Exercises in Algebra (master): Homological Algebra

Prof. Dr. Birgit Richter

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Exercise sheet no 3 for the exercise class on the 2nd of May 2018

1 (Kernels and friends) Let $\mathcal{C}$ be an abelian category and let $f \in \mathcal{C}(A, B)$.

(1) Show that
- the kernel of f is a monomorphism and
- the cokernel of f is an epimorphism.

(2) Prove that the image of f is a kernel of the cokernel of f and that the coimage is a cokernel of the kernel of f.

2 ((Co)Products in poset categories) Let $(X, \leqslant)$ be a poset and consider the associated category.
- What is an initial object in $(X, \leqslant)$ and what is a terminal object in $(X, \leqslant)$?
- What is a coproduct or product in $(X, \leqslant)$? Find examples where these do not exist.

3 (Initial, terminal and zero objects)
- If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor, does F map the zero object of $\mathcal{C}$, $0_{\mathcal{C}}$, to a zero object of $\mathcal{D}$, $0_{\mathcal{D}}$?
- Identify the initial and terminal objects in the following categories (if they exist):
  (1) Sets, the category of sets and functions,
  (2) Gr, the category of groups and homomorphisms of groups,
  (3) Rings, the category of rings and morphisms of rings,
  (4) R-mod, the category of R-modules, for R an arbitrary ring, and R-linear maps,
  (5) $\mathcal{E}_G$, the translation category of G. Here G is a group and we let the objects of $\mathcal{E}_G$ be the elements of G. As morphisms we have
$$\mathcal{E}_G(g, h) = \{hg^{-1}\}.$$