The chromatic splitting conjecture; the case of the prime 3 and chromatic level 2

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(joint work with Paul Goerss and Mark Mahowald)
Fix a prime $p$.

**Chromatic localizations**

The height filtration of the moduli stack of one dimensional formal groups laws over $p$-local rings has a counterpart in the category of $p$-local spectra known as the chromatic filtration.

- Let $E(n)$ be the $n$-th Johnson-Wilson spectrum
- Let $K(n)$ be the $n$-th Morava $K$-theory spectrum
- Let $L_n$ be Bousfield localization with respect to $E(n)_\ast$.
- Let $L_{K(n)}$ be Bousfield localization with respect to $K(n)_\ast$.

**Examples**

- $L_0$ is rationalization.
- $L_1$ is localization with respect to $p$-local $K$-theory.
- $L_{K(1)}$ is localization with respect to mod-$p$ $K$-theory.
The chromatic tower

These functors assemble into a tower known as the **chromatic tower**. Chromatic homotopy theory is the study of the \( p \)-local stable category via the tower of Bousfield localization functors \( L_n \).

\[
\begin{array}{ccc}
X & \longrightarrow & L_n X \\
\| & & \| \\
X & \longrightarrow & L_{n-1} X \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
X & \longrightarrow & L_0 X \\
\end{array}
\]
Chromatic convergence and the chromatic square

- Chromatic convergence says that for finite $p$-local spectra the right hand tower is pro-isomorphic to the constant left hand tower.

- $L_n$ is determined by $L_{n-1}$ and $L_{K(n)}$. More precisely, for every $X$ there is a homotopy pull back square (the “$n$-th chromatic square”)

$$
\begin{array}{ccc}
L_n X & \rightarrow & L_{K(n)} X \\
\downarrow & & \downarrow \\
L_{n-1} X & \rightarrow & L_{n-1} L_{K(n)} X
\end{array}
$$
The chromatic splitting conjecture - (Hopkins, popularized by Hovey in early 1990’s)

- If $X$ is $p$-complete finite then $L_{n-1}X \to L_{n-1}L_{K(n)}X$ is a split monomorphism (This is a weak form of the conjecture).
- There is a more precise form which describes the complementary factor of $L_{n-1}L_{K(n)}$ in terms of $L_k$’s for $k < n$. (See below in the cases $n \leq 2$)

An immediate application (cf. Hovey)

- Suppose the chromatic splitting conjecture is true and let $f : X \to Y$ be a map between two finite spectra with $L_{K(n)}f : L_{K(n)}X \to L_{K(n)}Y$ null for infinitely many $n$.
- Then $f$ is null.
What is known?

Fix \( n \) and \( p \). If \( L_{n-1}X_p \to L_{n-1}L_{K(n)}X \) is a split monomorphism for \( X = S^0 \) then it is true for every finite \( X \). The following is a complete list of all known cases.

- \( n = 1 \) and all primes (easy, see below)
- \( n = 2 \) and \( p \geq 5 \) (known as consequence of calculations by Shimomura-Yabe in Topology (1995))
- \( n = 2 \) and \( p = 3 \) (Goerss, H. and Mahowald)
The case $n = 1$

The case $n = 1$ follows immediately from the fibration (Adams-Baird, Bousfield, Ravenel ...)

\[(\ast)\]

\[L_{K(1)} S^0 \to K \mathbb{Z}_p h^{\mu_p} \psi^{p+1} - id \to K \mathbb{Z}_p h^{\mu_p}\]

where

- $\mu_p$ stands for the group of roots of unity in the $p$-adic integers $\mathbb{Z}_p$
- $\psi^{p+1}$ is the appropriate Adams operation.

In fact, $(\ast)$ implies immediately

\[L_0 L_{K(1)} S^0 \simeq L_0(S^0_p \vee S^{−1}_p)\]

and the map from $L_0 S^0_p$ is easily checked to be the inclusion as the first factor.
The case $n = 2$ and $p > 2$ - the strategy

The idea is to generalize the proof for $n = 1$ in the following way:

- Replace $(\ast)$ by a “resolution of spectra”

$$L_{K(2)}S^0 \to X_0 \to X_1 \to X_2 \to X_3 \to X_4$$

and use this resolution in order to show

- $$L_0L_{K(2)}S^0 \simeq L_0(S_p^0 \vee S_p^{-1} \vee S_p^{-3} \vee S_p^{-4})$$

- $$L_{K(1)}L_{K(2)}S^0 \simeq L_{K(1)}(S_p^0 \vee S_p^{-1})$$

- Use the chromatic square to deduce

$$L_1L_{K(2)}S^0 \simeq L_1(S_p^0 \vee S_p^{-1}) \vee L_0(S_p^{-3} \vee S_p^{-4})$$
Remarks

- In case $n = 2$ and $p > 3$ this can be deduced from the very complicated calculations of $\pi_* (L_{K(2)} S^0)$ due to Shimomura-Yabe.

