields. If R_* has global dimension at most 2 then every invertible R-module spectrum U has invertible coefficients U_*.*

Proof. As we can perform a local-to-global argument, we may as well assume that R_* is local and coherent. Let V be an inverse of U. The existence of a residue field κ which is a ring spectrum ensures that $\kappa^R_*(U)$ is a κ_*-vector space. The Künneth map

$$\kappa^R_*(U) \otimes_{\kappa_*} \kappa^R_*(V) \longrightarrow \kappa_*$$

has to be an isomorphism. Therefore we can set $\kappa^R_*(U) \cong \kappa_*$.

Together with the existence of a minimal resolution of V_* this guarantees that the E^2-term of the Künneth spectral sequence is given by

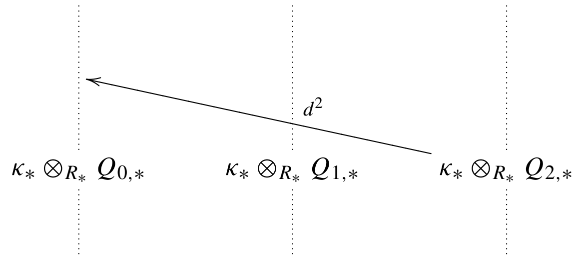
$$\mathrm{E}^2_{p,*} = \mathrm{Tor}^{R_*}_{p,*}(\kappa_*, V_*) \cong \kappa_* \otimes_{R_*} Q_{p,*}.$$

If the global dimension of R_* is at most 1, this spectral sequence is concentrated in two columns. Therefore there cannot be any non-trivial differentials. The abutment of the spectral sequence is κ_*. As $Q_{p,*}$ is a resolution of V_*, $Q_{0,*}$ cannot be trivial. If $Q_{1,*}$ were non-trivial a dimension count leads to a contradiction. Similarly, $Q_{0,*}$ must be free of rank one over R_* and therefore $V_* \cong \Sigma^k R_*$ for some $k \in \mathbb{Z}$.

In the case of global dimension 2 the E^2-term of the Künneth spectral sequence converging to κ_* with

$$\mathrm{Tor}^{R_*}_{p,*}(\kappa_*, V_*) \cong \kappa_* \otimes_{R_*} Q_{p,*}$$

has only three non-trivial columns



There are two possible cases. If the differential d^2 is trivial, then the E^2-term is the E^∞-term. As the spectral sequence has a one-dimensional abutment, we see as before that $V_* \cong \Sigma^k R_*$ for some $k \in \mathbb{Z}$. On the other hand, if d^2 is non-trivial, then we have a non-trivial map between the 0-column and the 2-column. But the entries in the 1-column are infinite cycles. So either they are trivial or at most one-dimensional. If they are trivial then the 2-column has to be trivial as well and we get a contradiction. If they are non-trivial, then we can conclude that the d^2-differential must be an isomorphism and that $Q_{1,*} \cong \Sigma^\ell R_*$ for some $\ell \in \mathbb{Z}$. Therefore up to suspensions the resolution $Q_{\bullet,*}$ is of the form

$$R_*^n \longleftarrow R_* \longleftarrow R_*^n.$$

If n were bigger than one, then this could not be a resolution. For $n = 1$ the differentials must be given by multiplication by some element in R_*. Therefore this gives no resolution either. $\square$

We can loosen the requirements on R a little bit by referring to Condition (B).

