Homological perturbation theory and the existence of the BRST differential

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The BRST setting

Algebras and derivations

Homology and cohomology

Differentials modulo differentials

Homological perturbation theory

The situation

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such that the homology of s still gives the gauge invariant functions.

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- The algebra A is associative and it is *supercommutative*: we have
$$uv = (-1)^{\varepsilon(u)\varepsilon(v)} vu \ \text{ for all } u, v.$$

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- Thus $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$. The $\mathbb{Z}/2\mathbb{Z}$-grading is compatible with composition:

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- We denote by $\mathrm{Der}(A)$ the graded sub-Lie algebra of $\mathrm{End}(A)$.

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- Then in homology, P kills the generator z and $(\mathcal{A}, \delta)$ is a resolution of $A = \mathbb{C}[x]$.

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- We define the cohomology of $\mathrm{Der}(A)$ with respect to D to be

$$\mathcal{H}^*(D) := \frac{D\text{-closed derivations}}{D\text{-exact derivations}}$$

Cohomology of $\mathrm{Der}(A)$ wrt a differential

- Let A be as above and $D \in \mathrm{Der}(A)$ a differential.
- We call a derivation D' D-closed, if
$$[D', D] = 0.$$
We call a derivation D' D-exact, if there is a derivation D'' such that
$$D' = [D'', D].$$

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Here, the grading $*$ corresponds to the degree of the derivations.

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- Then d induces a differential (which we still call d) on $H_*(\delta)$. We denote the cohomology of d on $H_*(\delta)$ by $H^*(d|H_*(\delta))$ and call d a differential modulo δ.

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- The class of x doesn't change if we modify x and consider

$$x' = x + d(u) + \delta(v)$$

for some u with $\delta(u) = 0$.

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In this generality, the answer is NO. But in BRST situations things will work.

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Remark: The differential s is highly non-unique.

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- We know that δ, d and $s^{(1)}$ are derivations, so the higher $s^{(k)}$ are derivations as well and so is s_n. Note, that $2s_n^2 = [s_n, s_n]$, because all summands in s_n have parity $\varepsilon(s^{(k)}) = 1$.

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- We claim that solving $2[\delta, s^{(n+1)}] + r_n = 0$ is equivalent to showing that $[\delta, r_n] = 0$

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- Let x be an element of a fixed ghost degree, say k. Expand x according to resolution degree

$$x = x^{(0)} + x^{(1)} + \dots$$

  with $r(x^{(n)}) = n$. Define π as $\pi(x) = x^{(0)}$.
- In $s(x)$ the term of resolution degree zero is $d(x^{(0)}) + \delta(x^{(1)})$. Therefore for a cycle x, $\pi(s(x))$ is equivalent to $d(x^{(0)})$ up to something in the image of δ and thus π induces a well defined map

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- If $x^{(0)}$ is a representative in $H^k(d, H^0(\mathcal{A}, \delta))$ then there is an $x^{(1)}$ with $dx^{(0)} + \delta x^{(1)} = 0$.

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- Assume that we constructed $y_n = x^{(0)} + \ldots + x^{(n)}$ such that sy_n starts with terms in resolution degree n, say

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That's it! □

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