

Properties of the homology of algebraic n -fold loop spaces

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Lille, October 2012

The operad of little n -cubes

The rational case

Structure in characteristic two

Little n -cubes

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Little n -cubes

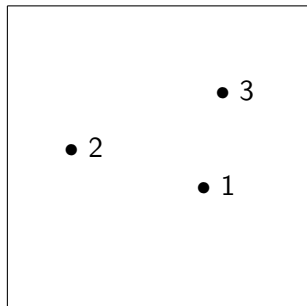
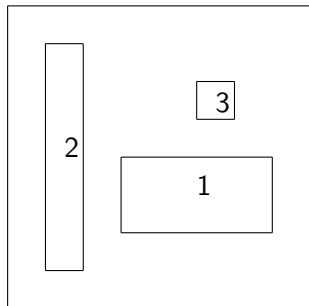
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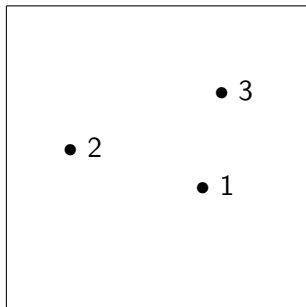
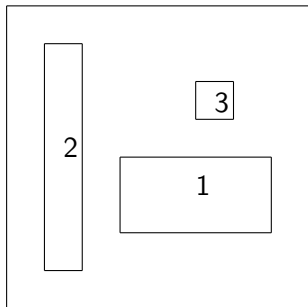
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This equivalence is Σ_r -equivariant.

Some history

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Araki-Kudo (1956): Pontrjagin rings $H_*(\Omega^N \mathbb{S}^m; \mathbb{F}_2)$, $0 < N < m$, definition of H_n -spaces, and some homology operations

$$Q_i: H_q(X; \mathbb{F}_2) \rightarrow H_{2q+i}(X; \mathbb{F}_2)$$

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Browder (1960): Description of $H_*(\Omega^n \Sigma^n Z; \mathbb{F}_2)$ as an algebra in terms of $H_*(Z; \mathbb{F}_2)$. Construction of a new operation (Browder operation).

Some history – continued

Dyer-Lashof (1962): Extension of (some of) the Q_i 's to odd primes. Partial results about $H_*(\Omega^n \Sigma^n Z; \mathbb{F}_p)$.

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Milgram (1966): $H_*(\Omega^n \Sigma^n Z; \mathbb{F}_p)$ as an algebra, depending only on the homology of Z and n .

Cohen (1976): Complete description of the homology operations on iterated loop spaces, and of $H_*(\Omega^n \Sigma^n Z; k)$ for $k = \mathbb{Q}$ and $k = \mathbb{F}_p$.

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But in general there is more, unless we have $k = \mathbb{Q} \dots$

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Cohen showed that the rational homology of any space $X = \Omega^{n+1}Y$ is an n -Gerstenhaber algebra and that

$$H_*(C_{n+1}Z; \mathbb{Q}) \cong nG(\bar{H}_*(Z; \mathbb{Q}))$$

for any space Z .

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Think of this as being 'half the circle' giving rise to 'half the Lie bracket $[x, x]$ ', aka the restriction on x .

Dyer-Lashof operations

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There are also relations between the Q^i 's and the action of the duals of the Sq^j 's (Nishida relations).

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Take the standard resolution of $\Sigma_2 = \mathbb{Z}/2\mathbb{Z}$, W_* , and compose

$$\begin{aligned}\theta_*: W_* \otimes C_*(X; \mathbb{F}_2)^{\otimes 2} &\rightarrow C_*(C_\infty(2); \mathbb{F}_2) \otimes C_*(X; \mathbb{F}_2)^{\otimes 2} \\ &\rightarrow C_*(C_\infty(2) \times X^2; \mathbb{F}_2) \rightarrow C_*(X; \mathbb{F}_2).\end{aligned}$$

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$Q_i(x) = \theta_*(e_i \otimes x \otimes x)$ ($e_i \in W_i$) is the induced map on homology and $Q^s(x) := Q_{s-|x|}(x)$ if $s - |x| \geq 0$ (0 otherwise).

At odd primes

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We get additional relations wrt the mod- p Bockstein.

Homology of $\Omega^{n+1}\Sigma^{n+1}Z$

Cohen: Complete descriptions of $H_*(C_{n+1}Z; \mathbb{F}_p)$ and $H_*(\Omega^{n+1}\Sigma^{n+1}Z; \mathbb{F}_p)$ as free objects built out of the reduced homology of Z .

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- ▶ a compatible coalgebra structure.

We get a Hopf algebra with a compatible Dyer-Lashof action and a restricted n -Lie algebra structure.

On chain level

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