Proposition 29. *Let R be a commutative $\mathbb{S}$-algebra such that for every maximal ideal $\mathfrak{m} \lhd R_*$ the ring $(R_*)_{\mathfrak{m}}$ is coherent and satisfies* Condition (B) *with residue fields which are ring spectra. If R_* has global dimension at most* 2 *then for every invertible R-module spectrum U, U_* is an invertible graded R_*-module.*

Proof. Let $R \longrightarrow R'$ be the unit of a suitable R-algebra as required in Condition (B) and let $V' = R' \wedge_R V$. Coherence of R guarantees the existence of a minimal resolution $Q_{\bullet,*} \longrightarrow V_* \to 0$ of V_*. Flatness of R' over R ensures that

$$Q'_{\bullet,*} \longrightarrow V'_* = R'_* \otimes_{R_*} V_* \to 0$$

is still a resolution and as we assumed the map $R_* \longrightarrow R'_*$ to be local, this resolution is minimal. Using the proof of Theorem 28 and an argumentation as in the proof of Proposition 20 for R' and $U' = R' \wedge_R U$, we see that $Q'_{p,*} = 0$ when $p > 0$. As $Q'_{p,*} = R'_* \otimes_{R_*} Q_{p,*}$ and $Q_{p,*}$ is free over R, we must have $Q_{p,*} = 0$ for $p > 0$. Thus $Q_{0,*} \cong V_*$ and the Künneth spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(U_*, V_*) \Longrightarrow \pi_{p+q}(U \wedge_R V) = R_{p+q}$$

collapses and the edge homomorphism $U_* \otimes_{R_*} V_* \longrightarrow R_*$ is an isomorphism, so U_* is invertible with inverse V_*. $\square$

Example 30. Theorem 28 and Proposition 29 cover the examples of the first two Lubin-Tate spectra E_1 and E_2 and their close relatives, the completed Johnson-Wilson spectra $\widehat{E(1)}$ and $\widehat{E(2)}$, as well as the Adams summand $E(1)$. Complex periodic K-theory and real periodic K-theory with 2-inverted, $KO[1/2]$, fulfills the requirements as well.

8. Noetherian complete local regular rings

In the following we extend the results of Section 7 to commutative $\mathbb{S}$-algebras whose coefficients have higher global dimension. However, we have to impose regularity conditions. The method of proof is adapted from that of [14, theorem 1.3, pp.117,118].

We will make use of the algebraic theory of Noetherian regular rings and their finite modules for which we refer to [7, 17]. We begin with some local results.

Assumption 31. Throughout this section, R will be a commutative $\mathbb{S}$-algebra for which $R_* = R_0[u, u^{-1}]$ with $|u| = 2$. We assume that R_0 is a complete Noetherian local regular ring whose maximal ideal $\mathfrak{m} \lhd R_0$ is generated by a regular sequence $u_1, \ldots, u_n$, where n is the Krull dimension of R_*. We could view R_* and its modules as $\mathbb{Z}/2$-graded R_0-modules.

Theorem 15 then applies.

Lemma 32. *For each prime ideal $\mathfrak{p} \lhd R_*$, there is an R-algebra realizing the R_*-algebra $R_*/\mathfrak{p}$. Hence the graded residue field*

$$\kappa(\mathfrak{p})_* = (R_*/\mathfrak{p})_{\mathfrak{p}} = (R_*)_{\mathfrak{p}}/(R_*)_{\mathfrak{p}}\mathfrak{p}$$

can be realized as an $R_{\mathfrak{p}}$-algebra and so $R_{\mathfrak{p}}$ has a residue field $\kappa(\mathfrak{p})$.

For R-modules M and N, $\kappa(\mathfrak{m}) \wedge_R M$ and $\kappa(\mathfrak{m}) \wedge_R N$ are $\kappa(\mathfrak{m})$-left modules, and we can consider $\kappa(\mathfrak{m}) \wedge_R M$ as a right $\kappa(\mathfrak{m})$-module spectrum via the action of $\kappa(\mathfrak{m})$ on itself by right multiplication. Since R is central in $\kappa(\mathfrak{m})$, this is well-defined.

Corollary 33. *Let M and N be R-modules. Then there is a Künneth isomorphism*

$$\kappa(\mathfrak{m})^R_*(M) \otimes_{\kappa(\mathfrak{m})_*} \kappa(\mathfrak{m})^R_*(N) \cong \kappa(\mathfrak{m})^R_*(M \wedge_R N).$$

If U is an invertible R-module then $\dim_{\kappa(\mathfrak{m})_*} \kappa(\mathfrak{m})^R_*(U) = 1$.

