Functor homology

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A biased overview
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In order to get functor homology interpretations we have to understand what something really is...

Some small categories

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3. Δ, the small category of finite ordered sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$ considered as an ordered set with the standard ordering $0 < 1 < \ldots < n$. Morphisms are order preserving, *i.e.*, for $f \in \Delta([n], [m])$ and $i < j$ in $[n]$ we require $f(i) \leq f(j)$.

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A covariant functor $F\colon \Gamma \to R$-mod is a Γ-module.

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for all $F \in \mathcal{C}$-mod and $\mathrm{Hom}_{\text{mod-}\mathcal{C}}(R\{\mathcal{C}(-,C)\}, G) \cong G(C)$ for all G in mod-$\mathcal{C}$.

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Then t can be written as the cokernel

$$R\{\Gamma(-, [2])\} \to R\{\Gamma(-, [1])\} \to t \to 0$$

where the map from $R\{\Gamma(-, [2])\}$ to $R\{\Gamma(-, [1])\}$ is induced by $f - p_1 - p_2$ with $f \colon [2] \to [1]$ being the fold map, sending $1, 2$ to 1 and $p_i(i) = 1$ and $p_i(j) = 0$ otherwise.

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Definition For any left $\mathcal{C}$-module F and any right $\mathcal{C}$-module G we define

$$G \otimes_{\mathcal{C}} F := \bigoplus_{C \in \mathcal{C}} G(C) \otimes_R F(C) / \sim$$

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Proposition The natural evaluation map induces isomorphisms

$$R\{\mathcal{C}(-, C)\} \otimes_{\mathcal{C}} F \cong F(C), \quad G \otimes_{\mathcal{C}} R\{\mathcal{C}(C, -)\} \cong G(C).$$

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then $H_i(F) \cong \mathrm{Tor}_i^{\mathcal{C}}(G, F)$ for all F.

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Here, $b = \sum_{i=0}^{n} (-1)^i d_i$ where
$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n$ for $i < n$ and
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Hochschild homology is André-Quillen homology for associative algebras up to a shift of degree. For a free algebra (a tensor algebra) it vanishes in degrees higher than one.

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$d_i\colon [n] \to [n-1]$,

$$d_i(j) = \begin{cases} j, & j < i \\ i, & j = i < n, \quad (0, \quad j = i = n), \\ j-1, & j > i. \end{cases}$$

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If we want to interpret Hochschild homology via functor homology on finite sets, A has to be commutative and M has to be a symmetric A-bimodule. Then we can define $\mathcal{L}(A; M)$ which sends $\Gamma \ni [n] \mapsto M \otimes A^{\otimes n}$.

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Interpreting $\mathbb{S}^1$ as a functor $\Delta^{op} \to \Gamma$ we get by composition $\mathcal{L}(A; M) \circ \mathbb{S}^1 \colon \Delta^{op} \to R$-mod and

$$HH_*(A; M) = \pi_* \mathcal{L}(A; M)(\mathbb{S}^1).$$

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A morphism $[n] \to [m]$ is a pointed map $f : [n] \to [m]$ together with a total ordering on the preimages $f^{-1}(j)$ for all $j \in [m]$.
Theorem [Pirashvili-R 2002] For any associative unital R-algebra A and any A-bimodule M

$$HH_*(A; M) \cong \mathrm{Tor}_*^{\Gamma(as)}(\bar{b}, \mathcal{L}(A; M)).$$

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Here, $\bar{b}$ is $\bar{b}(-) = \mathrm{coker}(R\{\Gamma(as)(-,[1])\} \to R\{\Gamma(as)(-,[0])\})$ where the map is induced by $d_0 - d_1$ where d_0 and d_1 send $0, 1$ to 0 but d_0 has $0 < 1$ as ordering on the preimage whereas d_1 has the ordering $1 < 0$ on $[1]$.

