Homological perturbation theory and the existence of the BRST differential

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The BRST setting

Algebras and derivations

Homology and cohomology

Differentials modulo differentials

Homological perturbation theory

The situation

Consider a surface Σ in a manifold M ("phase space"). View $C^\infty(\Sigma)$ as the zeroth homology of a differential graded algebra $(\mathcal{A}, \delta)$; here $\mathcal{A}$ is built out of $C^\infty(M)$ and δ is the Koszul-Tate differential. Consider a differential d on $C^\infty(\Sigma)$, such that its homology corresponds to the smooth functions on Σ that are constant along gauge orbits. Then d is called the longitudinal exterior derivative. Lift d to a derivation on $\mathcal{A}$.

HPT: Often, there is a differential s on $\mathcal{A}$ with

$$s = \delta + d + \text{higher term},$$

such that the homology of s still gives the gauge invariant functions.

Examples of superalgebras

- Consider $A = \mathbb{C}[x_1, \ldots, x_m] \otimes \Lambda(y_1, \ldots, y_n)$ where the x_i are even degree generators and the y_i are in odd degrees.
- Then A splits as $A = A_0 \oplus A_1$, where A_0 contains all elements of even degree and A_1 collects odd degree elements. We have $A_0 A_0 \subset A_0$, $A_1 A_0 \subset A_1 \supset A_0 A_1$ and $A_1 A_1 \subset A_0$. We define the parity of a homogeneous element $z \in A$ to be

$$\varepsilon(z) = \begin{cases} 0 & z \in A_0 \\ 1 & z \in A_1. \end{cases}$$

- The algebra A is associative and it is *supercommutative*: we have

$$uv = (-1)^{\varepsilon(u)\varepsilon(v)} vu \ \text{ for all } u, v.$$

Endomorphisms of super-vector spaces

- Let $V = V_0 \oplus V_1$ be a super-vector space over $\mathbb{C}$. For homogeneous elements $x \in V$ we define the parity to be

$$\varepsilon(x) = \begin{cases} 0 & x \in V_0 \\ 1 & x \in V_1 \end{cases}$$

- The endomorphism vector space of V, $\mathrm{End}(V)$ inherits a $\mathbb{Z}/2\mathbb{Z}$-grading from V. If $f \in \mathrm{End}(V)$, then $\varepsilon(f)$ is determined if

$$\varepsilon(f(x)) = \varepsilon(f) + \varepsilon(x) \mod 2$$

  for all homogeneous $x \in V$.
- Thus $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$. The $\mathbb{Z}/2\mathbb{Z}$-grading is compatible with composition:

$$\varepsilon(f_2 \circ f_1) = \varepsilon(f_2) + \varepsilon(f_1) \mod 2$$

Derivations

- For two endomorphisms f_1, f_2 of V we define

$$[f_1, f_2] = f_1 \circ f_2 - (-1)^{\varepsilon(f_1)\varepsilon(f_2)} f_2 \circ f_1.$$

  This product satisfies a graded version of anti-symmetry and of the Jacobi identity. $(\mathrm{End}(V), [-,-])$ is a graded Lie-algebra.

- Let A be a supercommutative, associative algebra. Consider endomorphisms $D \in \mathrm{End}(A)$ that satisfy the Leibniz rule

$$D(xy) = xD(y) + (-1)^{\varepsilon(D)\varepsilon(y)} D(x)y.$$

  Such D are derivations and if D_1, D_2 are derivations, then so is $[D_1, D_2]$.

- We denote by $\mathrm{Der}(A)$ the graded sub-Lie algebra of $\mathrm{End}(A)$.

Differentials and internal gradings

- Let A be again a supercommutative, associative algebra and $D \in \mathrm{Der}(A)$. We call D a differential on A, if $\varepsilon(D) = 1$ and $D^2 = 0$. Note, that $2D^2 = [D, D]$.
- Assume, that A has an additional internal grading $A = \bigoplus_{n \in J} A^n$ with $J = \mathbb{N}_0$ or $\mathbb{Z}$.
- Then $A^n = A^n_0 \oplus A^n_1$. We want that $A^n A^m \subset A^{n+m}$ and that the unit of the multiplication is contained in A^0.
- For a homogeneous element $x \in A^n$ we denote $\deg(x) = n$.
- $\mathrm{End}(A)$ and $\mathrm{Der}(A)$ inherit an internal grading from A via:

$$\deg(f(x)) = \deg(f) + \deg(x).$$

- **Example**: the degree derivation N is defined as

$$N(x) := \deg(x)x.$$

  Note, $\varepsilon(N) = 0 = \deg(N)$.