Proof. In our case, the Künneth spectral sequence of [10, theorem IV.4.1]

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{\kappa(\mathfrak{m})_*}_{p,q}(\kappa(\mathfrak{m})^R_*(M), \kappa(\mathfrak{m})^R_*(N)) \Longrightarrow \kappa(\mathfrak{m})^R_{p+q}(M \wedge_R N),$$

collapses because $\kappa(\mathfrak{m})_*$ is a graded field. When $U \wedge_R V \simeq R$, we have

$$\kappa(\mathfrak{m})^R_*(U) \otimes_{\kappa(\mathfrak{m})_*} \kappa(\mathfrak{m})^R_*(V) \cong \kappa(\mathfrak{m})_*,$$

hence

$$\dim_{\kappa(\mathfrak{m})_*} \kappa(\mathfrak{m})^R_*(U) = 1 = \dim_{\kappa(\mathfrak{m})_*} \kappa(\mathfrak{m})^R_*(V). \quad \square$$

We will need some technical results about killing regular sequences in R. We make use of the results of [10, lemma V.1.5].

Lemma 34. *For every sequence* $(u_1^{i_1}, \dots, u_n^{i_n})$ *with* $i_j > 1$*, there are cofibre sequences*

$$\begin{aligned} R/(u_1^{i_1}, \dots, u_n^{i_n}) &\xrightarrow{u_j} R/(u_1^{i_1}, \dots, u_n^{i_n}) \\ &\longrightarrow R/(u_1^{i_1}, \dots, u_j^1, \dots, u_n^{i_n}) \vee \Sigma R/(u_1^{i_1}, \dots, u_j^1, \dots, u_n^{i_n}) \end{aligned} \tag{5}$$

and

$$\begin{aligned} R/(u_1^{i_1}, \dots, u_j^{i_j-1}, \dots, u_n^{i_n}) &\longrightarrow R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}) \\ &\longrightarrow R/(u_1^{i_1}, \dots, u_j^1, \dots, u_n^{i_n}). \end{aligned} \tag{6}$$

Proof. The cofibre of the multiplication map by u_j on $R/(u_1^{i_1}, \dots, u_n^{i_n})$ can be identified as follows. As the variables behave independently we might just consider the case of one u_j. Then we get the following diagram of cofibre sequences.

$$\begin{tikzcd} R \arrow[r, "u_j^i"] \arrow[d, "u_j"'] & R \arrow[r] \arrow[d, "u_j"] & R/u_j^i \arrow[d, "u_j"] \\ R \arrow[r, "u_j^i"] \arrow[d] & R \arrow[r] \arrow[d] & R/u_j^i \arrow[d] \\ R/u_j \arrow[r, "u_j^i"] & R/u_j \arrow[r] & \mathrm{cofibre}(u_j) \end{tikzcd}$$

As multiplication by u_j^i is nullhomotopic on R/u_j, the cofibre of the multiplication by u_j splits as $R/u_j \vee \Sigma R/u_j$.
For the second sequence, consider

$$\begin{aligned} R/(u_1^{i_1}, \dots, u_{j-1}^{i_{j-1}}, u_{j+1}^{i_{j+1}}, \dots, u_n^{i_n}) &\xrightarrow{u_j^{i_j-1}} R/(u_1^{i_1}, \dots, u_{j-1}^{i_{j-1}}, u_{j+1}^{i_{j+1}}, \dots, u_n^{i_n}) \\ &\longrightarrow R/(u_1^{i_1}, \dots, u_{j-1}^{i_{j-1}}, u_j^{i_j-1}, u_{j+1}^{i_{j+1}}, \dots, u_n^{i_n}). \end{aligned}$$

