Homological perturbation theory and the existence of the BRST differential

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The BRST setting

Algebras and derivations

Homology and cohomology

Differentials modulo differentials

Homological perturbation theory

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HPT: Often, there is a differential s on \mathcal{A} with

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such that the homology of *s* still gives the gauge invariant functions.

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► The algebra A is associative and it is *supercommutative*: we have

$$uv = (-1)^{\varepsilon(u)\varepsilon(v)}vu$$
 for all u, v .

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▶ Thus $\operatorname{End}(V) = \operatorname{End}(V)_0 \oplus \operatorname{End}(V)_1$. The $\mathbb{Z}/2\mathbb{Z}$ -grading is compatible with composition:

$$\varepsilon(f_2 \circ f_1) = \varepsilon(f_2) + \varepsilon(f_1) \mod 2$$

▶ For two endomorphisms f_1 , f_2 of V we define

$$[f_1, f_2] = f_1 \circ f_2 - (-1)^{\varepsilon(f_1)\varepsilon(f_2)} f_2 \circ f_1.$$

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▶ We denote by Der(A) the graded sub-Lie algebra of End(A).

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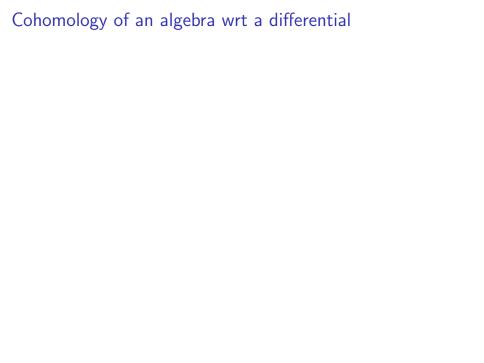
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$$H_*(\mathcal{A},\delta) = \left\{ egin{array}{ll} 0 & *
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▶ Let A be a supercommutative, associative algebra as before. A resolution of A is an \mathbb{N}_0 -graded supercommutative, associative algebra \mathcal{A} with a differential δ of internal degree -1 such that

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- ▶ Then in homology, P kills the generator z and (A, δ) is a resolution of $A = \mathbb{C}[x]$.



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- ▶ Then $A = H_0(A) \cong C^{\infty}(\Sigma)$. That is the typical input for homological perturbation theory. Bahns and Ribeiro will deal with actual examples in their talk.

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Here, the grading * corresponds to the degree of the derivations.

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▶ Then d induces a differential (which we still call d) on $H_*(\delta)$.

- ▶ Let $\delta \in \text{Der}(A)$ be a differential of internal degree 1. We consider its homology $H_*(A, \delta)$ and abbreviate this to $H_*(\delta)$.
- Let d be a derivation with $\varepsilon(d) = 1$. Assume d has internal degree one, satisfies

$$d\delta + \delta d = 0$$

and d^2 is δ -exact, i.e., there is a derivation D such that

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▶ Then d induces a differential (which we still call d) on $H_*(\delta)$. We denote the cohomology of d on $H_*(\delta)$ by $H^*(d|H_*(\delta))$ and call d a differential modulo δ .

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- ▶ The class of x doesn't change if we modify x and consider

$$x' = x + d(u) + \delta(v)$$

for some u with $\delta(u) = 0$.

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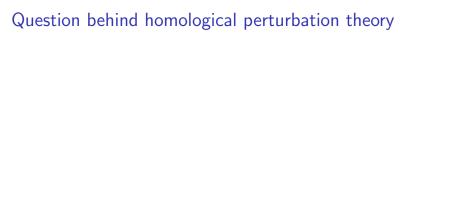
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In this generality, the answer is NO. But in BRST situations things will work.

▶ Throughout we will assume that there is a resolution (A, δ) of our supercommutative, associative algebra $A = H_0(A, \delta)$, so we will drop A and A from the notation (most of the time). It is common to denote A by $H_0(\delta)$.

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, $gh(d) = 1 - 0 = 1$.

In the setting above assume in addition that the homology of the Lie-algebra of derivations with respect to the differential δ is trivial but in degree zero, i.e., $\mathcal{H}^*(\delta) = 0$ for all $* \neq 0$.

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a) Then there is a differential s with gh(s) = 1 such that

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b) Any such differential s has the property that its cohomology is isomorphic to the cohomology of d on A:

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Remark: The differential *s* is highly non-unique.

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We know that δ , d and $s^{(1)}$ are derivations, so the higher $s^{(k)}$ are derivations as well and so is s_n . Note, that $2s_n^2 = [s_n, s_n]$, because all summands in s_n have parity $\varepsilon(s^{(k)}) = 1$.

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▶ We claim that solving $2[\delta, s^{(n+1)}] + r_n = 0$ is equivalent to showing that $[\delta, r_n] = 0$

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We still have to show $[\delta, r_n] = 0$. To this end expand $[[s_n, s_n], s_n]$. This term vanishes because of the Jacobi identity and $[\delta, r_n]$ is its term of resolution degree n-1. This proves existence of the differential s.

► Let *x* be an element of a fixed ghost degree, say *k*. Expand *x* according to resolution degree

$$x = x^{(0)} + x^{(1)} + \dots$$

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• We claim that π induces an isomorphism.

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• We claim that π induces an isomorphism. Surely $\pi(x)\pi(y)=x^{(0)}y^{(0)}=xy^{(0)}=\pi(xy)$ and π is additive. We'll prove surjectivity and skip injectivity because it's the same trick anyway.

▶ If $x^{(0)}$ is a representative in $H^k(d, H^0(\mathcal{A}, \delta))$ then there is an $x^{(1)}$ with $dx^{(0)} + \delta x^{(1)} = 0$.

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- Assume that we constructed $y_n = x^{(0)} + ... + x^{(n)}$ such that sy_n starts with terms in resolution degree n, say
- $sy_n = t_n + t_{n+1} + \dots$ with $r(t_i) = i$.

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- ► The equation $s^2y_n = 0$ has δt_n as term of lowest resolution degree n-1 and hence $\delta t_n = 0$.
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- If $x^{(0)}$ is a representative in $H^k(d, H^0(\mathcal{A}, \delta))$ then there is an $x^{(1)}$ with $dx^{(0)} + \delta x^{(1)} = 0$. Thus $s(x^{(0)} + x^{(1)})$ starts in resolution degree 1.
- Assume that we constructed $y_n = x^{(0)} + ... + x^{(n)}$ such that sy_n starts with terms in resolution degree n, say

$$sy_n = t_n + t_{n+1} + \dots$$
 with $r(t_i) = i$.

▶ The equation $s^2y_n = 0$ has δt_n as term of lowest resolution degree n-1 and hence $\delta t_n = 0$. But $H^*(\mathcal{A}, \delta) = 0$ in positive degrees and therefore there is an element $x^{(n+1)}$ with

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That's it!

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