A spectral sequence for the homology of a finite algebraic delooping

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joint work in progress with Stephanie Ziegenhagen

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2) Reduce this further to Quillen homology of graded Lie-algebras and of commutative algebras, aka André-Quillen homology.
3) Apply this for instance to the Hodge decomposition of higher order Hochschild homology (in the sense of Pirashvili).

E_n-homology

A resolution spectral sequence

A Blanc-Stover spectral sequence

Hodge decomposition

Little n-cubes

Let C_n denote the operad of little n-cubes.
Then $(C_* C_n(r))_r$, $r \geq 1$ is an operad in the category of chain complexes. Let E_n be a cofibrant replacement of $C_* C_n$.

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I.e., E_n-homology is the homology of an n-fold algebraic delooping.

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Can we gain information about $HH_*^{[n]}(A; k)$, at least rationally?
What is $H_*^{E_n}(\bar{A}_*)$ in other interesting cases such as Hochschild cochains, $A_* = C^*(B, B)$, or $A_* = C_*(\Omega^n X)$?

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Over $\mathbb{F}_2$: $n = 2$; for $\mathbb{Q}$: arbitrary n.

1-restricted Lie algebras

Definition

A 1-*restricted Lie algebra over* $\mathbb{F}_2$ is a non-negatively graded $\mathbb{F}_2$-vector space, $\mathfrak{g}_*$, together with two operations, a Lie bracket of degree one, $[-,-]$ and a restriction, ξ:

$$\begin{aligned} [-,-]\colon \quad & \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j+1}, \quad && i,j \geq 0, \\ \xi\colon \quad & \mathfrak{g}_i \to \mathfrak{g}_{2i+1} && i \geq 0. \end{aligned}$$

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These satisfy the relations

1. The bracket is bilinear, symmetric and satisfies the Jacobi relation

   $[a,[b,c]]+[b,[c,a]]+[c,[a,b]] = 0$ for all homogeneous $a,b,c \in \mathfrak{g}_*$.

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In particular, the bracket and the restriction annihilate squares: $[a, b^2] = 2b[a, b] = 0$ and $\xi(a^2) = 2a^2\xi(a) + a^2[a, a] = 0$. Thus if 1 denotes the unit of the algebra structure in G_*, then $[a, 1] = 0$ for all a and $\xi(1) = 0$.

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Note: $Q_{1rG}(G_*) = Q_{1rL}(Q_a(G_*))$.

Homology of free objects

Lemma

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Take X with $\bar{H}_*(X;\mathbb{F}_2) \cong H_*(\bar{A}_*)$, then

$$\begin{aligned}
H_*(E_2(\bar{A}_*)) &\cong \bigoplus_r H_*(E_2(r) \otimes_{\mathbb{F}_2[\Sigma_r]} H_*(\bar{A}_*)^{\otimes r}) \\
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&\cong \quad H_*(C_2(X);\mathbb{F}_2) \cong 1rG(H_*\bar{A}_*).
\end{aligned}$$

Resolution spectral sequence

Theorem

There is a spectral sequence

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{1rG}(H_*(\bar{A}_*)))_q \Rightarrow H^{E_2}_{p+q}(\bar{A}_*).$$

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d^1 takes homology wrt resolution degree.

Example

For X connected:
$(\mathbb{L}_p Q_{1rG}(H_*(C_*(\Omega^2\Sigma^2 X;\mathbb{F}_2)))_* = (\mathbb{L}_p Q_{1rG}(1rG(\bar{H}_*(X;\mathbb{F}_2)))_*.$

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This reduces to $\bar{H}_q(X;\mathbb{F}_2)$ in the $(p = 0)$-line and

$$H_q^{E_2}(\bar{C}_*(\Omega^2\Sigma^2 X;\mathbb{F}_2)) \cong \bar{H}_q(X;\mathbb{F}_2).$$

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$$H_*(E_{n+1}\bar{A}_*) \cong nG(H_*(\bar{A}_*)),$$

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We get:

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{nG}(H_*(\bar{A}_*)))_q \Rightarrow H^{E_{n+1}}_{p+q}(\bar{A}_*)$$

for every E_{n+1}-algebra $\bar{A}_*$ over the rationals.