There is a canonical projection map

$$R/(u_1^{i_1}, \dots, u_{j-1}^{i_{j-1}}, u_{j+1}^{i_{j+1}}, \dots, u_n^{i_n}) \longrightarrow R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n})$$

which we can compose with multiplication by u_j. When precomposed with multiplication by $u_j^{i_j-1}$, this map becomes nullhomotopic, thus by [10, lemma V.1.5] it factors through a map

$$R/(u_1^{i_1}, \dots, u_j^{i_j-1}, \dots, u_n^{i_n}) \longrightarrow R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n})$$

whose cofibre is easily identified with $R/(u_1^{i_1}, \dots, u_j^1, \dots, u_n^{i_n})$. $\square$

Lemma 35. *If U is an invertible R-module spectrum, then for all sequences $(u_1^{i_1}, \dots, u_n^{i_n})$ with $\sum_{k=1}^n i_k \geqslant n$ and each $i_k \geqslant 1$, $(R/(u_1^{i_1}, \dots, u_n^{i_n}))_*^R(U)$ is a cyclic R_*-module.*

Proof. By suspending U if necessary and appealing to Lemma 33, we may as well assume that $\kappa(\mathfrak{m})_*^R(U) \cong \kappa(\mathfrak{m})_*$ is concentrated in even degrees. We prove the claim by induction on $m = \sum_{k=1}^n i_k$. For $m = n$ the result is clear since

$$(R/(u_1^{i_1}, \dots, u_n^{i_n}))_*^R(U) = \kappa(\mathfrak{m})_*^R(U) \cong \kappa(\mathfrak{m})_*.$$

Now let $m > n$ and assume that the result for all sequences of the above form with $m > \sum_{k=1}^n i_k$. Using the cofibre sequence (6), we see that the module $R/(u_1^{i_1}, \dots, u_n^{i_n}) \wedge_R U$ has homotopy groups which are concentrated in even degrees. From the cofibre sequence (5) we can read off that multiplication by u_j on $(R/(u_1^{i_1}, \dots, u_n^{i_n}))_*^R(U)$ has quotient $(R/(u_1^{i_1}, \dots, u_j^1, \dots, u_n^{i_n}))_*^R(U)$ which is cyclic by assumption.

Notice that all three terms are finitely generated R_*-modules and the image of the multiplication by u_j is contained in the submodule generated by the maximal ideal of R_*, hence by Nakayama's Lemma, the module $(R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))_*^R(U)$ is cyclic. $\square$

Lemma 36. *If U is an invertible R-module spectrum, then for every sequence $(u_1^{i_1}, \dots, u_n^{i_n})$ with $\sum_{k=1}^n i_k \geqslant n$ and $i_k \geqslant 1$, up to suspension, there is an isomorphism*

$$(R/(u_1^{i_1}, \dots, u_n^{i_n}))_*^R(U) \cong R_*/(u_1^{i_1}, \dots, u_n^{i_n}).$$

Furthermore, these isomorphisms are compatible with the projection maps

$$(R/(u_1^{i_1}, \dots, u_j^{i_j+1}, \dots, u_n^{i_n}))_*^R(U) \longrightarrow (R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))_*^R(U).$$

Proof. There is a canonical cofibre sequence

$$\begin{aligned} R/(u_1^{i_1}, \dots, u_j, \dots, u_n^{i_n}) &\longrightarrow R/(u_1^{i_1}, \dots, u_j^{i_j+1}, \dots, u_n^{i_n}) \\ &\longrightarrow R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}). \end{aligned}$$

As everything in sight is concentrated in even degrees, for each $\ell \in \mathbb{Z}$ we get the two short exact sequences

$$\begin{aligned} 0 \to R_{2\ell}/(u_1^{i_1}, \dots, u_j, \dots, u_n^{i_n}) &\to R_{2\ell}/(u_1^{i_1}, \dots, u_j^{i_j+1}, \dots, u_n^{i_n}) \\ &\to R_{2\ell}/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \to (R/(u_1^{i_1}, \dots, u_j, \dots, u_n^{i_n}))_{2\ell}^R(U) &\to (R/(u_1^{i_1}, \dots, u_j^{i_j+1}, \dots, u_n^{i_n}))_{2\ell}^R(U) \\ &\to (R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))_{2\ell}^R(U) \to 0. \end{aligned}$$