Cohomology of an algebra wrt a differential

- For A as above we assume that the differential D has internal degree one, i.e.,

$$\deg(D(x)) = 1 + \deg(x)\,\forall\, x.$$

- The n-th cohomology of A with respect to D is then defined as

$$H^n(A, D) := \frac{\ker(D\colon A^n \to A^{n+1})}{\operatorname{im}(D\colon A^{n-1} \to A^n)}.$$

- $H^*(A, D) = \bigoplus_{n\in J} H^n(A, D)$ is an algebra again.

Homological setting

- Let A be as above and $D \in \mathrm{Der}(A)$ a differential but with internal degree -1, i.e.
$$D\colon A_n \to A_{n-1}.$$
- Then we define the *n-th homology* of A with respect to D to be
$$H_n(A, D) := \frac{\ker(D\colon A^n \to A^{n-1})}{\mathrm{im}(D\colon A^{n+1} \to A^n)}.$$
The homology of A with respect to D is an algebra again.

Resolutions

- Let A be a supercommutative, associative algebra as before. A resolution of A is an $\mathbb{N}_0$-graded supercommutative, associative algebra $\mathcal{A}$ with a differential δ of internal degree -1 such that

$$H_*(\mathcal{A},\delta) = \begin{cases} 0 & * \neq 0 \\ A & * = 0 \end{cases}$$

- The internal degree of $\mathcal{A}$ is called resolution degree and abbreviated by r.
- Baby-example: Consider the polynomial algebra $A = \mathbb{C}[x]$ on one generator x in degree zero. Let $\mathcal{A} = \mathbb{C}[x,z] \otimes \Lambda(P)$ be the supercommutative associative algebra with the following data: $r(x) = r(z) = 0$, $r(P) = 1$, $\varepsilon(P) = 1$ $\varepsilon(x) = \varepsilon(z) = 0$. We define the differential δ on the generators as $\delta(x) = 0 = \delta(z)$ and $\delta(P) = z$. As δ is a derivation, this defines it on $\mathcal{A}$.
- Then in homology, P kills the generator z and $(\mathcal{A},\delta)$ is a resolution of $A = \mathbb{C}[x]$.

The real world

- Let Σ be a surface that's embedded in a manifold M. (M is often called P for phase space.)
- Build $\mathcal{A} = \mathbb{C}[P_\alpha] \otimes C^\infty(M)$ and cook up a differential δ in such a way that the P_α kill all functions in $C^\infty(M)$ that vanish on Σ and such that $\mathcal{A}$ has no higher homology.
- Then $A = H_0(\mathcal{A}) \cong C^\infty(\Sigma)$. That is the typical input for homological perturbation theory. Bahns and Ribeiro will deal with actual examples in their talk.

Cohomology of $\mathrm{Der}(\mathrm{A})$ wrt a differential

- Let A be as above and $D \in \mathrm{Der}(A)$ a differential.
- We call a derivation D' D-closed, if
$$[D', D] = 0.$$
We call a derivation D' D-exact, if there is a derivation D'' such that
$$D' = [D'', D].$$

- D-exact derivations are also D-closed. The Jacobi-identity implies that D-closed derivations are a sub-Lie algebra of $\mathrm{Der}(A)$ and that D-exact derivations form a Lie-ideal in the Lie algebra of D-closed derivations.
- We define the cohomology of $\mathrm{Der}(A)$ with respect to D to be
$$\mathcal{H}^*(D) := \frac{D\text{-closed derivations}}{D\text{-exact derivations}}$$
Here, the grading $*$ corresponds to the degree of the derivations.

Differentials modulo differentials

- Let $\delta \in \mathrm{Der}(A)$ be a differential of internal degree 1. We consider its homology $H_*(A, \delta)$ and abbreviate this to $H_*(\delta)$.
- Let d be a derivation with $\varepsilon(d) = 1$. Assume d has internal degree one, satisfies

$$d\delta + \delta d = 0$$

and d^2 is δ-exact, i.e., there is a derivation D such that

$$d^2 = [D, \delta].$$

- Then d induces a differential (which we still call d) on $H_*(\delta)$. We denote the cohomology of d on $H_*(\delta)$ by $H^*(d|H_*(\delta))$ and call d a differential modulo δ.

