Exercises in Algebra (master): Homological Algebra

Prof. Dr. Birgit Richter

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Exercise sheet no 8

for the exercise class on the 13th of June 2018

1 (Free groups)

Let F_n be again a free group on n generators for $n \ge 1$ and let \mathbb{Z} be the trivial F_n -module. Calculate $H^k(F_n; \mathbb{Z})$ and $H_k(F_n; \mathbb{Z})$ for all $k \ge 0$.

2 (Induced maps)

- (1) Let G be a group and let $\tilde{g} \in G$ be a fixed element. Consider the group homomorphism $c: G \to G$, $g \mapsto \tilde{g}g\tilde{g}^{-1}$. For a G-module M we consider the map $f: M \to M$, $f(m) = \tilde{g}m$. Show that (c, f) induces a self-map on $H_*(G; M)$ and prove that this map is trivial.
- (2) Let $G_1 = \mathbb{Z}$ and $G_2 = \mathbb{Z}/n$ for some $n \ge 2$ and consider $M = \mathbb{Z}$ as the trivial G_1 and G_2 -module. We denote by $\pi \colon \mathbb{Z} \to \mathbb{Z}/n$ the canonical projection. What is

$$H_1(\pi; \mathbb{Z}): H_1(G_1; \mathbb{Z}) \to H_1(G_2; \mathbb{Z})?$$

3 (Extensions)

- (1) How many extensions are there of $G = \mathbb{Z}/2$ by $M = \mathbb{Z}/3$? What does that say about $H^2(\mathbb{Z}/2; \mathbb{Z}/3)$?
- (2) Let k be a field and let $n \ge 2$ be an integer. We consider the projective linear group $PGL_n(k)$ which is defined by the short exact sequence

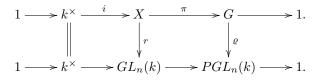
$$1 \to k^{\times} \hookrightarrow GL_n(k) \twoheadrightarrow PGL_n(k) \to 1.$$

Here, the units of k, k^{\times} , are embedded into $GL_n(k)$ as the diagonal copy of k^{\times} in $GL_n(k)$. A projective representation of G is a group homomorphism $\varrho: G \to PGL_n(k)$. Consider the diagram

Prove that its pullback

$$\begin{array}{c|c} X & \xrightarrow{\pi} & G \\ & & & \downarrow^{\rho} \\ GL_n(k) & \xrightarrow{\pi} & PGL_n(k) \end{array}$$

fits in a commutative diagram



whose rows are extensions. In particular, ρ gives rise to a representation $r: X \to GL_n(k)$. (So if $H^2(G; k^{\times}) = 0$, then the upper extension splits and we get a representation of G, so we can lift a projective representation to an 'honest' representation of G.)