We start with the isomorphism $\kappa(\mathfrak{m})^R_*(U) \cong \kappa(\mathfrak{m})_*$. As every module of the form $R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))^R_*(U)$ is cyclic we can choose epimorphisms

$$R_{2\ell}/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}) \twoheadrightarrow (R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))^R_{2\ell}(U)$$

which make the following diagram commute.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
R_{2\ell}/(u_1^{i_1}, \dots, u_j, \dots, u_n^{i_n}) \arrow[r] \arrow[d] & (R/(u_1^{i_1}, \dots, u_j, \dots, u_n^{i_n}))^R_{2\ell}(U) \arrow[d] \\
R_{2\ell}/(u_1^{i_1}, \dots, u_j^{i_j+1}, \dots, u_n^{i_n}) \arrow[r] \arrow[d] & (R/(u_1^{i_1}, \dots, u_j^{i_j+1}, \dots, u_n^{i_n}))^R_{2\ell}(U) \arrow[d] \\
R_{2\ell}/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}) \arrow[r] \arrow[d] & (R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))^R_{2\ell}(U) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Now an induction over $m = \sum_{j=1}^n i_j$ proves the claim. $\square$

Theorem 37. *If R satisfies* Assumption 31*, then every invertible R-module is equivalent to a suspension of R.*

Proof. Again, we may suspend U if necessary to ensure that $\kappa(\mathfrak{m})^R_*(U) \cong \kappa(\mathfrak{m})_*$.

Lemma 36 ensures that the identifications

$$R_*/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}) \cong (R/(u_1^{i_1}, \dots, u_j^{i_j}, \dots, u_n^{i_n}))^R_*(U)$$

are consistent with the projection maps in the inverse system defining holim $R/(u_1^{i_1}, u_2^{i_2}, \dots, u_n^{i_n}) \wedge_R U$. Since R_* is a Noetherian complete local ring $\varprojlim R_*/(u_1^{i_1}, u_2^{i_2}, \dots, u_n^{i_n}) = \varprojlim_\ell R_*/\mathfrak{m}^\ell$. As U is a finite cell R-module, we have

$$\operatorname{holim} R/(u_1^{i_1}, u_2^{i_2}, \dots, u_n^{i_n}) \wedge_R U \simeq R \wedge_R U \simeq U.$$

Using the above description of $(R/(u_1^{i_1}, u_2^{i_2}, \dots, u_n^{i_n})^R_*(U)$ we find that

$$\operatorname{holim} R/(u_1^{i_1}, u_2^{i_2}, \dots, u_n^{i_n}) \wedge_R U \simeq \operatorname{holim} R/(u_1^{i_1}, u_2^{i_2}, \dots, u_n^{i_n}) \simeq R.$$

Therefore we have $U \simeq R$. In the general case, U might be equivalent to ΣR. $\square$

The proof of the following more general result involves a standard local-to-global argument.

Theorem 38. *Let R be a commutative $\mathbb{S}$-algebra such that the localization of R at any maximal ideal $\mathfrak{m} \lhd R_*$ satisfies* Assumption 31. *Then for every invertible R-module U, U_* is an invertible R_*-module.*

Remark 39. Assumption 31 is not optimal. For example, one might loosen the requirement that R_* is 2-periodic and replace this by a periodicity of degree 2ℓ for some ℓ. One might also wish to allow that the generators of the maximal ideal then lie in degrees different from zero. This is no problem if one takes appropriate suspensions into account in the numerous cofibre sequences. Last but not least there might be cases of infinite Krull dimension that are tractable.

As an application we discuss invertible modules over certain abelian group rings $R[G]$.

