Applications of ring spectra to the geometry of flat bundles

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Flat bundles

A principal $U(n)$–bundle $E \to M^m$ is *flat* if

- $E = (\widetilde{M} \times U(n))/\pi_1 M$ for some $\rho\colon \pi_1 M \to U(n)$,

  or equivalently,

- E admits a *connection* with trivial curvature.

Chern–Weil Theory: E flat $\implies c_i(E)$ is torsion for all i.

Flat bundles are rare.

Given an S^k–*family* $\rho : S^k \to \mathrm{Hom}(G, U(n))$, set

$$E_\rho = (S^k \times \widetilde{M} \times U(n))/\pi_1 M \to S^k \times M.$$

Definition: k–flat

$x \in \widetilde{K}^i(M)$ ($i = 0, -1$) is k–flat if $\beta^k(x) \in \widetilde{K}^0(S^{2k-i} \wedge M)$ has the form $[E_\rho] - [E_\psi]$ for some S^{2k-i}–families ρ, ψ.

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Flat Realization Problem: Given $x \in \widetilde{K}^i(M)$ $(i = 0, -1)$, find the minimum k such that x is k–flat (could be $k = \infty$!).

A cohomological obstruction:

Theorem: (Baird–R., '11)

For every family $\rho\colon\ S^k \to \mathrm{Hom}(G, U(n))$, the Chern classes $c_i(E_\rho)$ are torsion for $i > k$.

The proof involves two main steps:

1. If ρ is *smooth*, apply Chern–Weil Theory.
2. All families are homotopic to smooth families. (**Hard!!**)

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Deformation K–theory

For a discrete group G, let $\mathrm{Rep}(G) = \coprod_n \mathrm{Hom}(G, U(n))$.

Definition: Deformation K–theory

$\widetilde{K}_m^{\mathrm{def}}(G) = Gr[S^m, \mathrm{Rep}(G)]/Gr(\pi_0 \mathrm{Rep}(G))$

Observation: $x \in K^i(BG)$ $(i = 0, -1)$ is k–flat if and only if $\beta^k(x)$ is in the image of the **Atiyah–Segal map**

$$\widetilde{K}_{2k-i}^{\mathrm{def}}(G) \xrightarrow{\alpha_{2k-i}} \widetilde{K}^{-2k+i}(BG)$$

$$[\rho] - [\psi] \longmapsto [E_\rho] - [E_\psi](+\text{ correction term})$$

Note: Previous result $\rightsquigarrow$ cohomological obstruction to surjectivity of α_k for $k \leqslant \mathbb{Q}\mathrm{cd}(G) - 2$.

Goal: Study the Flat Realization Problem using a *ring spectrum* version of the Atiyah–Segal map.

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The Deformation K–theory Spectrum

Let $\mathrm{Vect}_{\mathbb{C}} = \begin{cases} \mathrm{Ob}\mathrm{Vect}_{\mathbb{C}} = \mathbb{N} \\ \mathrm{Mor}\mathrm{Vect}_{\mathbb{C}} = \coprod_n U(n) \end{cases}$ denote the (topological) bipermutative category of Hermitian vector spaces.

$\mathcal{R}(G) = \mathrm{Funct}(\underline{G}, \mathrm{Vect}_{\mathbb{C}})$ is the category of unitary G–rep's.

Definition: (Carlsson)

$K^{\mathrm{def}}(G) = \mathbb{K}(\mathcal{R}(G))$ is the K–theory spectrum of $\mathcal{R}(G)$.

- Elmendorf–Mandell, May: $K^{\mathrm{def}}(G)$ is a ring spectrum, and in fact a **ku**–algebra. (Note that $\mathbf{ku} = \mathbb{K}(\mathrm{Vect}_{\mathbb{C}})$).

Theorem: (T. Lawson) The homotopy cofiber of the Bott map $\Sigma^2 K^{\mathrm{def}}(G) \xrightarrow{\beta} K^{\mathrm{def}}(G)$ is the spectrum

$$\mathbb{K}\left(\coprod_n \mathrm{Hom}(G, U(n))/U(n) \right) =: R^{\mathrm{def}}(G).$$

- In many cases, $\Omega^\infty R^{\mathrm{def}}(G) \simeq \mathbb{Z} \times \mathrm{Hom}(G, U)/U$.