Elements in $H^*(d|H_*(\delta))$

- What is a class in $H^*(d|H_*(\delta))$? We need a δ-closed element x (i.e., $\delta(x) = 0$), that is d-closed modulo the image of δ. Thus, there is an element y, such that $d(x) = \delta(y)$.
- The class of x doesn't change if we modify x and consider

$$x' = x + d(u) + \delta(v)$$

for some u with $\delta(u) = 0$.

THE example

Let s be a differential on some supercommutative associative algebra $\mathcal{A}$ and let us assume that there is yet another internal $\mathbb{N}_0$-grading on $\mathcal{A}$ (e.g. a resolution grading). Expand s with respect to this additional grading, and assume this gives

$$s = s^{(-1)} + s^{(0)} + s^{(1)} + \dots$$

such that the (new) degree of $s^{(i)}$ is i. Call $s^{(-1)}$ δ and $s^{(0)}$ d. Expand the equation $s^2 = 0$ up to the new internal degree 0. For degree -2 we obtain $\delta^2 = 0$, i.e., δ is a differential. The terms in degree -1 are $\delta d + d\delta$ and the ones in degree zero are $\delta s^{(1)} + d^2 + s^{(1)}\delta$. Therefore d is a differential modulo δ with $d^2 = -[\delta, s^{(1)}]$.

Question behind homological perturbation theory

- We considered the example of a differential s that we could expand with respect to some suitable internal grading. We were able to conclude that from this input the low degree terms of s contain the data of a differential δ and a differential relative to δ, d.
- One feature of homological perturbation theory (HPT) is to investigate the converse question. Given a differential δ and a differential relative to δ, d,
  Is there a suitable grading and a differential s, such that we can expand s as

$$s = \delta + d + \text{higher degree terms?}$$

In this generality, the answer is NO. But in BRST situations things will work.

The setting

- Throughout we will assume that there is a resolution $(\mathcal{A}, \delta)$ of our supercommutative, associative algebra $A = H_0(\mathcal{A}, \delta)$, so we will drop $\mathcal{A}$ and A from the notation (most of the time). It is common to denote A by $H_0(\delta)$.
- Assume we have a differential modulo δ, d, such that $s^{(1)}$ is a derivation of resolution degree 1 with
$$d^2 = -[\delta, s^{(1)}].$$
Then $r(d) = 0$. The derivation defines an internal degree if we declare $\deg(d) = 1$. We assume that δ doesn't change the d-degree, thus $\deg(\delta) = 0$.
- Define the total or ghost grading as
$$\mathrm{gh}(x) := \deg(x) - r(x).$$
Then we have
$$\mathrm{gh}(\delta) = 0 - (-1) = 1,\ \mathrm{gh}(d) = 1 - 0 = 1.$$

Main theorem of HPT

In the setting above assume in addition that the homology of the Lie-algebra of derivations with respect to the differential δ is trivial but in degree zero, i.e., $\mathcal{H}^*(\delta) = 0 \quad$ for all $* \neq 0$.

a) Then there is a differential s with $\mathrm{gh}(s) = 1$ such that

$$s = \delta + d + \sum_{k \geq 1} s^{(k)}$$

with $r(s^{(k)}) = k$ and $\mathrm{gh}(s^{(k)}) = 1$.

b) Any such differential s has the property that its cohomology is isomorphic to the cohomology of d on A:

$$H^k(\mathcal{A}, s) \cong H^k(d, A = H_0(\delta)).$$

Remark: The differential s is highly non-unique.

Proof of existence

- We have the starting terms of the differential we want to build: δ, d and $s^{(1)}$. Thus we have the terms up to resolution degree 1. If we consider $s_1 := \delta + d + s^{(1)}$, then we know that all terms of resolution degree up to zero vanish in s_1^2. This is the start of our inductive construction.
- If we have built s up to resolution degree n,

$$s_n = \delta + d + s^{(1)} + \ldots + s^{(n)},$$

  then we assume that there are no terms of resolution degree less than n in s_n^2. Denote $2s_n^2$ by

$$2s_n^2 = r_n + r_{n+1} + \ldots$$

  where the r_m are the terms of resolution degree m in s_n^2.
- We know that δ, d and $s^{(1)}$ are derivations, so the higher $s^{(k)}$ are derivations as well and so is s_n. Note, that $2s_n^2 = [s_n, s_n]$, because all summands in s_n have parity $\varepsilon(s^{(k)}) = 1$.