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Theorem [Pirashvili-R 2002]

$$HC_*(A) \cong \mathrm{Tor}_*^{\mathcal{F}(as)}(b, \mathcal{L}(A; A)).$$

Here, $\mathcal{F}(as)$ is the category of associative (unpointed) sets and b is the cokernel

$$b = \mathrm{coker}(R\{\mathcal{F}(as)(-, [1])\} \to R\{\mathcal{F}(as)(-, [0])\}).$$

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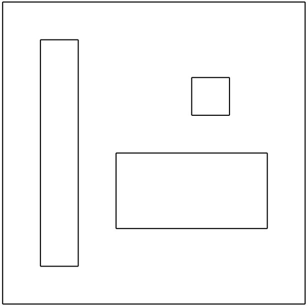

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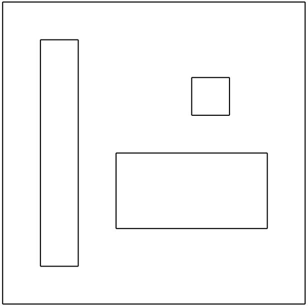

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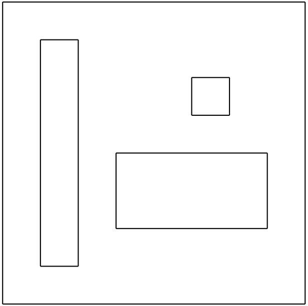

C_n acts on and detects n-fold based loop spaces.
$(C_* C_n(r))_r$, $r \geq 1$ is an operad in the category of chain complexes.
Let E_n be a cofibrant replacement of $C_* C_n$.

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For simplicity, let $A \to R$ be an augmented commutative R-algebra and $\bar{A}$ its augmentation ideal.

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Fresse's description in terms of iterated bar constructions gives a direct identification (in the commutative case over a field k) of $H_*^{E_n}(\bar{A})$ with $HH_{*+n}^{[n]}(A; k)$, that is Pirashvili's Hochschild homology of order n.

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'Proof' that $H_*^{E_n}(\bar{A}) \cong HH_{*+n}^{[n]}(A; k)$:

$$\begin{aligned} H_*^{E_n}(\bar{A}) &\cong H_*(\Sigma^{-n} B^n(\bar{A})) \cong H_{*+n} B^n(\bar{A}) \\ &\cong H_{*+n}(\mathbb{S}^n \bar{\otimes} A) \cong HH_{*+n}^{[n]}(A; k). \end{aligned}$$

The limit: Gamma homology

Fresse showed as well, that in the limiting case

$$H^{E_\infty}(\bar{A}) \cong H\Gamma_*(A;k).$$

Here, $H\Gamma_*(A;k)$ denotes Gamma homology of A with coefficients in k, as defined by Alan Robinson and Sarah Whitehouse.

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Gamma (co)homology plays an important role as the habitat for obstructions to E_∞-ring structures on ring spectra.

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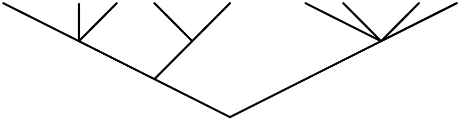
Here, I_n is the n-tree with only one leaf and Y_n is the tree that has two leaves at the top level.

The category Epi_n – an example



The category Epi_n – the definition

Objects are sequences

$$[r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1] \tag{1}$$

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A morphism to an object $[r'_n] \xrightarrow{f'_n} [r'_{n-1}] \xrightarrow{f'_{n-1}} \ldots \xrightarrow{f'_2} [r'_1]$ consists of surjective maps $\sigma_i \colon [r_i] \to [r'_i]$ for $1 \leq i \leq n$ such that σ_1 is order-preserving surjective and for all $2 \leq i \leq n$ the map σ_i is order-preserving on the fibres $f_i^{-1}(j)$ for all $j \in [r_{i-1}]$ and such that the diagram

$$\begin{tikzcd}
{[r_n]} \arrow[r, "f_n"] \arrow[d, "\sigma_n"] & {[r_{n-1}]} \arrow[r, "f_{n-1}"] \arrow[d, "\sigma_{n-1}"] & \cdots \arrow[r, "f_2"] & {[r_1]} \arrow[d, "\sigma_1"] \\
{[r'_n]} \arrow[r, "f'_n"] & {[r'_{n-1}]} \arrow[r, "f'_{n-1}"] & \cdots \arrow[r, "f'_2"] & {[r'_1]}
\end{tikzcd}$$

commutes.

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