On the theory of derivators

A few monoidal aspects

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Outline

Homotopy Categories Are Strong Truncations

$\mathcal{M}$: stable model category
$\mathrm{Ho}(\mathcal{M})$: associated homotopy category

$\mathcal{M} \rightsquigarrow \mathrm{Ho}(\mathcal{M})$ results in loss of information

Structure on $\mathrm{Ho}(\mathcal{M})$:

- fiber and cofiber sequences
- triangulated structure
- some homotopy colimits, but only non-canonically

Problem: 'Diagrams in $\mathrm{Ho}(\mathcal{M})$ do not carry enough information in order to canonically define their homotopy (co)limits'

Prederivators: Form Many Homotopy Categories

'Solution': Consider diagrams before passing to homotopy categories, i.e., replace $\mathrm{Ho}(\mathcal{M})^J$ by $\mathrm{Ho}(\mathcal{M}^J)$

- to a small category J associate $\mathrm{Ho}(\mathcal{M}^J) = \mathbb{D}_{\mathcal{M}}(J)$
- a functor $u\colon J \to K$ induces a *restriction functor* $\mathbb{D}_{\mathcal{M}}(u) = u^*\colon \mathrm{Ho}(\mathcal{M}^K) \to \mathrm{Ho}(\mathcal{M}^J)$
- we obtain that way a 2-functor $\mathbb{D}_{\mathcal{M}}\colon \mathrm{Cat}^{op} \to \mathrm{CAT}$

Definition

A *prederivator* is a 2-functor $\mathbb{D}\colon \mathrm{Cat}^{op} \to \mathrm{CAT}$. $\mathbb{D}(e)$ is the *underlying category*, where e is the terminal category.

Derivators: Homotopically Bicomplete Prederivators

A prederivator $\mathbb{D}$ is a *derivator* if:

- restriction functors $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ have adjoints on both sides: *homotopy Kan extensions* exist
- homotopy Kan extensions can be calculated as usual
- two more 'natural *properties*' hold true

Example

A model category $\mathcal{M}$ has an underlying derivator $\mathbb{D}_{\mathcal{M}}$. A category induces a *represented prederivator* which is a derivator if and only if the representing category is bicomplete.

A Hierarchy Of Such Structures

The theory of derivators admits the following hierarchy:

- derivators
- pointed derivators
- additive derivators
- stable derivators

In all three cases, one asks for additional *properties*, no additional *structure* is imposed.

Derivators themselves are organized in a (2-)category Der.

Monoidal Derivators

The 2-category Der of derivators has (2-)products, i.e., is *Cartesian monoidal* by the pointwise formula:

$$(\mathbb{D} \times \mathbb{E})(J) = \mathbb{D}(J) \times \mathbb{E}(J)$$

Definition

A *monoidal derivator* is a monoidal object in $(\mathrm{Der}, \times)$.

- A monoidal derivator $\mathbb{D}\colon \mathrm{Cat}^{op} \to \mathrm{CAT}$ factors canonically over the 2-category of monoidal categories.
- There is the notion of a *module over a monoidal derivator*.

Linear Structures Induced By Actions

Example

Monoidal model categories and bicomplete monoidal categories induce monoidal derivators.

Proposition

An additive $\mathbb{E}$-module is canonically linear over the ring of selfmaps of the monoidal unit of the underlying monoidal category $\mathbb{E}(e)$. There is also a stable variant of this result.

Example

Let E be a (cofibrant) symmetric ring spectrum, the derivator of E-module spectra is linear over $\pi_* THH(E)$.

A Conceptual Explanation Of Some Linear Structures

Theorem

The 2-category Der *is Cartesian closed. In particular, for every derivator* $\mathbb{D}$ *there is a monoidal derivator of endomorphisms* $\mathrm{END}(\mathbb{D})$.

Let now $\mathbb{D}$ be an additive $\mathbb{E}$-module derivator.

- There is a canonical additive morphism of monoidal derivators $\mathbb{E} \to \mathrm{END}(\mathbb{D})$.
- We obtain an underlying additive, monoidal functor $\mathbb{E}(e) \to \mathrm{Hom}(\mathbb{D}, \mathbb{D})$.
- Thus, there is a ring map $hom_{\mathbb{E}(e)}(\mathbb{S}, \mathbb{S}) \to \mathrm{Z}(\mathbb{D})$, where $\mathrm{Z}(\mathbb{D})$ denotes the *center of* $\mathbb{D}$.

As A Summary. . .

The theory of (pre)derivators is:

- simple, i.e., formal: everything reduces to (2-)category theory
- convenient: all axioms of a derivator ask for *properties*, the only actual *structure* is the prederivator
- a common generalization of category theory and homotopical algebra
- a theory of 'smart shadows of $(\infty, 1)$-categories': combinatorial models can be reconstructed (non-canonically) from their underlying derivators

I thank you for your attention!!!