Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 4 for the exercise class on the 9th of May 2018

1 (Quasi-isomorphisms)

(1) Is "being quasi-isomorphic" an equivalence relation on chain complexes?
(2) Let k be a field and consider chain complexes in k-vector spaces, $\mathsf{Ch}(k)$. For a $C_* \in \mathsf{Ch}(k)$ we consider $H_*(C_*)$ as a chain complex with zero differential. Prove that every $C_* \in \mathsf{Ch}(k)$ is quasi-isomorphic to $H_*(C_*)$.

2 (Projective and injective modules over group algebras)

Let G be a group and consider $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module, *i.e.*, every $g \in G$ acts as the identity on $\mathbb{Z}$.

(1) For which G is $\mathbb{Z}$ a projective $\mathbb{Z}[G]$-module?
(2) Let P be a projective $\mathbb{Z}[G]$-module. Show that P is also projective as an $\mathbb{Z}[H]$-module for all subgroups $H < G$.
(3) Assume now that G is finite. Prove that every module over $\mathbb{Q}[G]$ is projective. Does that also hold for injectivity? Do you really need $\mathbb{Q}$ or what is a condition on k such that the same thing works for $k[G]$?

3 (Cyclic groups)

(1) Let C_n be the cyclic group with n elements and let t be its multiplicative generators, *i.e.*, $C_n = \langle t | t^n = 1 \rangle$. Set $N := \sum_{i=0}^{n-1} t^i$. This is often called the *norm element*. Prove that in $\mathbb{Z}[C_n]$

$$N(1-t) = (1-t)N = 0.$$

(2) Construct a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[C_n]$-module. Here, the C_n-action on $\mathbb{Z}$ is trivial.