General Blanc-Stover setting

Let $\mathcal{C}$ and $\mathcal{B}$ be some categories of graded algebras (*e.g.*, Lie, Com, n-Gerstenhaber etc.) and let $\mathcal{A}$ be a concrete category (such as graded vector spaces) and $T\colon \mathcal{C} \to \mathcal{B}$, $S\colon \mathcal{B} \to \mathcal{A}$.

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If TF is is S-acyclic for every free F in $\mathcal{C}$, then there is a Grothendieck composite functor spectral sequence for all C in $\mathcal{C}$

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- $\mathcal{B} = Com$: $\pi_*(B)$ has divided power operations. $\mathcal{B} = rLie$: $\pi_* B$ inherits a Lie bracket and has some extra operations.

In our case

Theorem

- $k = \mathbb{F}_2$: For any $C \in 1rG$:

$$E^2_{s,t} = \mathbb{L}_s((\bar{Q}_{1rL})_t)(AQ_*(C|\mathbb{F}_2, \mathbb{F}_2)) \Rightarrow \mathbb{L}_{s+t}(Q_{1rG})(C).$$

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- For $k = \mathbb{Q}$ we get for all n-Gerstenhaber algebras C:

$$\mathbb{L}_s((\bar{Q}_{nL})_t)(AQ_*(C|\mathbb{Q}, \mathbb{Q})) \Rightarrow \mathbb{L}_{s+t}(Q_{nG})(C).$$

Hodge decomposition for $k = \mathbb{Q}$

Let $HH_*^{[n]}(A; \mathbb{Q})$ denote Hochschild homology of order n or A with coefficients in $\mathbb{Q}$.
Over $\mathbb{Q}$, $HH_*^{[n]}(A; \mathbb{Q})$ has a decomposition, the *Hodge decomposition*:

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Over $\mathbb{Q}$, $HH_*^{[n]}(A; \mathbb{Q})$ has a decomposition, the *Hodge decomposition*:
Theorem [Pirashvili 2000] For odd n we obtain

$$HH_{\ell+n}^{[n]}(A; \mathbb{Q}) = \bigoplus_{i+nj=\ell+n} HH_{i+j}^{(j)}(A; \mathbb{Q}).$$

Here $HH_*^{(j)}(A; \mathbb{Q})$ is the j-th Hodge summand of ordinary Hochschild homology.

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Here, $\theta^j[n]$ is the dual of the $\mathbb{Q}$-vector space that is generated by the $S \subset \{1, \ldots, n\}$ with $|S| = j$.

Relationship to Taylor towers

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Theorem [R,2000]

$$\mathrm{Tor}_i^\Gamma(\theta^j, \mathcal{L}(A, \mathbb{Q})) \cong H_i(D_j(\mathcal{L}(A, \mathbb{Q}))[1])$$

where D_j is the jth homogenous piece in the Taylor tower of $\mathcal{L}(A, \mathbb{Q})$

$$D_j(\mathcal{L}(A, \mathbb{Q}))_* = \mathrm{cone}_{*+1}(P_j(\mathcal{L}(A, \mathbb{Q})) \to P_{j-1}(\mathcal{L}(A, \mathbb{Q})))$$

with

$$\ldots P_n(\mathcal{L}(A, \mathbb{Q})) \to P_{n-1}(\mathcal{L}(A, \mathbb{Q})) \to \ldots \to P_1(\mathcal{L}(A, \mathbb{Q})) \to \mathbb{Q}.$$

Hodge summands as Quillen homology of Gerstenhaber algebras

Theorem Let A be a commutative augmented $\mathbb{Q}$-algebra. For all $\ell, k \geq 1$ and $m \geq 0$:

- $$HH^{(\ell)}_{m+1}(A;\mathbb{Q}) \cong (\mathbb{L}_m Q_{2kG}\bar{A})_{(\ell-1)2k}.$$

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Idea of proof

First we prove a stability result

$$(\mathbb{L}_m Q_{nG}\bar{A})_{qn} \cong (\mathbb{L}_m Q_{(n+2)G}\bar{A})_{q(n+2)}.$$

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The remaining argument is just a matching of the decomposition pieces in the Hodge decomposition and the resolution spectral sequence.