Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 2 for the exercise class on the 25th of April 2018

1 (Project modules and dual bases)

Let R be a ring. Prove that an R-module P is projective if and only if there exist elements $p_i, i \in I$, with $p_i \in P$ for some indexing set I and R-linear maps $\varphi_i \in \mathsf{Hom}_R(P, R)$ for $i \in I$ such that

- For all $x \in P$, $\varphi_i(x) = 0$ for almost all $i \in I$.
- For all $x \in P$, $x = \sum_{i \in I} \varphi_i(x) p_i$.

2 (Morita equivalence) Let R_1 and R_2 be two rings. Let M be a left R_1-module and a right R_2-module. Recall that we call M an R_1-R_2-bimodule if for all $r_1 \in R_1$, $r_2 \in R_2$ and all $m \in M$

$$(r_1 m) r_2 = r_1 (m r_2).$$

The rings R_1 and R_2 are called *Morita equivalent*, if there is an R_1-R_2-bimodule P and an R_2-R_1-bimodule Q such that $Q \otimes_{R_1} P \cong R_2$ as R_2-R_2-bimodules and $P \otimes_{R_2} Q \cong R_1$ as R_1-R_1-bimodules.

- Show that P is projective as a left R_1-module and as a right R_2-module.
- Prove that any ring R is Morita equivalent to the ring $M_n(R)$ of $n \times n$-matrices over R.

3 (Injective abelian groups)

- Prove that an abelian group A is injective if and only if it is divisible, *i.e.*, for all $0 \neq n \in \mathbb{Z}$ the map $\cdot n \colon A \to A$ is surjective.
- Let A be a finitely generated abelian group. Construct an injective abelian group I together with a monomorphism $i \colon A \to I$. (Such injective modules together with such monomorphisms exist in broader generality.)