- However, it is actually enough to know the comparatively much more comprehensible earlier calculation of $\pi_* (L_{K(2)} M(p))$ (by Shimomura).

- In case $n = 2$ and $p = 3$ Shimomura and Wang (Topology 2002) have calculated $\pi_* (L_{K(2)} S^0)$ and deduced from their calculation that the explicit form of the splitting conjecture given above does not hold. However, their calculation contains errors!!

- In the sequel we discuss the case $n = 2$ and $p = 3$ in a form which applies equally well to the previously known case $n = 2$ and $p \geq 5$. 
The starting point is given by the following result in which
- $G_n$ is the big (extended) Morava stabilizer group, $G_n = \mathbb{D}_n^\times / \langle p \rangle$, and
- $E_n$ is the Landweber exact 2-periodic Lubin Tate spectrum whose $\pi_0$ classifies deformations of the Honda formal group law of $\mathbb{F}_p [n]$, $E_n = \mathbb{W}_{\mathbb{F}_p}[n][[u_1, \ldots, u_{h-1}]][u^{\pm 1}]$.

**Theorem (Devinatz-Hopkins)**

- $L_{K(n)}S^0 \simeq E^n_{hG_n}$
- The associated descent spectral sequence

$$E_2^{s,t} \simeq H^s(G_n, (E_n)_t) \Rightarrow \pi_{t-s}(L_{K(n)}X)$$

can be identified with the Adams-Novikov spectral sequence for $L_{K(n)}S^0$. 
A decomposition of the Morava stabilizer group

We are thus interested in cohomological properties of the group $G_2$.

- Let $G_2$ be the big Morava stabilizer group, i.e.
  \[ G_2 = D_2^\times / < p >. \] The resolution arises from a resolution of $\mathbb{Z}_p$ considered as a trivial module for the group $G_2$.

- In fact, the problem can be reduced to a smaller group $G_2^1$, the kernel of the reduced norm which in this framework is a surjective homomorphism

  \[ D_n^\times / < p > \rightarrow \mathbb{Q}_p^\times / < p^n > \cong \mathbb{Z}_p^\times \times \mathbb{Z}/n \rightarrow \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times / \mu_p \cong \mathbb{Z}_p. \]

- For $n = 2$ and $p \geq 3$ there is an isomorphism $G_2 \cong G_2^1 \times \mathbb{Z}_p$. 
Theorem (Goerss, Henn, Mahowald, Rezk)

- There is an explicit exact complex of $\mathbb{Z}_3[[G_2^1]]$-modules
  
  $0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}_3$.

  with

  $C_0 = C_3 \cong \mathbb{Z}_3[[G_2^1/G_{24}]]$

  $C_1 = C_2 \cong \mathbb{Z}_3[[G_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \mathbb{Z}_3(\chi)$

  suitable permutation modules on finite subgroups of $G_2^1$.

- There is an explicit “exact” complex of spectra of the form

  $1 \to E_2^{hG_2^1} \to X_0 \to X_1 \to X_2 \to X_3 \to 1$.

  with

  $X_0 = \Sigma^{24} X_3 = (E_2)^{hG_{24}} = L_{K(2)} TMF$, $X_1 = X_2 = \Sigma^8 (E_2)^{SD_{16}}$. 
Analyzing the resulting spectral sequences

- By [Henn, Karamanov, Mahowald] the algebraic complex is sufficiently well under control so that we can evaluate the resulting spectral sequences after rationalization resp. $v_1^{-1}$-localization.

- In fact, the homotopy type of $L_1 L_{K(2)} X$ turns out to be “constant” on the exotic part of Hopkins’ Picard group consisting of $K(2)$-local spectra $X$ for which there is an isomorphism of Morava modules $(E_2)_* X \cong (E_2)_* S^0$. 
Theorem (Goerss, Henn, Mahowald)

Let $p = 3$ and suppose $X$ is a $K(2)$-local spectra satisfying $(E_2)_* X \cong (E_2)_* S^0$ as Morava modules.

- $L_0 L_{K(2)} (E_2^{hG_2^1} \wedge X) \cong L_0 (S^0_p \vee S^{-3}_p)$

- $L_0 X \cong L_0 (S^0_p \vee S^{-1}_p \vee S^{-3}_p \vee S^{-4}_p)$

- $L_{K(1)} (E_2^{hG_2^1} \wedge (X \wedge M(p))) \cong L_{K(1)} M(p)$

- $L_{K(1)} L_{K(2)} (X \wedge M(p)) \cong L_{K(1)} (M(p) \vee \Sigma^{-1} M(p))$

- $L_{K(1)} X \cong L_{K(1)} (S^0_p \vee S^{-1}_p)$