The chromatic splitting conjecture; the case of the prime 3 and chromatic level 2

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Structured ring spectra: Hamburg, August 5, 2011

(joint work with Paul Goerss and Mark Mahowald)

Fix a prime p.

Chromatic localizations

The height filtration of the moduli stack of one dimensional formal groups laws over p-local rings has a counterpart in the category of p-local spectra known as the chromatic filtration.

- Let $E(n)$ be the n-th Johnson-Wilson spectrum
- Let $K(n)$ be the n-th Morava K-theory spectrum
- Let L_n be Bousfield localization with respect to $E(n)_*$.
- Let $L_{K(n)}$ be Bousfield localization with respect to $K(n)_*$.

Examples

- L_0 is rationalization.
- L_1 is localization with respect to p-local K-theory.
- $L_{K(1)}$ is localization with respect to mod-p K-theory.

The chromatic tower

These functors assemble into a tower known as the chromatic tower. Chromatic homotopy theory is the study of the p-local stable category via the tower of Bousfield localization functors L_n.

$$\begin{array}{ccc} \vdots & & \vdots \\ X & \longrightarrow & L_n X \\ \| & & \downarrow \\ X & \longrightarrow & L_{n-1} X \\ \| & & \downarrow \\ \vdots & & \vdots \\ \| & & \downarrow \\ X & \longrightarrow & L_0 X \end{array}$$

Chromatic convergence and the chromatic square

- Chromatic convergence says that for finite p-local spectra the right hand tower is pro-isomorphic to the constant left hand tower.
- L_n is determined by L_{n-1} and $L_{K(n)}$. More precisely, for every X there is a homotopy pull back square (the "n-th chromatic square")

$$\begin{array}{ccc} L_n X & \to & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \to & L_{n-1} L_{K(n)} X \end{array}$$

The chromatic splitting conjecture - (Hopkins, popularized by Hovey in early 1990's)

- If X is p-complete finite then $L_{n-1}X \to L_{n-1}L_{K(n)}X$ is a split monomorphism (This is a weak form of the conjecture).
- There is a more precise form which decribes the complementary factor of $L_{n-1}L_{K(n)}$ in terms of L_k's for $k < n$. (See below in the cases $n \leq 2$)

An immediate application (cf. Hovey)

- Suppose the chromatic splitting conjecture is true and let $f : X \longrightarrow Y$ be a map between two finite spectra with $L_{K(n)}f : L_{K(n)}X \to L_{K(n)}Y$ null for infinitely many n
- Then f is null.

What is known?

Fix n and p. If $L_{n-1}X_p \to L_{n-1}L_{K(n)}X$ is a split monomorphism for $X = S^0$ then it is true for every finite X. The following is a complete list of all known cases.

- $n = 1$ and all primes (easy, see below)
- $n = 2$ and $p \geq 5$ (known as consequence of calculations by Shimomura-Yabe in Topology (1995))
- $n = 2$ and $p = 3$ (Goerss, H. and Mahowald)

The case $n = 1$

The case $n = 1$ follows immediately from the fibration (Adams-Baird, Bousfield, Ravenel ...)

$$(*) \qquad L_{K(1)}S^0 \to K\mathbb{Z}_p^{h\mu_p} \stackrel{\psi^{p+1}-id}{\longrightarrow} K\mathbb{Z}_p^{h\mu_p}$$

where

- μ_p stands for the group of roots of unity in the p-adic integers $\mathbb{Z}_p$
- ψ^{p+1} is the appropriate Adams operation.

In fact, $(*)$ implies immediately

$$L_0 L_{K(1)} S^0 \simeq L_0(S_p^0 \vee S_p^{-1})$$

and the map from $L_0 S_p^0$ is easily checked to be the inclusion as the first factor.

The case $n = 2$ and $p > 2$ - the strategy

The idea is to generalize the proof for $n = 1$ in the following way:

- Replace $(*)$ by a "resolution of spectra"

$$L_{K(2)}S^0 \to X_0 \to X_1 \to X_2 \to X_3 \to X_4$$

  and use this resolution in order to show

- $$L_0 L_{K(2)}S^0 \simeq L_0(S_p^0 \vee S_p^{-1} \vee S_p^{-3} \vee S_p^{-4})$$

- $$L_{K(1)}L_{K(2)}S^0 \simeq L_{K(1)}(S_p^0 \vee S_p^{-1})$$

- Use the chromatic square to deduce

$$L_1 L_{K(2)}S^0 \simeq L_1(S_p^0 \vee S_p^{-1}) \vee L_0(S_p^{-3} \vee S_p^{-4})$$

Remarks

- In case $n = 2$ and $p > 3$ this can be deduced from the very complicated calculations of $\pi_*(L_{K(2)}S^0)$ due to Shimomura-Yabe.
- However, it is actually enough to know the comparatively much more comprehensible earlier calculation of $\pi_*(L_{K(2)}M(p))$ (by Shimomura).
- In case $n = 2$ and $p = 3$ Shimomura and Wang (Topology 2002) have calculated $\pi_*(L_{K(2)}S^0)$ and deduced from their calculation that the explixit form of the splitting conjecture given above does not hold. However, their calculation contains errors!!
- In the sequel we discuss the case $n = 2$ and $p = 3$ in a form which applies equally well to the previously known case $n = 2$ and $p \geq 5$.