Proposition 40. *Let R be a commutative $\mathbb{S}$-algebra which satisfies* Assumption 31. *Let G be a finite abelian group for which all the primes dividing the order of G are not contained in the maximal ideal $\mathfrak{m} \lhd R_0$. Suppose that R_0 contains a primitive d-th root of unity where d is the exponent of G. Then for every invertible $R[G]$-module U, U_* is an invertible $R_*[G]$-module.*

Remark 41. For a connective commutative $\mathbb{S}$-algebra R with coherent coefficients and an arbitrary finite abelian group G, the group ring $R_*[G]$ is coherent [13, corollary 1.2] and therefore in this case invertible $R[G]$-modules have invertible coefficients.

Proof. For sake of simplicity we will first give the proof for G being a cyclic group of order p^ℓ where p is a prime which is invertible in R_0. As R_0 contains enough roots of unity, there is a complete set of orthogonal idempotents e_i with $i = 1, \dots, p^\ell$ corresponding to the distinct characters $\chi_i \colon C_{p^\ell} \longrightarrow R_0^\times$. These idempotents can be realized as elements of $\pi_0 R[C_{p^\ell}] = R_0[C_{p^\ell}]$ and the localization $R_0[C_{p^\ell}][e_i^{-1}]$ can be realized as the homotopy of a commutative $\mathbb{S}$-algebra $R[C_{p^\ell}][e_i^{-1}]$ as in [22]. There is an isomorphism of rings

$$R_0[C_{p^\ell}][e_i^{-1}] \cong R_0,$$

and an equivalence of R-modules

$$R[C_{p^\ell}][e_i^{-1}] \sim R.$$

By assumption on R, $R[C_{p^\ell}][e_i^{-1}]$ admits a residue field and so satisfies Assumption 31. There is a splitting of rings

$$R_0[C_{p^\ell}] \cong \prod_i R_0[C_{p^\ell}][e_i^{-1}]$$

which is realized by an equivalence of commutative $\mathbb{S}$-algebras

$$R[C_{p^\ell}] \sim \prod_i R[C_{p^\ell}][e_i^{-1}].$$

Also the maximal ideals of $R_0[C_{p^\ell}]$ are precisely those induced from the maximal ideals of the factors $R_0[C_{p^\ell}][e_i^{-1}]$. Now Theorem 38 applies to show that for each invertible $R[C_{p^\ell}]$-module U, U_* is an invertible $R_*[C_{p^\ell}]$-module.

For a finite abelian group G for which all the primes dividing the order of the group are not contained in the maximal ideal $\mathfrak{m} \lhd R_0$ there exists a more general decomposition into eigenspaces of characters $\chi \in \mathrm{Hom}(G, R_0^\times)$ and the proof works analogously. $\square$

In general, the invertible modules in this result need not be concentrated in odd or even degree. However, when this is true, an invertible module is free of rank 1 over the group ring. An important instance of this is provided by the following application to Galois theory of commutative $\mathbb{S}$-algebras (see [21, 3] for background on this).

Example 42. Consider the 2-periodic Lubin-Tate spectrum E_n with

$$(E_n)_0 = W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]].$$

This ring is Noetherian, complete, local and regular. Let $W\mathbb{F}_{p^n}^{\mathrm{nr}}$ denote the maximal unramified extension of $W\mathbb{F}_{p^n}$. There is a commutative E_n-algebra E_n^{nr} with coefficient ring

$$(E_n^{\mathrm{nr}})_* = W\mathbb{F}_{p^n}^{\mathrm{nr}}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}] \quad \text{with } |u| = -2.$$

