Exercises in Algebra (master): Homological Algebra

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Summer term 2018

Exercise sheet no 6 for the exercise class on the 30th of May 2018

1 (Tor-calculations)
Let A and B be finitely generated abelian groups. Describe $\mathrm{Tor}_n^{\mathbb{Z}}(A, B)$ for all $n \geqslant 0$.

2 (5-Lemma) Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5.
\end{array}$$

Under which assumptions on f_1, f_2, f_4, f_5 can we deduce that the map f_3 is a monomorphism or an epimorphism? In particular, what assumptions are needed in order to ensure that f_3 is an isomorphism?

3 (Snake Lemma) `https://www.youtube.com/watch?v=etbcKWEKnvg`

(1) Use the long exact sequence in homology to prove the weak form of the Snake Lemma: If

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \\
0 & \longrightarrow & N' & \xrightarrow{h} & N & \xrightarrow{k} & N'' & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \longrightarrow \mathrm{coker}(\alpha) \longrightarrow \mathrm{coker}(\beta) \longrightarrow \mathrm{coker}(\gamma) \longrightarrow 0.$$

Specify the maps.

(2) Prove the stronger form: If

$$\begin{array}{ccccccccc}
 & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \\
0 & \longrightarrow & N' & \xrightarrow{h} & N & \xrightarrow{k} & N'' & &
\end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \longrightarrow \mathrm{coker}(\alpha) \longrightarrow \mathrm{coker}(\beta) \longrightarrow \mathrm{coker}(\gamma).$$