The starting point is given by the following result in which

- $\mathbb{G}_n$ is the big (extended) Morava stabilizer group, $\mathbb{G}_n = \mathbb{D}_n^\times / \langle p \rangle$, and
- E_n is the Landweber exact 2-periodic Lubin Tate spectrum whose π_0 classifies deformations of the Honda formal group law of $\mathbb{F}_{p^n}$, $E_n = \mathbb{W}_{\mathbb{F}_{p^n}}[[u_1, \ldots, u_{h-1}]][u^{\pm 1}]$.

Theorem (Devinatz-Hopkins)

- $$L_{K(n)} S^0 \simeq E_n^{h\mathbb{G}_n}$$
- The associated descent spectral sequence
$$E_2^{s,t} \cong H^s(\mathbb{G}_n, (E_n)_t) \Rightarrow \pi_{t-s}(L_{K(n)} X)$$
can be identified with the Adams-Novikov spectral sequence for $L_{K(n)} S^0$.

A decomposition of the Morava stabilizer group

We are thus interested in cohomological properties of the group $\mathbb{G}_2$.

- Let $\mathbb{G}_2$ be the big Morava stabilizer group, i.e. $\mathbb{G}_2 = \mathbb{D}_2^\times / < p >$. The resolution arises from a resolution of $\mathbb{Z}_p$ considered as a trivial module for the group $\mathbb{G}_2$.
- In fact, the problem can be reduced to a smaller group $\mathbb{G}_2^1$, the kernel of the reduced norm which in this framework is a surjective homomorphism

$$\mathbb{D}_n^\times / < p > \to \mathbb{Q}_p^\times / < p^n > \cong \mathbb{Z}_p^\times \times \mathbb{Z}/n \to \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times / \mu_p \cong \mathbb{Z}_p .$$

- For $n = 2$ and $p \geq 3$ there is an isomorphism $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_p$.

Theorem (Goerss, Henn, Mahowald, Rezk)

- There is an explicit exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}_3 \ .$$

with

$$C_0 = C_3 \cong \mathbb{Z}_3[[\mathbb{G}_2^1 / G_{24}]]$$

$$C_1 = C_2 \cong \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \mathbb{Z}_3(\chi)$$

suitable permutation modules on finite subgroups of $\mathbb{G}_2^1$.

- There is an explicit "exact" complex of spectra of the form

$$1 \to E_2^{h\mathbb{G}_2^1} \to X_0 \to X_1 \to X_2 \to X_3 \to 1 \ .$$

with

$$X_0 = \Sigma^{24} X_3 = (E_2)^{hG_{24}} = L_{K(2)} TMF, \ X_1 = X_2 = \Sigma^8 (E_2)^{SD_{16}} \ .$$

Analyzing the resulting spectral sequences

- By [Henn, Karamanov, Mahowald] the algebraic complex is sufficiently well under control so that we can evaluate the resulting spectral sequences after rationalization resp. v_1^{-1}-localization.
- In fact, the homotopy type of $L_1 L_{K(2)} X$ turns out to be "constant" on the exotic part of Hopkins' Picard group consisting of $K(2)$-local spectra X for which there is an isomorphism of Morava modules $(E_2)_* X \cong (E_2)_* S^0$.

Theorem (Goerss, Henn, Mahowald)

Let $p = 3$ and suppose X is a $K(2)$-local spectra satisfying $(E_2)_*X \cong (E_2)_*S^0$ as Morava modules.

- $$L_0 L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \simeq L_0(S_p^0 \vee S_p^{-3}))$$
- $$L_0 X \simeq L_0(S_p^0 \vee S_p^{-1} \vee S_p^{-3} \vee S_p^{-4})$$
- $$L_{K(1)}(E_2^{h\mathbb{G}_2^1} \wedge (X \wedge M(p))) \simeq L_{K(1)} M(p)$$
- $$L_{K(1)} L_{K(2)}(X \wedge M(p)) \simeq L_{K(1)}(M(p) \vee \Sigma^{-1} M(p))$$
- $$L_{K(1)} X \simeq L_{K(1)}(S_p^0 \vee S_p^{-1})$$