Every invertible E_n-module has invertible coefficients. Since $(E_n^{\mathrm{nr}})_*$ contains enough roots of unity, whenever G is a finite abelian group whose order is not divisible by p, every invertible module over $E_n^{\mathrm{nr}}[G]$ has invertible coefficient module over $E_n^{\mathrm{nr}}[G]_* = (E_n^{\mathrm{nr}})_*[G]$. In particular, for a G-Galois extension B/E_n^{nr}, $B_*/(E_n^{\mathrm{nr}})_*$ is an algebraic G-Galois extension. There cannot be any odd degree elements in B_* for the following reason: In an algebraic Galois extension of graded commutative algebras B_* over A_* with Galois group G, the group G acts on B_* degree-perserving. In particular if B_* is a G-Galois extension of $(E_n^{\mathrm{nr}})_*$, $B_{2\ell+1}$ is a G-representation. Under the assumptions we made, $B_{2\ell+1}$ has a decomposition into character eigenspaces. If there were odd-degree elements and if p is an odd prime, then the map

$$h\colon B_* \otimes_{(E_n^{\mathrm{nr}})_*} B_* \to \prod_G B_*, \quad b_1 \otimes b_2 \mapsto (b_1 g(b_2))_{g \in G}$$

would have a non-trivial kernel, since for every element $x \in B_{2\ell+1}$ in an eigenspace, $x \otimes x$ would map to zero, contradicting the fact that h has to be an isomorphism. For $p = 2$, since every irreducible character χ has odd order, an odd degree element of the corresponding summand is nilpotent and a similar argument applies.

Therefore B_* is a free $(E_n^{\mathrm{nr}})_*[G]$-module, hence it has a normal basis.

9. Examples

We will now restate our earlier results in terms of Picard groups.

Theorem 43. *For a commutative $\mathbb{S}$-algebra R, there is a monomorphism of abelian groups*

$$\Phi\colon \operatorname{Pic}(R_*) \longrightarrow \operatorname{Pic}(R).$$

Furthermore, if R satisfies the conditions of Theorem 21*,* Theorem 25 *or of* Theorem 38*, then Φ is an isomorphism.*

Thus $\operatorname{Pic}(R_*) \cong \operatorname{Pic}(R)$ in all of the following cases.

- $R = HA$, where A is a commutative ring.
- $R = MU/I$, where $I \lhd MU_*$ is a finitely generated ideal for which MU/I is a commutative $\mathbb{S}$-algebra.
- KU, $KO[1/2]$, ku, and ko.
- tmf at a prime p, $E(1)$, $BP\langle 1\rangle$, $\widehat{E(1)}$, and $\widehat{E(2)}$. See [2] for the existence of commutative $\mathbb{S}$-algebra structures on some of these.
- MSp, $MSpin$ and MSU.
- E_n for any n and p.

We close with a counterexample which originates from Galois theory of commutative $\mathbb{S}$-algebras. In [3, theorem 2.5.1], we show that every finite abelian Galois extension B/A with Galois group G gives rise to an element in the Picard group of the group ring $A[G]$.

Example 44. Complex periodic K-theory, KU, is a Galois extension of the real periodic K-theory spectrum KO, whose Galois group is C_2, the cyclic group of order 2 (see [21] or [3, example 1.4.8]). Therefore we obtain

$$KU \in \operatorname{Pic}(KO[C_2]).$$

But KU_* is not projective over KO_*, therefore it cannot be projective over $KO_*[C_2]$. In particular, the coefficient module KU_* is not an element in $\operatorname{Pic}(KO_*[C_2])$.

Acknowledgements. We would like to thank Halvard Fausk, John Greenlees, Peter Kropholler, and Stefan Schwede. We are grateful that John Rognes spotted an embarrassing blunder in an earlier version of this paper. The first author thanks the Mathematics Departments in Bonn, Hamburg and Oslo for their hospitality. The second author thanks the Mathematics Departments in Bonn, Glasgow and Oslo, she was partially supported by the *Strategisk Universitetsprogram i Ren Matematikk* (SUPREMA) of the Norwegian Research Council.