- We have to show that we can add a term $s^{(n+1)}$ to s_n that will leave only terms of resolution degree higher than n in $(s_n + s^{(n+1)})^2$.
- Expanding the term $[s_n + s^{(n+1)}, s_n + s^{(n+1)}]$ gives

$$2[\delta, s^{(n+1)}] + r_n + \text{ terms of higher resolution degree :}$$

$$\begin{aligned}
&[s_n, s_n] + [s_n, s^{(n+1)}] + [s^{(n+1)}, s_n] + [s^{(n+1)}, s^{(n+1)}] \\
&\qquad = r_n + r_{n+1} + \ldots + [\delta, s^{(n+1)}] + [d, s^{(n+1)}] + \ldots \\
&\qquad\qquad + [s^{(n+1)}, \delta] + [s^{(n+1)}, d] + \ldots + [s^{(n+1)}, s^{(n+1)}]
\end{aligned}$$

- We claim that solving $2[\delta, s^{(n+1)}] + r_n = 0$ is equivalent to showing that $[\delta, r_n] = 0$

We can apply $[\delta, -]$ to $2[\delta, s^{(n+1)}] + r_n = 0$ an obtain

$$2[\delta, [\delta, s^{(n+1)}]] + [\delta, r_n] = 0.$$

The first summand is trivial due to the Jacobi identity and therefore $[\delta, r_n] = 0$ is a consequence of $2[\delta, s^{(n+1)}] + r_n = 0$.
For the other direction: $[\delta, r_n] = 0$ says that r_n is δ-closed, but $\mathcal{H}^*(\delta) = 0$ for $* \neq 0$ and therefore r_n has to be δ-exact. Writing r_n as $[\Delta, \delta]$ we get that we can choose $s^{(n+1)}$ to be $\pm\frac{1}{2}\Delta$.

We still have to show $[\delta, r_n] = 0$. To this end expand $[[s_n, s_n], s_n]$. This term vanishes because of the Jacobi identity and $[\delta, r_n]$ is its term of resolution degree $n - 1$. This proves existence of the differential s.

Proof that s respects cohomology.

- Let x be an element of a fixed ghost degree, say k. Expand x according to resolution degree
$$x = x^{(0)} + x^{(1)} + \dots$$
with $r(x^{(n)}) = n$. Define π as $\pi(x) = x^{(0)}$.
- In $s(x)$ the term of resolution degree zero is $d(x^{(0)}) + \delta(x^{(1)})$. Therefore for a cycle x, $\pi(s(x))$ is equivalent to $d(x^{(0)})$ up to something in the image of δ and thus π induces a well defined map
$$\pi \colon H^k(\mathcal{A}, s) \to H^k(d, H_0(\mathcal{A}, \delta)) = H^k(d, A).$$
- We claim that π induces an isomorphism. Surely $\pi(x)\pi(y) = x^{(0)}y^{(0)} = xy^{(0)} = \pi(xy)$ and π is additive. We'll prove surjectivity and skip injectivity because it's the same trick anyway.

- If $x^{(0)}$ is a representative in $H^k(d, H^0(\mathcal{A}, \delta))$ then there is an $x^{(1)}$ with $dx^{(0)} + \delta x^{(1)} = 0$. Thus $s(x^{(0)} + x^{(1)})$ starts in resolution degree 1.
- Assume that we constructed $y_n = x^{(0)} + \ldots + x^{(n)}$ such that sy_n starts with terms in resolution degree n, say

$$sy_n = t_n + t_{n+1} + \ldots \text{ with } r(t_i) = i.$$

- The equation $s^2 y_n = 0$ has δt_n as term of lowest resolution degree $n-1$ and hence $\delta t_n = 0$. But $H^*(\mathcal{A}, \delta) = 0$ in positive degrees and therefore there is an element $x^{(n+1)}$ with

$$t_n = -\delta x^{(n+1)}.$$

That's it! $\square$

References

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