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- $\mathcal{R}(G) \cong \widetilde{\mathcal{R}}(G)^G$, where $\widetilde{\mathcal{R}}(G)$ is the category of unitary G–representations and *non-equivariant* isometries.
- $K^{\mathrm{def}}(G) = K^{\mathrm{def}}(\mathcal{R}(G)) \cong \mathbb{K}(\widetilde{\mathcal{R}}(G))^G \hookrightarrow \mathbb{K}(\widetilde{\mathcal{R}}(G))^{hG}$.
- Inclusion of the trivial rep's: $\mathrm{Vect}_{\mathbb{C}} \overset{\simeq}{\hookrightarrow} \widetilde{\mathcal{R}}(G)$
  $\implies F(BG_+, \mathbf{ku}) = \mathbb{K}(\mathrm{Vect}_{\mathbb{C}})^{hG} \overset{\simeq}{\leftarrow} \mathbb{K}(\widetilde{\mathcal{R}}(G))^{hG}$.

Main Theorem: (R., '11)

The map of **ku**–*algebras*

$$\alpha_G \colon K^{\mathrm{def}}(G) = \mathbb{K}(\widetilde{\mathcal{R}}(G))^G \hookrightarrow \mathbb{K}(\widetilde{\mathcal{R}}(G))^{hG} \overset{\simeq}{\longleftarrow} F(BG_+, \mathbf{ku})$$

induces the Atiyah–Segal map on π_*, $* \geqslant 0$.

Proof: Relate both maps to

$$B : \mathrm{Hom}(G, U(n)) \to F(BG, BU(n)).$$

Note: $B\rho$ classifies E_ρ.

The Homotopy Limit Problem in Deformation K–theory

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induces the Atiyah–Segal map on π_*, $* \geqslant 0$.

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Surface Groups:

Theorem: (R., '09, '11)

S an aspherical surface. Then $\alpha_{\pi_1 S}$ is a w.e. on (0)–c'td covers

Proof uses Morse theory for the Yang–Mills functional.

Remark: Lawson's Bott cofiber sequence $\rightsquigarrow \mathbb{Q}$–isomorphism $\pi_*(\mathbb{Z} \times \mathrm{Hom}(G, U)/U) \cong_{\mathbb{Q}} H^*(G; \mathbb{Z})$ for these groups.

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The Heisenberg manifold

Let

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} \subset \mathbb{R}^3$$

denote the integral Heisenberg group.

Let $N^3 = \mathbb{R}^3 / H$ be the Heisenberg manifold (a closed, aspherical, orientable, 3-dimensional nil-manifold).

Theorem: (T. Lawson)

The Bott map $K_*^{\mathrm{def}} H \xrightarrow{\beta} K_{*+2}^{\mathrm{def}} H$ is an isomorphism for $* \geqslant 1$.

Corollary:

If $x \in K^{-1}(M)$ has $c_2(x) = m[N^3]$ ($m \neq 0$), then x is not k–flat for any k.

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- Commutes because α is a **ku**–module map.
- If $y \in \mathrm{Im}(\alpha_1)$, then y is 0–flat and hence $c_i(y) =_{\mathbb{Q}} 0$ for $i > 1$.
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Example: Products of surface groups

$\mathcal{S} = S_1 \times \cdots \times S_m$, a product of aspherical surfaces.

Theorem: (R., '11)

The Atiyah–Segal map $\alpha\colon K^{\mathrm{def}}(\pi_1\mathcal{S}) \to F(B\pi_1\mathcal{S}_+, \mathbf{ku})$ *is an equivalence on* $(\mathbb{Q}\mathrm{cd}(\mathcal{S}) - 2)$*–connected covers.*

Notice: Analogous to Quillen–Lichtenbaum conjectures.

Components of proof:

- Lawson's Product Formula:
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Methods from previous slide $\rightsquigarrow$ calculation of

$$\mathrm{Im}\left(\alpha_k \colon K_k^{\mathrm{def}}(\pi_1 \mathcal{S}) \longrightarrow K^{-k}(\mathcal{S})\right).$$

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Further Applications:

- *Free abelian groups (spaces of commuting matrices)*
- *Crystallographic groups (rational results)*

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$$x \in K^i(B\Gamma) \text{ is } k\text{–flat} \iff c_l(x) = 0 \text{ for } l > 2k - i \ (i = 0, -1).$$

- Non-orientable products: not all torsion classes are 0–flat.

Further Applications:

- *Free abelian groups (spaces of commuting matrices)*
- *Crystallographic groups (rational results)*

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