References

[1] Angeltveit, V.: A_∞-obstruction theory and the strict associativity of E/I, available at `http://hopf.math.purdue.edu/`
[2] Baker, A. and Richter, B.: Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory, to appear in Comment. Math. Helv., `math.AT/0304473`
[3] Baker, A. and Richter, B.: Realizability of algebraic Galois extensions by strictly commutative ring spectra, `math.AT/0406314`
[4] Bauer, T.: Computation of the homotopy of the spectrum tmf, `math.AT/0311328`.

[5] Benson, D., Krause, H. and Schwede, S.: Realizability of modules over Tate cohomology, Trans. Amer. Math. Soc. **356**, 3621–3668 (2004)

[6] Bourbaki, N.: Éléments de Mathematique, Fasc. XXVII: Algèbre Commutative, Hermann, 1961

[7] Bruns, W. and Herzog, J.: Cohen-Macaulay rings, rev. ed., Cambridge University Press. 1998

[8] Cohen, J. M.: Coherent graded rings and the non-existence of spaces of finite stable homotopy type, Comment. Math. **44**, 217–228 (1969)

[9] Eilenberg, S.: Homological dimension and syzygies, Ann. of Math. **64**, 328–36 (1956)

[10] Elmendorf, A., Kriz, I., Mandell, M. and May, J. P.: Rings, modules, and algebras in stable homotopy theory, Mathematical Surveys and Monographs **47** (1997)

[11] Fausk, H.: Picard groups of derived categories, J. Pure Appl. Algebra **180**, 251–261 (2003)

[12] Fausk, H., Lewis, L. G. and May, J. P.: The Picard group of equivariant stable homotopy theory, Adv. in Math. **163**, 17–33 (2001)

[13] Harris, M.E.: Some results on coherent rings, Proc. Am. Math. Soc. **17**, 474–479 (1966)

[14] Hopkins, M. J., Mahowald, M. and Sadofsky, H.: Constructions of elements in Picard groups, In: Friedlander E. M., and Mahowald, M. E. (eds) Topology and representation theory (Evanston, IL, 1992), Contemp. Math. **158**, 89–126 (1994)

[15] Hovey, M., Palmieri, J. H. and Strickland, N. P.: Axiomatic stable homotopy theory, Mem. Amer. Math. Soc. **128** (1997), no. 610.

[16] Lam, T. Y.: Lectures on Modules and Rings, Graduate Texts in Mathematics **189**, Springer-Verlag, (1999)

[17] Matsumura, H.: Commutative Ring Theory, Cambridge University Press, 1989

[18] May, J. P.: Equivariant Homotopy and Cohomology theories, CBMS Regional Conference Series in Mathematics **91** (1996)

[19] May, J. P.: Picard groups, Grothendieck rings, and Burnside rings of categories, Adv. in Math. **163**, 1–16 (2001)

[20] Rezk, C. : Notes based on a course on topological modular forms, (2001), `http://www.math.uiuc.edu/~rezk/papers.html`

[21] Rognes, J.: Galois extensions of structured ring spectra, `math.AT/0502183`

[22] Schwänzl, R., Vogt, R. M. and Waldhausen, F.: Adjoining roots of unity to E_∞ ring spectra in good cases – a remark, Homotopy invariant algebraic structures. A conference in honor of J. Michael Boardman. AMS special session on homotopy theory, Baltimore, MD, USA, January 7-10, 1998, eds.: J.-P. Meyer et al., Contemp. Math. AMS **239**, 245–249 (1999)

[23] Serre, J.-P.: Faisceaux algébriques cohérents, Ann. of Math. **61**, 197–278 (1955)

[24] Strickland, N. P.: On the p-adic interpolation of stable homotopy groups, Adams Memorial Symposium on Algebraic Topology, **2** (Manchester, 1990), In: Ray N., and Walker, G. (eds) London Math. Soc. Lecture Note Ser. Cambridge University Press **176**, 45–54 (1992)

[25] Strickland, N. P.: Products on MU-modules, Trans. Amer. Math. Soc. **351**, 2569–2606 (1999)